

Steering the Chaplygin Sleigh by a Moving Mass

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Abstract—In this paper we design a steering control algorithm for the Chaplygin sleigh with a moving mass. Our strategy is to only use the controlled dynamics to initiate short-time transitions between the various uncontrolled modes of the system in order to achieve the desired direction of motion.

I. INTRODUCTION

The objective of this paper is to use a moving mass to control the direction of motion of the Chaplygin sleigh—a rigid body on a horizontal plane constrained by a blade. The blade limits the velocity of the body-plane contact point to a direction fixed in the body. This constraint is *nonholonomic* as it imposes a velocity restriction on the system which is not derivable from a position constraint. Rand and Ramani [13] and Ruina [14] point out that blade constraints similar to this have been used to model an underwater missile with fins.

In recent years much work has been done in using geometric structures to both formulate the equations and to address aspects of control of constrained mechanical systems. We summarize some such works. For a more complete list of references on the dynamics and control of nonholonomic systems see [1].

In the seminal paper by Bloch, Krishnaprasad, Marsden, and Murray [2] (hereafter referred to as BKMM), nonholonomic mechanical systems with symmetry are studied. For a system with symmetry, it is natural to split the configuration variables into the *group variables* g which describe the overall position (attitude) of the system and the *shape variables* r which describe the positions of the system's components relative to each other. In the case of the Chaplygin sleigh with a moving mass, the variable g is the element of the group of Euclidean transformations of the two-dimensional plane and the variable r is the position of the moving mass relative to the contact point of the body and the plane.

The dynamics of a nonholonomic system with symmetry are governed by the system of equations

$$\ddot{r} = f(r, \dot{r}, p) + u, \quad (1)$$

$$\dot{p} = \langle \alpha(r)p, p \rangle + \langle \beta(r)p, \dot{r} \rangle + \langle \gamma(r)\dot{r}, \dot{r} \rangle, \quad (2)$$

$$\dot{g} = g(J(r)p - A(r)\dot{r}), \quad (3)$$

where p is the *nonholonomic momentum*, which in general is no longer conserved, and u represents control forces.¹ Note that equations (1) and (2) decouple from the the group

dynamics (3). See BKMM [2] for details and formulae that define various coefficients in equations (1)–(3). Equations (1), (2), and (3) are referred to as the *shape equation*, *momentum equation*, and *reconstruction equation*, respectively.

Utilizing the perturbation methods of [6] to study equations (1)–(3), Ostrowski [12] determines relations between the cyclic control inputs $u(t)$, resultant *momentum generation*, and ultimately, motion. These relations are critical in designing momentum generating and steering algorithms.

In Lewis and Murray [8], a *symmetric bracket* is introduced and used to formulate sufficient conditions for various types of configuration controllability of *simple* mechanical control systems. The general equations analyzed are of geodesic type with both external forces and control input terms.

Simple mechanical control systems *with constraints* are treated in Lewis [7]. The symmetric bracket was shown to address configuration controllability questions in this constraint setting also.

In Bullo, Leonard, and Lewis [4], simple mechanical control systems on a Lie group G are investigated. Here the dynamics of interest are of the form

$$\dot{p} = \langle \alpha p, p \rangle + u^a F_a, \quad (4)$$

$$\dot{g} = g(Jp), \quad (5)$$

where u^a are the control inputs and F_a are the directions in which they act. Implementing the perturbation approach of [6] and using the symmetric bracket technique, Bullo, Leonard, and Lewis [4] design steering control algorithms for equations (5) and (4).

The physical system of particular interest to us is the Chaplygin sleigh with a fully actuated moving mass. The dynamics of this system are of the form (1)–(3) (details are given in Section II).

Unlike Ostrowski, in this paper we are not concerned with motion generation. We assume that the Chaplygin sleigh is already in motion and concentrate on *steering* the system using the movable mass. Since the mass is fully actuated, we can *assign* its position relative to the sleigh as a function of time. Therefore, the dynamics reduce to equations (2) and (3), where the moving mass position r relative to the contact point is interpreted as the control parameter. We emphasize that the dependence of the right-hand sides of (2) and (3) on r is *inherently nonlinear*, and thus the control design of Bullo, Leonard, and Lewis [4] is not applicable.

