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Curve Shortening and its Application to Multi-Agent Systems

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***Abstract*— If a smooth, closed, and embedded curve is deformed along its normal vector field at a rate proportional to its curvature, it shrinks to a circular point. This curve evolution is called Euclidean curve shortening, and the result is known as the Gage-Hamilton-Grayson Theorem. Motivated by the rendezvous problem in multi-agent systems, we address the problem of creating a polygon shortening flow. A simple linear scheme is proposed that exhibits several properties similar to Euclidean curve shortening. The polygon shrinks to an elliptical point; convex polygons remain convex; and, the perimeter of the polygon is monotonically decreasing.**

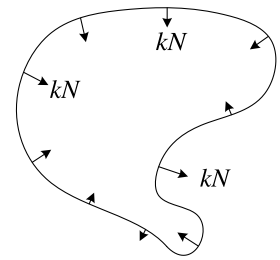

Fig. 1. The Euclidean curve shortening flow.

I. INTRODUCTION

A problem which has received considerable interest within the multi-agent systems literature is called the *rendezvous*, *consensus*, or *agreement* problem. The general problem is to develop a local control strategy for a group of agents such that they achieve consensus, or agreement, on some information. A version of this problem that is commonly studied is the following; given a group of n agents (vehicles) whose positions are represented in the complex plane by $z_i = x_i + jy_i$, $i = 1, \dots, n$, $j = \sqrt{-1}$, and whose dynamics are given by $\dot{z}_i = u_i$, find a local control strategy that will ensure convergence of all z_i's to a point. Solutions to this problem have been developed for both the case where the communication topology is fixed, and where it is time-varying (see [22] and the references therein). In this paper we propose a strategy which solves this problem with a fixed communication topology. The strategy is motivated by the theory of curve shortening. Because of this, the formation of the group of agents as they converge to a common point has properties which are analogous to the curve shortening theory.

Consider a family of smooth, closed curves $\mathbf{x}(p, t)$ lying in the plane. Here, p parameterizes the points along each individual curve, and t parameterizes the family (i.e., the initial curve $\mathbf{x}(p, 0)$ evolves as a function of time to $\mathbf{x}(p, t)$). The *Euclidean curve shortening* flow is given by

$$\frac{\partial \mathbf{x}}{\partial t}(p, t) = k(p, t)\mathbf{N}(p, t), \qquad (1)$$

where $k(p, t)$ is the Euclidean curvature and $\mathbf{N}(p, t)$ is the inner Euclidean normal. Intuitively, the curvature at a point on a curve is the inverse of the radius of the largest tangent circle to the curve (on the concave side) at the point. The Euclidean curve shortening flow is depicted in Fig. 1. We also define $L(t)$ to be the length of the curve at time t and $A(t)$ the area enclosed by the curve. The isoperimetric inequality [19] states that

$$\frac{L(t)^2}{A(t)} \geq 4\pi.$$

Equality is achieved if and only if the curve is a circle. Therefore, the ratio L^2/A gives a measure of "how circular" the curve is. In 1983, Gage [9] showed that when a convex curve evolves according to (1), $A(t) \to 0$, and the ratio L^2/A is decreasing. In 1984 Gage [10] showed that if the curvature does not blow up prematurely (i.e., a cusp does not form) then, for a convex curve, $L^2/A \to 4\pi$ and the curve shrinks to a circular point. The term "circular point" means that the curve is collapsing to a point, and if we zoom in on the curve as it is collapsing, the curve is becoming circular. In 1986 Gage and Hamilton [11] showed that for convex curves the curvature does not blow up prematurely, and in 1987 Grayson [12] showed that any embedded (non-self-intersecting) curve shrinks to a circular point. The Euclidean curve shortening flow also has the property that it shrinks the length of the curve $L(t)$ as fast as possible using only local information [13]. The notion of shrinking "as fast as possible" will be clarified later.

The Euclidean curve shortening flow is defined in terms of the Euclidean curvature and the Euclidean normal. These quantities are invariant under Euclidean transformations (i.e., rotations, translations, and reflections). Sapiro and Tannenbaum [21] created a new curve shortening flow which is defined in terms of quantities that are invariant under affine transformations. We say that a flow is invariant under a transformation if the flow and transformation commute. This flow, which is called *affine curve shortening*, has applications in image processing [5] and computer vision [4]. In [21] it is shown that if a smooth convex curve evolves according to the affine curve shortening flow, the curve shrinks to an elliptical point. In [1] this result is extended to smooth embedded curves. For a complete account of many of the results of curve shortening see [6].

