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TRANSFER FUNCTIONS OF SAMPLED-DATA SYSTEMS IN THE LIFTED DOMAIN

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***Abstract*— In this paper a novel representation of transfer functions of sampled-data systems in the lifted domain is proposed. The main idea is to express these transfer functions by the STPBC (systems with two-point boundary conditions) machinery avoiding the appeal to the state space in the lifted domain. This produces a compact LTI description of sampled-data systems in which the intersample dynamics are driven solely by the open-loop dynamics of continuous-time parts of the system and discrete-time dynamics shows up through a reshaping of the boundary conditions. The proposed representation simplifies manipulations over sampled-data systems and enables one to keep track of their structure under multi-level algebraic manipulations.**

I. INTRODUCTION

Consider a continuous-time LTI system $\mathcal{G}$, described by the following state equation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \tag{1}$$

This equation, quite naturally, characterizes $\mathcal{G}$ as a relation between continuous-time signals $u(t)$ and $y(t)$.

Yet one can look at $\mathcal{G}$ form a different perspective/viewpoint. To this end, introduce a transformation $\mathcal{W}_h$, which maps a continuous-time signal $\zeta(t)$ into the discrete-time sequence $\{\breve{\zeta}_k\}$ such that each $\breve{\zeta}_k$ is a *function* in $[0, h]$ (throughout the paper the space of such functions is referred to as $\mathcal{K}_h$) satisfying $\breve{\zeta}_k(\tau) = \zeta(kh + \tau)$. This transformation, called the *lifting transformation* or simply *lifting*, can be visualized as shown in Fig. 1. Clearly, ζ and $\breve{\zeta}$ describe the same signal. One thus can think of the system $\mathcal{G}$ above not as a relation between u and y, but rather as a relation between their lifted versions $\breve{u}$ and $\breve{y}$. The latter can be easily derived from the solution

$$x(kh + \tau) = e^{A\tau}x(kh) + \int_0^\tau e^{A(\tau-\theta)}Bu(kh + \theta)d\theta$$

of (1), which holds for all $k \in \mathbb{Z}^+$ and $\tau \in \mathbb{R}^+$. This equation suggests that (1) can be rewritten in the following form:

$$\begin{cases} \bar{x}_{k+1} = e^{Ah}\bar{x}_k + \int_0^h e^{A(h-\theta)}B\breve{u}_k(\theta)d\theta, \\ \breve{y}_k(\tau) = Ce^{A\tau}\bar{x}_k + D\breve{u}_k(\tau) + C\int_0^\tau e^{A(\tau-\theta)}B\breve{u}_k(\theta)d\theta, \end{cases}$$

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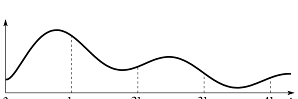
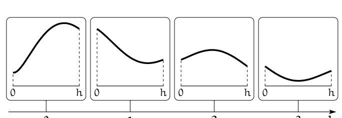


(a) Continuous-time signal ζ



(b) Its lifted equivalent $\breve{\zeta} = \mathcal{W}_h\zeta$

Fig. 1. Lifting continuous-time signals

where $\bar{x}_k \doteq x(kh)$, or, equivalently, in the form

$$\begin{cases} \bar{x}_{k+1} = \bar{A}\bar{x}_k + \acute{B}\breve{u}_k, \\ \breve{y}_k = \grave{C}\bar{x}_k + \breve{D}\breve{u}_k, \end{cases} \tag{2}$$

where[1]

$$\bar{A} : \mathbb{R}^n \to \mathbb{R}^n \qquad \bar{x}_k \mapsto e^{Ah}\bar{x}_k, \tag{3a}$$

$$\acute{B} : \mathcal{K}_h \to \mathbb{R}^n \qquad \breve{u}_k \mapsto \int_0^h e^{A(h-\theta)}B\breve{u}_k(\theta)d\theta, \tag{3b}$$

$$\grave{C} : \mathbb{R}^n \to \mathcal{K}_h \qquad \bar{x}_k \mapsto Ce^{A\tau}\bar{x}_k, \tag{3c}$$

$$\breve{D} : \mathcal{K}_h \to \mathcal{K}_h \qquad \breve{u}_k \mapsto D\breve{u}_k + C\int_0^\tau e^{A(\tau-\theta)}B\breve{u}_k(\theta)d\theta. \tag{3d}$$

This is a discrete-time state equation with a finite-dimensional state space and infinite-dimensional input and output spaces. The discrete-time system described by this equation is called the *lifting* of $\mathcal{G}$ and denoted as $\breve{\mathcal{G}} = \mathcal{W}_h\mathcal{G}\mathcal{W}_h^{-1}$.

The rationale behind the use of the lifting transformation becomes apparent when sampled-data systems are analyzed. These are systems containing both continuous- and discrete-time elements connected by A/D (sampler) and D/A (hold) converters, see [4]. Analyzing sampled-data systems is complicated since their dynamics are hybrid (continuous/discrete) and their continuous-time behavior is periodically time varying even when both continuous and discrete parts are time invariant. Both these problems can be overcome by lifting all continuous-time signals, thus converting continuous-time elements to equivalent discrete-time ones. This puts all elements of sampled-data systems on an equal footing, turning them into discrete time-invariant systems. This idea was introduced independently in [22], [20], [18], [3] and has proven to be a powerful tool for analyzing many aspects of sampled-data systems [2], [5], [23] as well as of the sampled-data controller design [18], [1], [13].