¹Observe that the controls appear only in the shape equation.

The perturbation control techniques and associated algorithm design of Ostrowski [12] and Bullo, Leonard, and Lewis [4] mentioned above are extremely useful and have a wide range of application. However, it is our philosophy that the dynamics of the *uncontrolled* system, which are not explicitly addressed in any of the above references, should play a critical role in the design of control algorithms.

Our control philosophy can be outlined as follows: We first study the variety of trajectories of equations (2) and (3) in the *uncontrolled* setting (*i.e.*, constant r). We then use the *controlled* dynamics (*i.e.*, equations (2) and (3) with non-constant r) to switch between the various types of *uncontrolled* dynamics which then lead to the goal configuration. This approach proved to be useful in various situations (see, e.g., [3]). We emphasize that the transfer is very short in duration and hence the system remains uncontrolled for most of the steering procedure.

On a technical note, we assume that, except for the short time that the actuators must implement the change in shape configuration, they are at rest. That is, the actuators are engineered to maintain the constancy of r when inactive. For example, in the Chaplygin sleigh, we can view the mass as sliding on a rod where the friction between the rod and sliding mass, not the actuator, is applied to keep the mass fixed.

The exposition is organized as follows: In Section II we summarize the properties of the uncontrolled dynamics of the Chaplygin sleigh. In particular, we list all possible types of trajectories of the contact point of the sleigh on the plane. In Section III we study “control primitives” which implement the transitions between the uncontrolled trajectories of the sleigh. These control primitives, when applied in the proper order, result in the desired reorientation of the system. Simulations are presented in Section IV.

II. THE DYNAMICS OF THE CHAPLYGIN SLEIGH

A. The Configuration Variables

The Chaplygin sleigh is a rigid body sliding on a horizontal plane. The body is supported at three points, two of which slide freely without friction while the third is a knife edge, a constraint that allows no motion orthogonal to this edge. This mechanical system was introduced and studied in 1911 by Chaplygin [5] (although the work was actually finished in 1906).

The configuration space of this system is the group of Euclidean motions of the two-dimensional plane which we parameterize with coordinates (θ, x, y) . As Figure 1 indicates, θ and (x, y) are the angular orientation of the blade (shown as the bold segment in the Figure) and position of the contact point of the blade on the plane, respectively. We view the sleigh as a platform whose center of mass is at the contact point. The mass and moment of inertia of the platform relative to the contact point are M and I , respectively. There is also a point mass m positioned at (a, b) relative to the platform, see Figure 1. In the classical Chaplygin sleigh this mass is motionless relative to the platform; in Section III we

will control its position in order to steer the sleigh on the plane. The constraint imposed by the blade reads

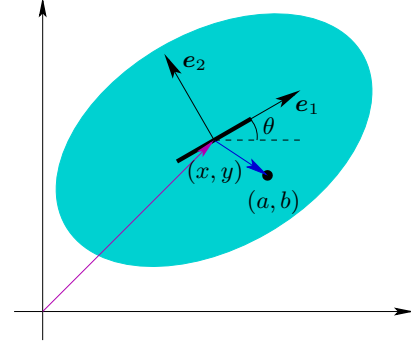


Fig. 1. The Chaplygin sleigh.

$$-\dot{x} \sin \theta + \dot{y} \sin \theta = 0. \quad (6)$$

This constraint is *nonholonomic*, whereby we mean it is not possible to derive the velocity constraint (6) from a position constraint $G(\theta, x, y) = 0$.

B. The Momentum Dynamics and Reconstruction

Let Ω^1 be the angular velocity of the platform and Ω^2 , Ω^3 be the components of linear velocity of the contact point along and orthogonal to the blade, respectively. Constraint (6) implies $\Omega^3 = 0$.