The elegant results obtained in the curve shortening literature have motivated research in creating discrete analogues of the flows; that is, to create a shortening flow for polygons which exhibits similar attributes to the Euclidean (or affine) curve shortening flow. This research has been driven by both theoretical interest [3], [15], [18], and by applications such as crystal growth [20]. In [3], the evolution of planar polygons is studied in discrete time, and an affine polygon shortening scheme is proposed which shrinks polygons to elliptical points. This means that the vertices are collapsing to a point, and if we zoom in on the collapsing polygon, the vertices are converging to an ellipse. In addition, [3] proposes a Euclidean polygon shortening scheme based on the Menger-Melnikov curvature [17]. In [15] this scheme is studied and it is shown that most quadrilaterals shrink to circular points. In [18] a discrete curve shortening equation is formulated such that the the area inclosed by the polygon shrinks at a rate of 2π (the same as in Euclidean curve shortening). It is also shown that the perimeter of the polygon is monotonically decreasing.

Unlike in [15], [18], where nonlinear schemes are developed that approach (1) as the polygon tends to a smooth curve, we attempt to achieve analogues to the curve shortening theory using a linear scheme. By exploiting the linear structure we are able to obtain analytical results that mimic those of the Gage-Hamilton-Grayson Theorem. We study a polygon consisting of vertices $z_1, \dots, z_n$ as it evolves according to

$$\dot{z}_i = \frac{1}{2}(z_{i+1} - z_i) + \frac{1}{2}(z_{i-1} - z_i). \tag{2}$$

Intuitively, (2) dictates that each vertex chases the centroid of its two neighboring vertices. Notice that each agent need only compute the relative position of its two neighbors, and thus this scheme is based on local information. This scheme is studied in discrete time in [3], where it is referred to as an affine polygon shortening scheme. We show the following properties of this scheme: 1) Polygons shrink to elliptical points, 2) Convex polygons remain convex, 3) If vertices are arranged in a star formation about their centroid, they will remain in a star formation for all time (i.e, the vertices (agents) will not collide) 4) The perimeter of the polygon is a monotonically decreasing function of time.

This paper is organized as follows. In Section II we give a more detailed development of Euclidean curve shortening. In Section III we introduce the linear scheme given in (2). In Sections IV and V we show that under (2) star formations are invariant, and convex polygons remain convex. Finally, in Section VI we derive the optimal direction for perimeter shortening and show that under (2) the perimeter of the polygon is monotonically decreasing. Most proofs are omitted due to space limitations. However, a full paper is currently in preparation.

II. Background

In this section we will give some background on Euclidean geometry and derive the Euclidean curve shortening flow. The sections on Euclidean geometry and Euclidean curve shortening follow the development of [2] and [14] respectively.

A. Euclidean geometry

A *Euclidean transformation* of $\mathbb{R}^2$ is a function $L : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$L(\mathbf{x}) = U\mathbf{x} + a,$$

where U is an orthogonal 2×2 matrix and $a \in \mathbb{R}^2$. Recall that a matrix U is orthogonal if $U^{-1} = U^T$ (where ^{T} denotes transpose), which is equivalent to saying that the columns of U are orthonormal. The set of all Euclidean transformations of $\mathbb{R}^2$ is denoted $E(2)$ and is the set of all rotations, translations, and reflections of a figure in $\mathbb{R}^2$. Roughly speaking, *Euclidean geometry* is the study of properties of figures which remain unchanged by Euclidean transformations. These properties are called *Euclidean properties*, and include distance, angle, curvature, and collinearity of points. From this we can introduce the concept of congruence.

Definition 1: A figure F_1 is *Euclidean-congruent* to a figure F_2 if there is a Euclidean transformation which maps F_1 onto F_2.

A few examples of sets of figures which are Euclidean-congruent to each other are: The set of all line segments of a fixed length, and the set of all squares of a fixed area. It can easily be verified that Euclidean-congruence is an equivalence relation, and thus the previous sets are equivalence classes.