Nonetheless, the advantages in using lifting come at a high price. Since the I/O spaces of systems in the lifted domain are infinite dimensional, most parameters of the lifted state-space realization (2) are operators over the infinite-dimensional space $\mathcal{K}_h$. This fact complicates manipulation over these parameters, especially when they are presented in an integral form (3) which, although appearing quite natural, complicates manipulations over these operators.

Many of these difficulties can be alleviated by replacing the integral representation (3) with one based on STPBC (see Appendix A for relevant definitions). The fact that the representation using differential equations simplifies manipulations over a class of integral operators on $[0, h]$ was apparently first recognized on an abstract level in [6]. Applying to the "D" part of the lifted systems, this idea was exploited in [1]. Then, a general framework

[1]To improve the readability of formulae involving both finite- and infinite-dimensional components, the following operator nomenclature is used hereafter: a bar, like $\bar{O}$, indicates an operator whose input and output spaces are both finite dimensional; a grave accent, $\grave{O}$, indicates a finite-dimensional input space and a distributed output space, like $\mathcal{K}_h$; an acute accent, $\acute{O}$, indicates a finite-dimensional output space and a distributed input space; finally, a breve accent, $\breve{O}$, indicates that both the input and output take distributed values.

incorporating the impulse and sampling operators was proposed in [10]. This representation proved a powerful tool, enabling solution of some open problems in sampled-data control and estimation [13], [11], [14]. For example, the last two references successfully handled the computation of matrices like $\grave{C}^*(I-\breve{D}\breve{\Pi}_B\breve{D}^*)^{-1}\grave{C}:\mathbb{R}^n\mapsto\mathbb{R}^n$, where $\breve{\Pi}_B \doteq I-\acute{B}^*(\acute{B}\acute{B}^*)^{-1}\acute{B}$ is the orthogonal projection onto the null space of $\acute{B}$.

The STPBC representation above, however, is only of limited help when multilevel computations (i.e., those involving intermediate manipulations) have to be performed. Examples include adaptation of the discrete formulae of [19] or the procedure of [15] to the sampled-data H^∞ smoothing problem. In these cases, the solutions are not carried out directly in terms of the problem data, but rather in terms of the solution of the corresponding Kalman filtering problem, or, equivalently, coprime factors of the data with the co-inner numerator. When existing lifting techniques are used, the transformed problem data loses its structure and cannot be readily handled.

In this respect, alternative representations of the lifted transfer functions, which would make it possible to maintain the structure of sampled-data systems under manipulations over their lifted transfer functions, would be of value. The purpose of this paper is to introduce such a representation. This paper conforms to [10] in its use of the STPBC machinery as the basic tool. Here, however, the STPBC machinery is not applied to the parameters of the state-space realization (2), but rather directly to the corresponding transfer function $\breve{G}(z)$. It turns out that the resulting expression for $\breve{G}(z)$ is compact and nicely interpretable, bearing some resemblance to the ideas of [17], [21] (see Remark 2.1). Yet, unlike the latter approach (which leads to a time-varying representation of sampled-data systems), the proposed representation makes it possible to stay in the time-invariant framework and use well-understood LTI tools.

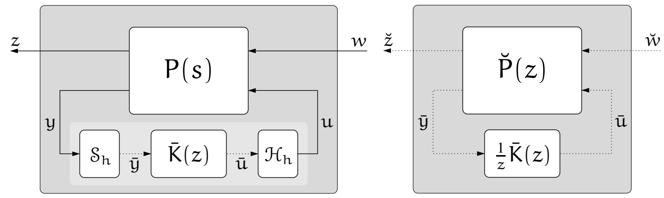
Arguably, a major drawback[2] of the available representations of sampled-data systems in the lifted domain is that all of them hinge upon the discrete state equation (2) and thus inherit technical problems associated with it. These problems are apparently caused by the non-closeness of this model to inverse and adjoint operations. The problems are manifested either in significantly more cumbersome derivations and solutions compared to their continuous-time counterparts or in the need to impose restrictive assumptions on system parameters (in fact, some of these assumptions never hold in the sampled-data case). In this respect, one of the advantages of the proposed approach can be seen in its moving away from (2) as the ultimate model for analyzing lifted systems. I believe that the proposed representation could become a basis for more efficient methods (see [16] for some examples).

The paper is organized as follows. In Section II the representation is introduced and some of its basic properties are discussed. In Section III the application of the proposed representation to the construction of sampled-data observer-based controllers is addressed. Section IV is devoted to the conversion of the general sampled-data estimation problem to that with stable data. Concluding remarks are provided in Section V. The paper contains an Appendix, in which some basic facts about STPBCs are presented.

II. The Compact Representation

A. The idea

The main idea of the proposed representation is to avoid the direct use of the state-space realization (2). Instead, the transfer function



(a) In the time domain (b) In the lifted domain

Fig. 2. General sampled-data setup.

in the lifted domain, i.e., $\breve{G}(z)=\breve{D}+\grave{C}(zI-\bar{A})^{-1}\acute{B}$, is written in a more compact form. To this end, note that for all $\breve{u}\in\mathcal{K}_h$

$$\big(\breve{G}(z)\breve{u}\big)(\tau)=D\breve{u}(\tau)+C\int_0^\tau e^{A(\tau-\theta)}B\breve{u}(\theta)d\theta \\ +Ce^{A\tau}(zI-e^{Ah})^{-1}\int_0^h e^{A(h-\theta)}B\breve{u}(\theta)d\theta.$$

The comparison of this expression with (16a) prompts the following representation of $\breve{G}(z)$:

Theorem 2.1: Given $G(s)=D+C(sI-A)^{-1}B$, its lifted version has the transfer function

$$\breve{G}(z)=\left(\begin{array}{c|c|c} A & \boxed{zI \rightleftharpoons -I} & B \\ C & & D \end{array}\right) \tag{4}$$

for all $z\in\mathbb{C}$.