B. Euclidean curve shortening

Consider a family of smooth closed curves $\mathbf{x}(p, t) : [0, 1] \times [0, \tau] \to \mathbb{R}^2$, where $p \in [0, 1]$ parameterizes the curve and $t \in [0, \tau]$ the family. For now we fix t and study a single curve $\mathbf{x}(p)$. The tangent vector to the curve is given by $d\mathbf{x}/dp =: \dot{\mathbf{x}}$ and thus we define the unit tangent as $\mathbf{T}(p) := \dot{\mathbf{x}}/\|\dot{\mathbf{x}}\|$. Introducing coordinates in $\mathbb{R}^2$ we can write $\mathbf{x}(p) = (x_1(p), x_2(p))$. The unit tangent is $(\dot{x}_1, \dot{x}_2)/\|\dot{\mathbf{x}}\|$ and the unit normal is then given by $\mathbf{N}(p) := (-\dot{x}_2, \dot{x}_1)/\|\dot{\mathbf{x}}\|$. When $\mathbf{T}$ runs in the counterclockwise direction around the curve, $\mathbf{N}$ is the inner unit normal.

It is convenient and customary to use arc-length to describe distance around the curve instead of p. The Euclidean arc-length s is defined via $ds := \|\dot{\mathbf{x}}\|dp$. We can re-parameterize the curve by s as $\mathbf{x}(s) = (x_1(s), x_2(s))$. The unit tangent and normal vectors can be written in terms of s as

$$\mathbf{T}(s) = (x_1', x_2'), \quad \text{and} \quad \mathbf{N}(s) = (-x_2', x_1'),$$

where $'$ denotes differentiation with respect to s. Using column vector notation we define

$$A(s) := \begin{bmatrix} \mathbf{T}^T \\ \mathbf{N}^T \end{bmatrix} = \begin{bmatrix} x_1' & x_2' \\ -x_2' & x_1' \end{bmatrix}.$$

The matrix $A(s)$ is a rotation matrix which rotates the standard basis to the coordinate frame $\mathbf{T}, \mathbf{N}$ attached to the curve. Differentiating $A(s)$ we get

$$A' = A'A^{-1}A =: C(s)A. \tag{3}$$

Now, since A is a rotation matrix it is orthogonal and so $A^T A = AA^T = I$. Using this, we can show that $C(s)$ is skew-symmetric as follows. Differentiating the expression $I = AA^T$ we obtain

$$0 = (AA^T)' = A'A^T + A(A^T)' = A'A^{-1} + (A^{-1})^T (A')^T = A'A^{-1} + (A'A^{-1})^T,$$

which implies that $C(s) = -C(s)^T$. Computing $C(s)$ and using the fact that it is a skew symmetric matrix we get

$$C(s) = \begin{bmatrix} 0 & k(s) \\ -k(s) & 0 \end{bmatrix}, \tag{4}$$

where the curvature $k(s)$ is given by

$$k(s) := x_1' x_2'' - x_1'' x_2' = \det(\mathbf{x}', \mathbf{x}''),$$

and $\det(\cdot, \cdot)$ denotes the determinant of the 2×2 matrix created by the two 2×1 vectors. Since $ds = \|\dot{\mathbf{x}}\| dp$ we can also write k in terms of the parameter p as $k(p) = (\dot{\mathbf{x}}, \ddot{\mathbf{x}})/\|\dot{\mathbf{x}}\|^3$. From (3) and (4) we obtain the Frenet equation:

$$\frac{d\mathbf{T}}{ds} = k\mathbf{N}, \quad \frac{d\mathbf{N}}{ds} = -k\mathbf{T}.$$

The *curvature* of the curve $\mathbf{x}(s)$ is given by $k(s)$ and the *radius of curvature* is defined to be $1/|k(s)|$.

In the Euclidean curve shortening flow the curve $\mathbf{x}(p,t)$ is deformed along its unit normal vector $\mathbf{N}(p,t)$ at a rate proportional to its curvature $k(p,t)$. From (1) this can be written as

$$\frac{\partial \mathbf{x}}{\partial t}(p,t) = k(p,t)\mathbf{N}(p,t).$$

Using the Frenet equation we have $k\mathbf{N} = d\mathbf{T}/ds = d^2\mathbf{x}/ds^2$. Therefore, the Euclidean curve shortening flow can be written as

$$\frac{\partial \mathbf{x}}{\partial t}(p,t) = \frac{\partial^2 \mathbf{x}}{\partial s^2}(p,t). \tag{5}$$

This equation is called the *heat equation* or *diffusion equation*. (Note: on the right hand side of (5), p is a function of s.)