Formula (4) has quite a neat interpretation: the intersample behavior is completely determined by the continuous-time dynamics of $G(s)$ and the discrete-time dynamics appears in the boundary conditions only. In fact, the boundary conditions in this case are $zx(0)=x(h)$, which simply means the concatenation of the pieces in Fig. 1(b). Remarkably, an addition of pure discrete-time dynamics to the representation of Theorem 2.1 shows up only through a reshaping of the boundary conditions of the STPBC in (4) and, quite naturally, does not affect the intersample dynamics.

To see this, consider the standard sampled-data system in Fig. 2(a), where

$$P(s)=\left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & 0 & 0 \end{array}\right]$$

(D_{yw} and D_{yu} are both taken zero to guarantee the boundedness of the sampling operation [4]) is a continuous-time plant, $\bar{K}(z)$ is a discrete-time part of the controller, $\mathcal{S}_h$ is the ideal sampler (i.e., $\bar{y}_k=y(kh^-)$), and $\mathcal{H}_h$ is the zero-order hold ($u(kh+\tau)=\bar{u}_k$). The application of the lifting transformation to all continuous-time signals converts the system in Fig. 2(a) to an equivalent discrete-time LTI system in the lifted domain[3] shown in Fig. 2(b).

Conventionally, the lifted generalized plant, $\breve{P}(z)$, is expressed in terms of its state-space realization

$$\breve{P}(z)=\left[\begin{array}{c|cc} \bar{A} & \acute{B}_w & \bar{B}_u \\ \hline \grave{C}_z & \breve{D}_{zw} & \grave{D}_{zu} \\ \bar{C}_y & \acute{D}_{yw} & \bar{D}_{yu} \end{array}\right] \tag{5}$$

with appropriately defined [4, Sec. 10.3] parameters, some of which are operators over infinite-dimensional I/O spaces (see Footnote 1). The problem here is that the structure of the hybrid system in Fig. 2(a) is blurred in representation (5). The situation further

[2]At least, from the frequency-domain analysis point of view.

[3]Note that the sampler in the lifted domain always contains a unit delay part [12], which is attached to the discrete controller. The rest of the lifted sampler and the lifted hold are absorbed into the lifted generalized plant $\breve{P}$.

deteriorates as the loop is closed by the discrete controller $\frac{1}{z}\bar{K}(z)$, so that the structure of the original system is somewhat lost in algebraic constructions involving infinite-dimensional operators.

The following theorem proposes an alternative form of the closed-loop lifted transfer function:

Theorem 2.2: The closed-loop transfer function from $\breve{w}$ to $\breve{z}$ for the lifted system in Fig. 2(b) is

$$\breve{T}_{zw}(z) = \left(\begin{array}{cc|c|c} A & B_u & \boxed{z\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \rightleftharpoons -\left[\begin{smallmatrix} I & 0 \\ \bar{K}(z)C_y & 0 \end{smallmatrix}\right]} & B_w \\ 0 & 0 & & 0 \\ \hline C_z & D_{zu} & & D_{zw} \end{array}\right). \qquad (6)$$

Proof: It is known [12] that the sampler and hold in the lifting domain become the following transfer functions:

$$\acute{S}_h(z) = \tfrac{1}{z}\mathcal{J}_h^* \quad \text{and} \quad \grave{H}_h(z) = \left(\begin{array}{c|c|c} 0 & \boxed{I \rightleftharpoons 0} & I \\ \hline I & & 0 \end{array}\right)\mathcal{J}_0.$$

Augmenting $z\acute{S}_h(z)$ and $\grave{H}_h(z)$ to the lifting of $P(s)$ results then in the following lifted generalized plant:

$$\breve{P}(z) = \begin{bmatrix} I & 0 \\ 0 & \mathcal{J}_h^* \end{bmatrix} \left(\begin{array}{cc|c|cc} A & B_u & \boxed{\left[\begin{smallmatrix} zI & 0 \\ 0 & I \end{smallmatrix}\right] \rightleftharpoons -\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]} & B_w & 0 \\ 0 & 0 & & 0 & I \\ \hline C_z & D_{zu} & & D_{zw} & 0 \\ C_y & 0 & & 0 & 0 \end{array}\right) \begin{bmatrix} I & 0 \\ 0 & \mathcal{J}_0 \end{bmatrix} \qquad (7)$$

(this is an alternative to (5)). This STPBC has two inputs, one of which is of the form $\mathcal{J}_0\bar{u}$. The application of Proposition A.2 enables one to eliminate this input at the expense of the following reshape of the boundary conditions:

$$\begin{bmatrix} zI & 0 \\ 0 & I \end{bmatrix}\left(x(0) - \begin{bmatrix} 0 \\ I \end{bmatrix}\bar{u}\right) - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}x(h) = 0.$$

Now, taking into account that $\bar{u} = \frac{1}{z}\bar{K}(z)\bar{y}$ and $\bar{y} = \begin{bmatrix} C_y & 0 \end{bmatrix}x(h)$, the relation above rewrites as

$$\begin{bmatrix} zI & 0 \\ 0 & I \end{bmatrix}x(0) - \begin{bmatrix} I & 0 \\ \frac{1}{z}\bar{K}(z)C_y & 0 \end{bmatrix}x(h) = 0,$$

from which (6) follows by multiplying the last row by z. □

Remark 2.1: Note that the intersample (continuous-time) behavior of the closed-loop transfer function (6) is comprised of the dynamics of the continuous-time plant (the modes of A) and the zero-order hold (the modes at the origin) and is not affected by the discrete controller $\bar{K}$. The latter affects only the boundary conditions, i.e., the way in which the consecutive pieces of lifted signals are tailored together. This property shows some resemblance with the approach based on the representation of sampled-data systems by continuous-time dynamics with jumps [17], [21]. There is, however, an important conceptual difference between that approach and the approach proposed in this paper. In the "jump system" representation of sampled-data dynamics, one ends up with a time-varying system and, consequently, time-varying machinery should be adopted. On the contrary, the purpose of the representation proposed here is to end up with a time-invariant representation, for which standard frequency-domain methods could be taken up.

B. General sampled-data transfer function

Motivated by Theorem 2.2, define the STPBC

$$\breve{G}(z) = \left(\begin{array}{c|c|c} A & \boxed{\Omega(z) \rightleftharpoons \Upsilon(z)} & B \\ \hline C & & D \end{array}\right), \qquad (8)$$

where $\Omega(z)$ and $\Upsilon(z)$ are some square discrete transfer matrices such that $\begin{bmatrix} \Omega(z) & \Upsilon(z) \end{bmatrix}$ has full normal row rank, as a *general sampled-data transfer function* (in many cases $\Omega(z) = zI$). The STPBC (8) is not well-posed if

$$\det\bigl(\Omega(z) + \Upsilon(z)e^{Ah}\bigr) = 0.$$

It then appears natural to define all $z \in \mathbb{C}$ for which this condition holds as *poles of the representation*. For example, the poles of representation (4) are the eigenvalues of e^{Ah}. The poles of the representation, however, are not necessarily the poles of the corresponding transfer function. Indeed, any left common factor of $\Omega(z)$ and $\Upsilon(z)$ does not affect the operator $\breve{G}(z)$. This is clearly seen from (16a), where the boundary condition matrices appear only through the term $(\Omega(z)+\Upsilon(z)e^{Ah})^{-1}\Upsilon(z)$, in which this common factor is canceled. Thus, throughout the paper common factors of $\Omega(z)$ and $\Upsilon(z)$ are not counted as poles of $\breve{G}(z)$. An example is the $\dim(u)$ poles at the origin of realization (6), which are not the poles of $\breve{T}_{zw}(z)$.

The concept of the conjugate transfer function plays an important role in the frequency-domain analysis of linear systems. Given a transfer function $G(z)$, its conjugate is defined as $G^{\sim}(z) \doteq G'(z^{-1})$, where $G'(z)$ is the transpose (dual) transfer function [24, p. 67]. Since $G^{\sim}(z) = [G(z^{-*})]^*$, it follows from (17) that for the general sampled-data transfer function

$$\breve{G}^{\sim}(z) = \left(\begin{array}{c|c|c} -A' & \boxed{\Upsilon_d^{\sim}(z) \rightleftharpoons \Omega_d^{\sim}(z)} & C' \\ \hline -B' & & D' \end{array}\right),$$

where $\Omega_d(z)$ and $\Upsilon_d(z)$ are any square transfer matrices satisfying

$$\Omega(z)\Upsilon_d(z) = \Upsilon(z)\Omega_d(z)$$

and such that $\left[\begin{smallmatrix} \Omega_d(z) \\ \Upsilon_d(z) \end{smallmatrix}\right]$ has full normal rank (these transfer matrices always exist because $\begin{bmatrix} \Omega(z) & \Upsilon(z) \end{bmatrix}$ has full normal rank). It is worth emphasizing that the conjugate system always exists within the class of operators defined by (8). This is not always true for discrete-time systems described by the standard state equation.

III. Sampled-Data Observer-Based Controller

In this section some properties of sampled-data observer-based controllers are discussed. First, the basic building blocks, namely the sampled-data state feedback (§III-A) and the sampled-data state observer (§III-B), will be addressed and then the output feedback case will be studied in §III-C.

A. State feedback

The sampled-data state feedback corresponds to the choices $C_y = I$ and $\bar{K}(z) = F$ in the system in Fig. 2. In this case Theorem 2.2 yields the following closed-loop transfer function:

$$\breve{T}_{zw}(z) = \left(\begin{array}{cc|c|c} A & B_u & \boxed{z\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \rightleftharpoons -\left[\begin{smallmatrix} I & 0 \\ F & 0 \end{smallmatrix}\right]} & B_w \\ 0 & 0 & & 0 \\ \hline C_z & D_{zu} & & D_{zw} \end{array}\right).$$

The poles of this realization are the roots of

$$\det\left(zI - \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix}\begin{bmatrix} \bar{A} & \bar{B}_u \\ 0 & I \end{bmatrix}\right) = 0,$$

where

$$\begin{bmatrix} \bar{A} & \bar{B}_u \\ 0 & I \end{bmatrix} \doteq \exp\left(\begin{bmatrix} A & B_u \\ 0 & 0 \end{bmatrix}h\right) = \begin{bmatrix} e^{Ah} & \int_0^h e^{A\tau}d\tau\, B_u \\ 0 & I \end{bmatrix}. \qquad (9)$$

Excluding the common factor at the origin, the poles of the closed-loop transfer function are then the eigenvalues of the matrix $\bar{A} + \bar{B}_u F$.