C. Shrinking the length optimally

In [13] it is stated that the length, $L(t)$, of a curve which is evolving according to the Euclidean curve shortening flow is shrinking as fast as possible using only local information. To see why and in what sense this is true, consider the following. We can write the length at a fixed time t as

$$L(t) = \int_0^{L(t)} ds = \int_0^1 \left\| \frac{\partial \mathbf{x}}{\partial p} \right\| dp, \tag{6}$$

To take the time derivative of this expression we differentiate $\|\partial \mathbf{x}/\partial p\|^2$ and obtain

$$\frac{\partial}{\partial t} \left\| \frac{\partial \mathbf{x}}{\partial p} \right\| = \frac{1}{\|\partial \mathbf{x}/\partial p\|} \left\langle \frac{\partial \mathbf{x}}{\partial p}, \frac{\partial}{\partial p}\frac{\partial \mathbf{x}}{\partial t} \right\rangle.$$

Substituting this into $dL(t)/dt$ and integrating by parts, the following expression can be obtained:

$$\frac{dL}{dt} = -\int_0^L \left\langle k\mathbf{N}, \frac{\partial \mathbf{x}}{\partial t} \right\rangle ds. \tag{7}$$

Therefore, the direction of $\partial \mathbf{x}/\partial t$ in which $L(t)$ is decreasing most rapidly is $\partial \mathbf{x}/\partial t = k\mathbf{N}$, which is Euclidean curve shortening (see (1)). Note that this flow is optimal in the sense that the velocity of the curve at each point always points in the direction which maximizes the rate of decrease of $L(t)$. However, the magnitude of the velocity of the curve at each point is not in general the speed which maximizes the rate of decrease of $L(t)$.

III. POLYGON SHORTENING

We can consider a group of n agents lying in the plane to be the vertices of an n-sided polygon. We then attempt to create a polygon shortening scheme analogous to that of curve shortening. With an accurate analogue, the vertices, and thus the agent's positions, will converge to a point. In addition, the shape of the polygon as it shrinks to a point will have properties similar to the curve shortening theory. In this section we will formally define a polygon and introduce the polygon shortening scheme.

A. n-gons

We begin by formally defining a polygon and a non-self-intersecting polygon in $\mathbb{R}^2$ (or equivalently $\mathbb{C}$), which we will refer to as an n-gon and a simple n-gon respectively [7].

Definition 2: An *n-gon* (n-sided polygon) is a (possibly intersecting) circuit of n line segments $z_1z_2, z_2z_3, \ldots, z_nz_1$, joining consecutive pairs of n distinct points $z_1, z_2, \ldots, z_n$. The segments are called *sides* and the points are called *vertices*.

Definition 3: A *simple n-gon* is a non-self-intersecting n-gon.

We denote the counterclockwise *internal angle* between consecutive sides z_iz_{i+1} and $z_{i-1}z_i$ of an n-gon as β_i, where $i = 1, \ldots, n$ modulo n. For a simple n-gon these angles satisfy $\sum_{i=1}^n \beta_i = (n-2)\pi$.

Definition 4: An n-gon is *convex* if it is simple and its internal angles satisfy $0 < \beta_i \leq \pi$, $\forall i = 1, \ldots, n$.

Definition 5: An n-gon is *strictly convex* if it is simple and its internal angles satisfy $0 < \beta_i < \pi$, $\forall i = 1, \ldots, n$.

B. Linear Scheme

In this section we will introduce the linear polygon shortening scheme which will be the focus of the remainder of the paper. We will be able to exploit the linear structure of this scheme to demonstrate several of its polygon shortening properties. The scheme can be described as follows. A group of n agents are modeled as point masses and numbered from 1 to n. The position of each agent can be described in the complex plane by the point $z_i = x_i + jy_i$, $i = 1, \ldots, n$. These agents make up the vertices of an n-gon. The strategy is for agent i to chase the centroid of agents $i-1$ and $i+1$. The i^{th} agent's velocity points in the direction of the centroid of its neighbors and the magnitude of the velocity is equal to the distance from agent i to the centroid. This scheme is described in (2), where all indices are evaluated modulo n.

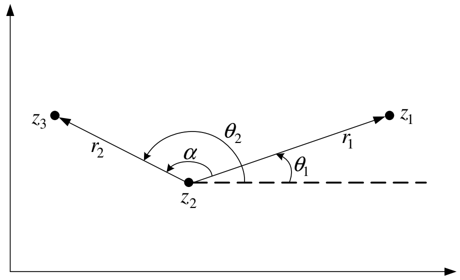

Fig. 2. The setup for the definition of the function F.