B. State observer

The sampled-data state observer for the state vector x of the plant in Fig. 2(a) is given by the following differential equation with jumps:

$$\dot{\xi} = A\xi + B_u u, \quad \xi(kh) = \xi(kh^-) + L\big(C_y \xi(kh^-) - \bar{y}_k\big)$$

for some gain L. In the lifted domain this equation becomes an LTI system having the following transfer function from $\left[\begin{smallmatrix}\breve{u}\\ \bar{y}\end{smallmatrix}\right]$ to $\breve{\xi}$:

$$\breve{K}_{\text{obs}}(z) = \left(\begin{array}{c|c|cc} A & \boxed{zI \rightleftharpoons -(I+LC_y)} & B_u & -L \\ \hline I & & 0 & 0 \end{array}\right) \begin{bmatrix} I & 0 \\ 0 & \mathcal{J}_0 \end{bmatrix} \tag{10}$$

or, when $\breve{u} = \grave{\mathcal{H}}_h \bar{u}$, the following transfer function from $\left[\begin{smallmatrix}\bar{u}\\ \bar{y}\end{smallmatrix}\right]$ to $\breve{\xi}$:

$$\grave{K}_{\text{obs}}(z) = \left(\begin{array}{cc|c|cc} A & B_u & \boxed{z\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \rightleftharpoons -\left[\begin{smallmatrix} I+LC_y & 0 \\ 0 & 0 \end{smallmatrix}\right]} & 0 & -L \\ 0 & 0 & & I & 0 \\ \hline I & 0 & & 0 & 0 \end{array}\right) \mathcal{J}_0. \tag{10$'$}$$

To derive the transfer function of the error equation, let us write the dynamics of the continuous-time state vector as

$$\dot{x} = Ax + B_w w + B_u u, \quad x(kh) = x(kh^-).$$

The error equation for $\epsilon \doteq x - \xi$ is then

$$\dot{\epsilon} = A\epsilon + B_w w, \quad \epsilon(kh) = (I + LC_y)\epsilon(kh^-),$$

where the fact that $\bar{y}_k = C_y x(kh^-)$ was exploited, which does not depend on u. This leads to the following error transfer function from $\breve{w}$ to $\breve{\epsilon}$:

$$\breve{T}_{\epsilon w}(z) = \left(\begin{array}{c|c|c} A & \boxed{zI \rightleftharpoons -(I+LC_y)} & B_w \\ \hline I & & 0 \end{array}\right),$$

the poles of which are the eigenvalues of $(I + LC_y)\bar{A}$, where $\bar{A}$ is given by (9).

C. Output feedback

Sampled-data observer-based controller is obtained from the state feedback control law by replacing the sampled plant state $\bar{x}_k$ with the sampled output $\bar{\xi}_k = \xi(kh^-)$ of observer (10$'$). Thus, the control law is $\bar{u}_k = F\bar{\xi}_k$. Using (10$'$) and Proposition A.2, it can now be shown that the transfer function of this controller is

$$\bar{K}(z) = -\frac{1}{z}\mathcal{J}_h^* \left(\begin{array}{cc|c|c} A & B_u & \boxed{z\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \rightleftharpoons -\left[\begin{smallmatrix} I+LC_y & 0 \\ F & 0 \end{smallmatrix}\right]} & L \\ 0 & 0 & & 0 \\ \hline F & 0 & & 0 \end{array}\right) \mathcal{J}_0 \tag{11a}$$

from which, with the help of Proposition A.1 and the Matrix Inversion Lemma, the following more conventional expression can be derived:

$$= -F\big(zI - ((I + LC_y)\bar{A} + \bar{B}_u F)\big)^{-1} \bar{A} L. \tag{11b}$$

Note that (11b) can in principle be easily obtained using the conventional sampled-data methods. These methods, however, are limited in revealing the structure of the closed-loop sampled-data system.

To derive the closed-loop transfer function, combine the transfer functions of the lifted generalized plant and the observer error. Taking into account the equality $\bar{\xi}_k = x(kh^-) - \epsilon(kh^-)$, the transfer function from $\left[\begin{smallmatrix}\breve{w}\\ \bar{u}\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}\breve{z}\\ \bar{\xi}\end{smallmatrix}\right]$ is

$$\begin{bmatrix} I & 0 \\ 0 & \frac{1}{z}\mathcal{J}_h^* \end{bmatrix} \left(\begin{array}{ccc|c|cc} A & B_u & 0 & \boxed{zI \rightleftharpoons -\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I+LC_y \end{smallmatrix}\right]} & B_w & 0 \\ 0 & 0 & 0 & & 0 & I \\ 0 & 0 & A & & B_w & 0 \\ \hline C_z & D_{zu} & 0 & & D_{zw} & 0 \\ I & 0 & -I & & 0 & 0 \end{array}\right) \begin{bmatrix} I & 0 \\ 0 & \mathcal{J}_0 \end{bmatrix}.$$

Now, closing the loop $\bar{u} = F\bar{\xi}$, the closed-loop transfer function is

$$\breve{T}_{zw}(z) = \left(\begin{array}{ccc|c|c} A & B_u & 0 & \boxed{z\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right] \rightleftharpoons -\left[\begin{smallmatrix} I & 0 & 0 \\ F & 0 & -F \\ 0 & 0 & I+LC_y \end{smallmatrix}\right]} & B_w \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & A & & B_w \\ \hline C_z & D_{zu} & 0 & & D_{zw} \end{array}\right).$$

It is readily seen that the poles of this realization are the solution of

$$\det\left(z\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ F & 0 & -F \\ 0 & 0 & I + LC_y \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B}_u & 0 \\ 0 & I & 0 \\ 0 & 0 & \bar{A} \end{bmatrix} \right) = 0,$$

where $\bar{A}$ and $\bar{B}_u$ are defined in (9). One can see that excluding $\dim(u)$ roots at the origin, which are the common rots of the "Ω" and "Υ" parts, the poles of the closed-loop transfer function $\breve{T}_{zw}(z)$ are comprised of the poles of the state feedback (eigenvalues of $\bar{A} + \bar{B}_u F$) and those of the state observer (eigenvalues of $(I + LC_y)\bar{A}$).

Remark 3.1: The arguments above can be easily adopted to the case, when the hold function is not the zero-order hold, but rather a free design parameter. In this case one should use (10) combined with the control law $\breve{u}_k = F\breve{\xi}_k$. This results in the controller

$$\grave{K}(z) = -\left(\begin{array}{c|c|c} A + B_u F & \boxed{zI \rightleftharpoons -(I+LC_y)} & L \\ \hline F & & 0 \end{array}\right) \mathcal{J}_0,$$

which actually (cf. (16b)) consists of the discrete-time part

$$\bar{K}(z) = -z\big(zI - (I + LC_y)e^{(A+B_u F)h}\big)^{-1} L$$

and the generalized hold acting as follows:

$$u(kh + \tau) = Fe^{(A+B_u F)\tau}\, \bar{u}_k.$$

The closed-loop transfer function is then derived in complete analogy with its derivation in the zero-order hold case taking into account the fact that the observer error transfer function $\breve{T}_{\epsilon w}(z)$ does not depend on the control signal. The result is

$$\breve{T}_{zw}(z) = \left(\begin{array}{cc|c|c} A + B_u F & -B_u F & \boxed{z\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \rightleftharpoons -\left[\begin{smallmatrix} I & 0 \\ 0 & I+LC_y \end{smallmatrix}\right]} & B_w \\ 0 & A & & B_w \\ \hline C_z + D_{zu} F & 0 & & D_{zw} \end{array}\right)$$

with the poles at the eigenvalues of $e^{(A+B_u F)h}$ and $(I + LC_y)\bar{A}$.

IV. "Stabilification" of General Estimation Problem

In this section an application of the proposed representation to the reduction of the general sampled-data estimation problem to the estimation problem for a stable system is addressed. The general sampled-data estimation problem can be described as follows. Given continuous-time LTI systems $\mathcal{G}_v$ and $\mathcal{G}_y$ having the transfer matrices

$$G(s) = \begin{bmatrix} G_v(s) \\ G_y(s) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_v & D_v \\ C_y & 0 \end{array}\right]$$

and the ideal sampler $\mathcal{S}_h$, find a stable $\mathcal{K}$ (the estimator) having a discrete-time input and a continuous-time output such that the error system

$$\mathcal{T}_\epsilon \doteq \mathcal{G}_v - \mathcal{K}\mathcal{S}_h\mathcal{G}_y$$

is stable as well and its norm (typically, either H^2 or H^∞) is minimized. In the lifted domain this can be equivalently recast as an the LTI problem of stabilizing (by a stable lifted estimator $\grave{K}(z)$) the error transfer function

$$\breve{T}_\epsilon(z) = \breve{G}_v(z) - \grave{K}(z)\acute{G}_y(z)$$

and minimizing $\|\breve{T}_\epsilon\|$.

In general neither $\mathcal{G}_v$ nor $\mathcal{G}_y$ needs to be stable. Yet the analysis is greatly simplified, especially in the fixed-lag smoothing version of the estimation problem [9], [15], when they are stable. It turns out, that the general problem can always (under the natural detectability assumption) be converted to an equivalent problem with stable data. The conversion, which will be referred to as the *stabilification* procedure, was proposed in [9] using earlier ideas from [8]. The stabilification procedure exploits the fact that the detectability is equivalent to the existence of a coprime factorization of the form

$$\breve{G}(z) = \begin{bmatrix} \breve{G}_v(z) \\ \acute{G}_y(z) \end{bmatrix} = \begin{bmatrix} I & \grave{M}_v(z) \\ 0 & \bar{M}_y(z) \end{bmatrix}^{-1} \begin{bmatrix} \check{N}_v(z) \\ \acute{N}_y(z) \end{bmatrix}. \tag{12}$$

In this case

$$\check{T}_\epsilon(z) = \check{N}_v(z) - \grave{K}_a(z)\acute{N}_y(z) \tag{13}$$

for $\grave{K}_a = (\grave{K} - \grave{M}_v)\bar{M}_y^{-1}$, which is stable (i.e., belongs to H^∞) iff so is $\grave{K}$ provided the error $\check{T}_\epsilon$ is stable as well. The original problem is converted thus to an equivalent problem with stable data, (13), and the solution of the latter is then used to generate $\grave{K}$ as follows:

$$\grave{K}(z) = \grave{M}_v(z) + \grave{K}_a(z)\bar{M}_y(z).$$

The stabilification procedure above can in principle be carried out using conventional representations of sampled-data systems in the lifted domain. This, however, would lead to a structure loss in the factors and effectively make (13) a dead end. As will be shown below, the use of the representation introduced in Section II makes it possible to circumvent these difficulties and end up with the factors, which have similar structure to that of $\breve{G}_v$ and $\acute{G}_y$.