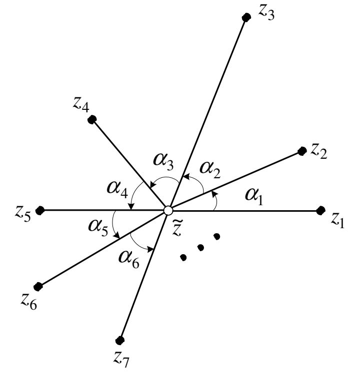

Fig. 3. A counterclockwise star formation.

This system can also be written in the form $\dot{z} = Az$ where the matrix A is circulant (see [8]) and is given by

$$A = \operatorname{circ}\left(-1, \frac{1}{2}, 0, \ldots, 0, \frac{1}{2}\right).$$

The matrix A can be written in terms of the polynomial

$$q_A(s) = \frac{1}{2}s^{n-1} + \frac{1}{2}s - 1,$$

and the matrix $P = \operatorname{circ}(0, 1, 0, \ldots, 0)$, as $A = q_A(P)$. By the spectral mapping theorem we obtain

$$\operatorname{eigs}(A) = \{q_A(1), q_A(\omega), q_A(\omega^2), \ldots, q_A(\omega^{n-1})\},$$

where $\omega = \mathrm{e}^{2\pi j/n}$. Therefore, denoting $\lambda_i := q_A(\omega^{i-1})$, we have $\operatorname{eigs}(A) = \{\lambda_i : i = 1, \ldots, n\}$. Evaluating λ_i we get

$$\begin{aligned}\lambda_i &= \frac{1}{2}(\mathrm{e}^{2\pi j(n-1)(i-1)/n} + \mathrm{e}^{2\pi j(i-1)/n}) - 1 \\ &= \cos(2\pi(i-1)/n) - 1,\end{aligned}$$

where $i = 1, \ldots, n$. Hence, the eigenvalues of A are real, with one eigenvalue at zero, and all others on the negative real line. The zero eigenvalue dictates that the agents converge to their stationary centroid rather than to the origin.

The following theorem describes the geometrical shape of the points, $z_i(t)$, as they converge to their centroid. This theorem is proved for discrete time in [3].

Theorem 6: Consider n points, $z_1(t), \ldots, z_n(t)$ evolving in the complex plane according to (2). As $t \to \infty$ these points converge to an ellipse. That is, $z_1(t), \ldots, z_n(t)$ collapse to an elliptical point.

IV. Star formations

In this section we will show that a group of agents, arranged in a star formation about their centroid (see Figure 3), remain in a star formation for all time. We require some preliminary tools, which are introduced in the following Lemmas.

Lemma 7 (Lin et al. [16]): Let z_1, z_2, and z_3 be three points in the complex plane, as shown in Fig. 2. Let $r_1 = |z_1 - z_2|$, $r_2 = |z_3 - z_2|$ and

$$F = \Im\{\overline{(z_1 - z_2)}(z_3 - z_2)\}.$$

Then (i) $0 < \alpha < \pi$, $r_1 > 0$, and $r_2 > 0$ iff $F > 0$. (ii) $\pi < \alpha < 2\pi$, $r_1 > 0$, and $r_2 > 0$ iff $F < 0$. (iii) the points are collinear iff $F = 0$.

Proof: Introducing the polar form

$$z_1 - z_2 = r_1 \mathrm{e}^{j\theta_1}, \quad z_3 - z_2 = r_2 \mathrm{e}^{j\theta_2}$$

where θ_1, θ_2 are the angles shown in Fig 2. Then

$$\begin{aligned}F = \Im\{\overline{(z_1 - z_2)}(z_3 - z_2)\} &= \Im\{r_1 \mathrm{e}^{-j\theta_1} r_2 \mathrm{e}^{j\theta_2}\} \\ &= r_1 r_2 \sin(\alpha)\end{aligned}$$

Thus, $0 < \alpha < \pi$, $r_1 > 0$, and $r_2 > 0$ iff $F > 0$; and $\pi < \alpha < 2\pi$, $r_1 > 0$, and $r_2 > 0$ iff $F < 0$. Also, the points are collinear iff $F = 0$. ■

Now consider our system of n agents, whose positions, not all collinear, are denoted by $z_1, \ldots, z_n$. Let $\tilde{z}$ be the centroid and r_i be the distance from the centroid to z_i. Let α_i denote the counterclockwise angle from $\tilde{z}z_i$ to $\tilde{z}z_{i+1}$ for $i = 1, \ldots, n-1$, and α_n denote the counterclockwise angle from $\tilde{z}z_n$ to $\tilde{z}z_1$.

Definition 8 (Lin et al. [16]): The n points are said to be arranged in a counterclockwise star formation if $r_i > 0$ and $0 < \alpha_i < \pi$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = 2\pi$. They are said to be arranged in a clockwise star formation if $r_i > 0$ and $-\pi < \alpha_i < 0$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = -2\pi$.

This formation is shown in Fig. 3. In what follows we will consider only counterclockwise star formations, since the treatment for clockwise star formations is analogous. Also, the case $n = 2$ is trivial, so it is omitted.

Lemma 9 (Lin et al. [16]): If n points $z_1, \ldots, z_n$, which are evolving according to (2), are collinear at some time t_1, then they are collinear for all $t < t_1$ and $t > t_1$.

Theorem 10: Suppose that n distinct points, with $n > 2$, are initially arranged in a counterclockwise star formation. If these points evolve according to (2) they will remain in a counterclockwise star formation for all time.

Proof: Sketch of Proof. We begin by considering the function

$$\begin{aligned}F_i(t) &= \Im\{\overline{(z_i(t) - \tilde{z})}(z_{i+1}(t) - \tilde{z})\} \\ &= r_i r_{i+1} \sin(\alpha_i).\end{aligned}$$

By the definition of a counterclockwise star formation we have $r_i(0) > 0$, and $0 < \alpha_i(0) < \pi$, $\forall i$. Hence by Lemma 7, $F_i(0) > 0$, $\forall i$. We want to show that $F_i(t) > 0$, $\forall i$

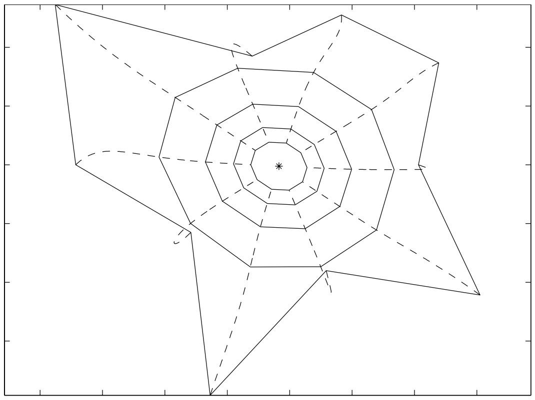

Fig. 4. The evolution of a polygon whose vertices start in a star formation about their centroid $*$. The dashed lines show the trajectories of each vertex.

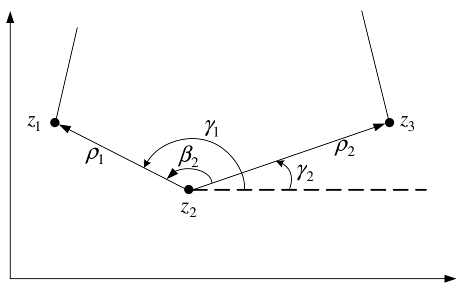

Fig. 5. The setup for the definition of the function H.

and $\forall t$, which by Lemma 7 shows that the vertices are in a counterclockwise star formation for all time.

Suppose by way of contradiction that t_1 is the first time that an F_i becomes zero. We can select $i = m$ such that $F_m(t_1) = 0$, and $F_{m+1}(t_1) > 0$, for if all the F_i's are zero at t_1, then the points are collinear, which by Lemma 9 is a contradiction. Hence, we have $F_i(t) > 0$, $\forall\ t \in [0, t_1)$ and $i = 1, \ldots, n$, $F_m(t_1) = 0$, and $F_{m+1}(t_1) > 0$. Now, if $F_m(t_1) = 0$, by Lemma 7, one of the four following conditions must be satisfied. (i) $\alpha_m(t_1) = \pi$ and $r_m(t_1), r_{m+1}(t_1) > 0$. (ii) $\alpha_m(t_1) = 0$ and $r_m(t_1), r_{m+1}(t_1) > 0$. (iii) $r_m(t_1) = 0$. (iv) $r_{m+1}(t_1) = 0$. The proof then follows by showing that a contradiction results if any of these four conditions are satisfied. ■

Fig. 4 shows the evolution of a polygon which starts in a star formation about its centroid. Notice that the polygon remains in a star formation, becomes convex, and collapses to an elliptic point.