To start with, bring in the lifted transfer functions

$$\begin{bmatrix} \breve{G}_v(z) \\ \acute{G}_y(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \frac{1}{z}\mathcal{J}_h^* \end{bmatrix} \left(\begin{array}{c|c|c} A & zI \rightleftharpoons -I & B \\ \hline C_v & & D_v \\ C_y & & 0 \end{array}\right). \tag{14}$$

Taking into account that (12) is equivalent to

$$\begin{bmatrix} \breve{v} \\ 0 \end{bmatrix} = \begin{bmatrix} \check{N}_v(z) & \grave{M}_v(z) \\ \acute{N}_y(z) & \bar{M}_y(z) \end{bmatrix} \begin{bmatrix} \breve{w} \\ -\bar{y} \end{bmatrix},$$

the equality $\left[\begin{smallmatrix} \breve{z} \\ \bar{y} \end{smallmatrix}\right] = \breve{G}(z)\breve{w}$ can be rewritten as

$$\begin{cases} \dot{x} = Ax + B\breve{w}, \quad zx(0) = x(h) \\ \breve{v} = C_v x + D_v \breve{w} \\ 0 = \frac{1}{z}\mathcal{J}_h^* C_y x - \bar{y} \end{cases}$$

The left coprime factorization is conventionally constructed by adding the last equation, premultiplied by the gain L, to the first equation (thus, in a sense, constructing a state observer). In the sampled-data case the state observer is constructed via adding the correction term to the boundary condition (see §III-B). Motivated by this observation, the boundary condition above can be rewritten as

$$z(x(0) + L\bar{y}) = (I + LC_y)x(h),$$

from which, using Proposition A.2, the following expressions for the factors in (12) can be inferred:

$$\begin{bmatrix} \check{N}_v(z) \\ \acute{N}_y(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \frac{1}{z}\mathcal{J}_h^* \end{bmatrix} \left(\begin{array}{c|c|c} A & zI \rightleftharpoons -(I+LC_y) & B \\ \hline C_v & & D_v \\ C_y & & 0 \end{array}\right)$$

and

$$\begin{bmatrix} \grave{M}_v(z) \\ \bar{M}_y(z) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \frac{1}{z}\mathcal{J}_h^* \end{bmatrix} \left(\begin{array}{c|c|c} A & zI \rightleftharpoons -(I+LC_y) & L \\ \hline C_v & & D_v \\ C_y & & 0 \end{array}\right)\mathcal{J}_0.$$

Provided $(I + LC_y)e^{Ah}$ is Schur, the coprimeness of these factors in RH^∞ can be shown by the direct construction of the corresponding Bézout factors. This part is omitted because of the space limitation.

Having the factorization above, the estimation problem reduces to that for the stable data $\check{N}_v$ and $\acute{N}_y$. An important observation is that these transfer functions have similar structure to that of the original problem data in (14), so that the stabilification procedure does not lead to any structure impairment. This property is exploited in [16] to solve the sampled-data H^2 smoothing problems.

V. Concluding Remarks

In this paper a novel representation of transfer functions of sampled-data systems in the lifted domain has been introduces. Unlike the existing approaches, the proposed representation avoids the explicit use of the state-space realization of lifted systems. Instead, the STPBC (systems with two-point boundary conditions) formalism is used to express transfer functions directly, in a compact form. The continuous-time dynamics of this STPBC representation reflect the intersample behavior of the sampled-data systems and the discrete-time dynamics are reflected by the boundary conditions. It has been shown that the proposed compact representation makes it possible to maintain the structure of sampled-data systems under manipulations over their lifted transfer functions, thus extending the class of problems which can be solved by the lifting technique.

It is worth emphasizing that this paper is devoted to the exposition of the main idea of the proposed formalism only. Many aspects of its application to the analysis and design of sampled-data systems are either left unaddressed or only outlined here. The reader is referred to [16] for the application of the proposed representation to the sampled-data H^2 smoothing.

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Appendix A
Systems with Two-Point Boundary Conditions

Systems with two-point boundary conditions (STPBC) are systems operating over the interval $[0, h]$ and driven by the following dynamics [7], [6]:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \qquad \Omega x(0) + \Upsilon x(h) = 0, \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{15}$$

where the square matrices Ω and Υ shape the boundary conditions of the state vector x. The boundary conditions are said to be well-posed if $\det(\Omega + \Upsilon e^{Ah}) \neq 0$. If this condition holds, the mapping $y = \check{\mathcal{O}}u$ is well defined $\forall u \in \mathcal{K}_h$ with

$$\begin{aligned} y(t) &= Du(t) + C\int_0^t e^{A(t-\theta)}Bu(\theta)d\theta \\ &\quad - Ce^{At}(\Omega + \Upsilon e^{Ah})^{-1}\Upsilon\int_0^h e^{A(h-\theta)}Bu(\theta)d\theta \qquad (16a) \\ &= Du(t) - C\int_t^h e^{A(t-\theta)}Bu(\theta)d\theta \\ &\quad + Ce^{At}(\Omega + \Upsilon e^{Ah})^{-1}\Omega\int_0^h e^{-A\theta}Bu(\theta)d\theta. \qquad (16b) \end{aligned}$$