V. Convex stays convex

In this section we will show that as a convex n-gon evolves according to (2), it remains convex. To do this we require a function similar to that in Lemma 7 which measures the counterclockwise internal angle between two sides of a n-gon.

Lemma 11: Consider a simple n-gon lying in the complex plane, whose vertices z_i are numbered counterclockwise around the n-gon. Let z_1, z_2, and z_3 be three vertices of the

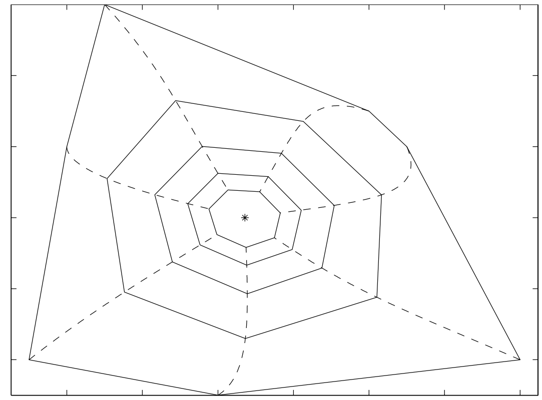

Fig. 6. The evolution of a convex n-gon. The dashed lines show the trajectories of each vertex.

n-gon as shown in Fig. 5. Let β_2 denote the counterclockwise angle from the side $z_2 z_3$ to the side $z_1 z_2$, and define $\rho_1 = |z_1 - z_2|$, $\rho_2 = |z_3 - z_2|$, and $H = \Im\{(z_1 - z_2)\overline{(z_3 - z_2)}\}$. Then (i) $0 < \beta_2 < \pi$, $\rho_1 > 0$, and $\rho_2 > 0$ iff $H > 0$. (ii) $\pi < \beta_2 < 2\pi$, $\rho_1 > 0$, and $\rho_2 > 0$ iff $H < 0$. (iii) the points are collinear iff $H = 0$.

This is proved in the same manner as Lemma 7.

Lemma 12: If an n-gon is convex, with its vertices z_i, $i = 1, \ldots, n$, numbered counterclockwise around the n-gon, then these vertices are in a counterclockwise star formation about their centroid.

Theorem 13: Consider a strictly convex n-gon at time $t = 0$ whose vertices z_i, $i = 1, \ldots, n$, are numbered counterclockwise around the n-gon. If these vertices evolve according to (2), the n-gon will remain strictly convex for all time.

The proof of this theorem proceeds in the same manner as the proof of Theorem 10 by using the function H_i. It also utilizes the fact that by Theorem 10 and Lemma 12 a strictly convex n-gon is in a star formation and thus remains in a star formation for all time.

Corollary 14: Consider an n-gon which is convex at $t = 0$. If the vertices evolve according to (2), then for any $t > 0$, the n-gon will be strictly convex.

This proof follows directly from the proof of Theorem 13. Fig. 6 shows the evolution of a convex n-gon. Notice that the polygon remains convex and collapses to a point.

VI. Shrinking a polygon's perimeter

Given a polygon with vertices $z_1, \ldots, z_n$ and sides $z_1 z_2, \ldots, z_n z_1$ we can write the perimeter of the polygon as

$$P(t) = \sum_{i=1}^{n} |z_{i+1} - z_i|. \tag{8}$$

We would like to compute an expression for $\dot{P}(t)$ analogous to that in Section II-C. In order to take the time derivative of this expression consider taking the derivative of $|z_{i+1} -$

$z_i|^2 = \langle z_{i+1} - z_i, z_{i+1} - z_i \rangle$, which yields

$$\begin{aligned}\frac{d}{dt}|z_{i+1} - z_i|^2 &= \frac{d}{dt}\langle z_{i+1} - z_i, z_{i+1} - z_i\rangle \\ &= 2\Re\left\{\langle z_{i+1} - z_i, \dot{z}_{i+1} - \dot{z}_i\rangle\right\}.\end{aligned}$$

But also,

$$\frac{d}{dt}|z_{i+1} - z_i|^2 = 2|z_{i+1} - z_i|\frac{d}{dt}|z_{i+1} - z_i|.$$

Letting $\dot{z}_i = u_i$ for $i = 1, \ldots, n$ and rearranging we have

$$\frac{d}{dt}|z_{i+1} - z_i| = \Re\left\{\left\langle \frac{z_{i+1} - z_i}{|z_{i+1} - z_i|}, u_{i+1} - u_i\right\rangle\right\}.$$

Therefore

$$\dot{P}(t) = \sum_{i=1}^{n} \Re\left\{\left\langle \frac{z_{i+1} - z_i}{|z_{i+1} - z_i|}, u_{i+1} - u_i\right\rangle\right\}.$$

Since all indices are evaluated modulo n this can be rewritten as

$$\dot{P}(t) = -\sum_{i=1}^{n} \Re\left\{\left\langle \frac{z_{i-1} - z_i}{|z_{i-1} - z_i|} + \frac{z_{i+1} - z_i}{|z_{i+1} - z_i|}, u_i\right\rangle\right\}. \quad (9)$$

To maximize the rate of decrease of $P(t)$ the two vectors in the inner product must point in the same direction. This implies that u_i should point in the direction of

$$\frac{z_{i-1} - z_i}{|z_{i-1} - z_i|} + \frac{z_{i+1} - z_i}{|z_{i+1} - z_i|}.$$

That is, u_i should point in the direction which bisects the internal angle β_i of the polygon. In general, neither the linear scheme (2) nor the shortening by Menger-Melnikov curvature point in this direction. However, this direction does not ensure that the polygon becomes circular (or elliptical); in simulation, adjacent vertices may capture each other and the polygon may collapse to a line.

Using (9) and the linear scheme (2) we can determine $\dot{P}(t)$. For $\dot{P}(t)$ to be defined we require that adjacent vertices be distinct. This is ensured, for example, if the vertices start in a star formation about their centroid.

Theorem 15: Consider an n-gon whose distinct vertices evolve according to (2). If adjacent vertices remain distinct, the perimeter $P(t)$ of the n-gon (defined in (8)) monotonically decreases to zero.

Proof: Substituting (2) into (9) and expanding we have

$$\dot{P}(t) = \frac{1}{2}\sum_{i=1}^{n} \Re\Big\{ -|z_i - z_{i-1}| - |z_{i+1} - z_i| + \left\langle \frac{z_i - z_{i-1}}{|z_i - z_{i-1}|}, z_{i+1} - z_i\right\rangle + \left\langle \frac{z_{i+1} - z_i}{|z_{i+1} - z_i|}, z_i - z_{i-1}\right\rangle\Big\}.$$

Each term in this summation has the form $\Re\{-|u| - |v| + \langle u/|u|, v\rangle + \langle v/|v|, u\rangle\}$. From the Cauchy-Schwarz inequality we have $\Re\{\langle u/|u|, v\rangle\} \leq |v|$, $\Re\{\langle v/|v|, u\rangle\} \leq |u|$, and thus $\Re\{-|u| - |v| + \langle u/|u|, v\rangle + \langle v/|v|, u\rangle\} \leq 0$. Therefore, $\dot{P}(t) \leq 0$. Equality is achieved if and only if $u/|u| = v/|v|$ for each term in the summation; that is, if and only if

$$\frac{z_i - z_{i-1}}{|z_i - z_{i-1}|} = \frac{z_{i+1} - z_i}{|z_{i+1} - z_i|}, \quad \forall i. \quad (10)$$

However, assume by way of contradiction that (10) is satisfied. Rotate the coordinate system such that z_1 and z_2 lie on the real axis and $z_2 - z_1 > 0$. Setting $i = 2$ in (10) we have $z_3 - z_2 > 0$, setting $i = 3$ we have $z_4 - z_3 > 0$, and so on. Hence $z_{i+1} - z_i > 0$, $\forall i = 1, \ldots, n-1$, which implies that $z_n > z_1$. But setting $i = n$ in (10) we have $z_1 - z_n > 0$, a contradiction. Therefore (10) cannot be satisfied, $\dot{P}(t) < 0$, and since the vertices converge to their stationary centroid, $P(t)$ monotonically decreases to zero. ■

VII. Conclusions

Motivated by the rendezvous problem in multi-agent systems we reviewed the theory of curve shortening. We proposed a simple linear scheme for polygon shortening and showed that it exhibits several properties similar to the curve shortening theory. In terms of future work, it would be interesting to study a nonlinear polygon shortening scheme to try to create an even closer analogy.

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