STPBC are denoted by using the compact block notation:

$$\left(\begin{array}{c|c|c} A & \Omega \rightleftharpoons \Upsilon & B \\ \hline C & & D \end{array}\right).$$

When $\mathcal{K}_h$ is a Hilbert space (e.g., $L^2[0,h]$), the adjoint of this system is given by [10]

$$\left(\begin{array}{c|c|c} A & \Omega \rightleftharpoons \Upsilon & B \\ \hline C & & D \end{array}\right)^* = \left(\begin{array}{c|c|c} -A' & \Upsilon_d' \rightleftharpoons \Omega_d' & C' \\ \hline -B' & & D' \end{array}\right), \tag{17}$$

where Υ_d and Ω_d are any square matrices satisfying $\Omega\Upsilon_d = \Upsilon\Omega_d$ and such that $\left[\,\Omega_d' \;\Upsilon_d'\,\right]$ has full row rank.

To make the STPBC formalism applicable to the representation of sampled-data systems in the lifted domain, the following two operators are also required:

- *The impulse operator* $\mathcal{I}_\theta$, which transforms a vector $\eta \in \mathbb{R}^n$ into a modulated δ-impulse as follows:

$$\big(\mathcal{I}_\theta\eta\big)(t) = \delta(t-\theta)\eta.$$

- *The sampling operator* $\mathcal{I}_\theta^*$, which transforms a continuous function $\zeta \in \mathcal{K}_h$ into a vector from $\mathbb{R}^n$:

$$\mathcal{I}_\theta^*\zeta = \zeta(\theta).$$

As the sampling operator is unbounded on $L^2[0,h]$, it can only be applied to continuous signals. If this is the case, then the sampling operation is continuous as a function of θ and there is no problem in using $\mathcal{I}_\theta^*$ to mean $\mathcal{I}_{\theta-}^*$.

The manipulations over STPBC can be performed in the state space, much like the manipulations over standard finite-dimensional state-space systems, see [6]. Moreover, as shown in [10], the sampling and impulse operators fit well into the STPBC formalism. For example, the following result can be formulated:

Proposition A.1 ([10]): Let $\det(\Omega + \Upsilon e^{Ah}) \neq 0$. Then the equality

$$\begin{bmatrix} \mathcal{I}_h^* C_h \\ \mathcal{I}_0^* C_0 \end{bmatrix} \left(\begin{array}{c|c|c} A & \Omega \rightleftharpoons \Upsilon & I \\ \hline I & & 0 \end{array}\right) \left[\, B_h\mathcal{I}_h \;\; B_0\mathcal{I}_0 \,\right] = \begin{bmatrix} C_h e^{Ah} \\ C_0 \end{bmatrix} (\Omega + \Upsilon e^{Ah})^{-1} \left[\, -\Upsilon B_h \;\; \Omega B_0 \,\right]$$

holds provided that $C_h B_h = 0$ and $C_0 B_0 = 0$.

Furthermore, the proposition below shows how impulse inputs can be recast as non-zero boundary conditions.

Proposition A.2: Let $\det(\Omega + \Upsilon e^{Ah}) \neq 0$. Then the systems

$$\begin{cases} \dot{x}_1 = Ax_1 + B_u u + B_\eta \mathcal{I}_0 \bar{\eta}, & \Omega x_1(0) + \Upsilon x_1(h) = 0 \\ y_1 = Cx_1 \end{cases} \tag{18a}$$

and

$$\begin{cases} \dot{x}_2 = Ax_2 + B_u u, & \Omega\big(x_2(0) - B_\eta\bar{\eta}\big) + \Upsilon x_2(h) = 0 \\ y_2 = Cx_2 \end{cases} \tag{18b}$$

are equivalent as mappings from $\mathcal{K}_h \times \mathbb{R}$ to $\mathcal{K}_h$.

Proof: Equation (18a) is a standard STPBC, so that (16) can be used to obtain

$$y_1(t) = C\int_0^t e^{A(t-\theta)}B_u u(\theta)d\theta - Ce^{At}(\Omega + \Upsilon e^{Ah})^{-1} \times \left(\Upsilon\int_0^h e^{A(h-\theta)}B_u u(\theta)d\theta - \Omega B_\eta\bar{\eta}\right).$$

On the other hand, the state equation of (18b) leads to

$$x_2(h) = e^{Ah}x_2(0) + \int_0^h e^{A(h-\theta)}B_u u(\theta)d\theta.$$

Substituting this to the boundary condition yields

$$\Omega\big(x_2(0) - B_\eta\bar{\eta}\big) + \Upsilon\left(e^{Ah}x_2(0) + \int_0^h e^{A(h-\theta)}B_u u(\theta)d\theta\right) = 0,$$

so that, assuming the well-posedness,

$$x_2(0) = -(\Omega + \Upsilon e^{Ah})^{-1}\left(\Upsilon\int_0^h e^{A(h-\theta)}B_u u(\theta)d\theta - \Omega B_\eta\bar{\eta}\right).$$

It is now readily seen that $y_1 \equiv y_2$. □

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