Denote the nonholonomic momentum by (p_1, p_2) . The components p_1 and p_2 satisfy the equations

$$\begin{aligned} \Omega^1 &= \frac{(M + m)p_1 + mbp_2}{(M + m)(I + ma^2) + Mmb^2}, \\ \Omega^2 &= \frac{mbp_1 + (I + ma^2 + mb^2)p_2}{(M + m)(I + ma^2) + Mmb^2}, \end{aligned}$$

see [2] and [16] for details and definitions. If $b = 0$, the components p_1 and p_2 equal the angular momentum of the sleigh relative to the contact point, and the projection of the linear momentum along the direction of the blade, respectively.

The dynamics of the Chaplygin sleigh is governed by the *momentum equations*

$$\dot{p}_1 = -ma\Omega^1\Omega^2, \quad \dot{p}_2 = ma(\Omega^1)^2, \quad (7)$$

coupled with the *reconstruction equations*

$$\dot{\theta} = \Omega^1, \quad \dot{x} = \Omega^2 \cos \theta, \quad \dot{y} = \Omega^2 \sin \theta, \quad (8)$$

(see, e.g., [16]). This representation of the equations of motion allows one to first solve (7) and then find the trajectory of the sleigh by integrating equations (8). We refer the reader to BKMM [2] and Zenkov and Bloch [16] for the details on the derivation of these equations.

The dynamics of the Chaplygin sleigh depends drastically on the value of a . This dependence is critical in the design of our control algorithm in Section III.

If $a = 0$, the momentum components p_1 and p_2 are preserved. Equations (8) then imply that the trajectory of

the contact point is either a circle or a straight line. In both cases the contact point is moving at a constant rate. The existence of circular trajectories is very important for our steering control algorithm.

If $a \neq 0$, the trajectories in the momentum plane are either equilibria situated on the line $(M + m)p_1 + mbp_2 = 0$, or elliptic arcs, as shown in Figure 2. Assuming $a > 0$, the equilibria located in the upper half plane are asymptotically stable (filled dots in Figure 2) whereas the equilibria in the lower half plane are unstable (empty dots). The elliptic arcs form heteroclinic connections between the pairs of equilibria. The trajectories of the contact point that correspond to the

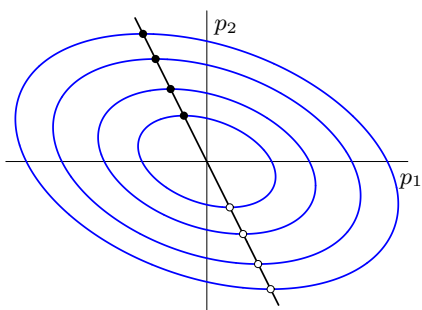


Fig. 2. The momentum dynamics of the unbalanced sleigh.

momentum equilibria are straight lines in the xy -plane. They are stable if the mass m precedes the contact point and unstable otherwise.

The trajectories of the contact point reconstructed from the heteroclinic momentum trajectories should be regarded as the transfer solutions from an unstable straight line motion to a stable one. A typical transfer trajectory is shown in Figure 3. The shape of these transfer trajectories is predetermined by the inertia of the body and the position of the center of mass relative to the contact point, and is independent of the initial conditions. The angle between the asymptotic directions of

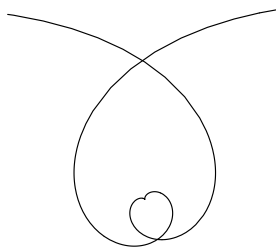


Fig. 3. A generic trajectory of the contact point.

a trajectory of the contact point in the xy -plane is evaluated in [11] for the case $b = 0$.²

²The formula obtained in [11] stays correct in the general case $b \neq 0$.

III. CONTROLLABILITY OF THE CHAPLYGIN SLEIGH WITH A MOVING MASS

A. The Reduced Controlled Dynamics

We now allow the point mass to change its position relative to the rigid body. That is, the quantities (a, b) are now dynamic variables. Assuming that the mass degrees of freedom are fully actuated, the system's dynamics are given by equations (2) and (3), where $r = (a, b)$ is viewed as the control parameter. Recall that the controller is active only when $\dot{a}^2 + \dot{b}^2 \neq 0$ —see the discussion of the physical implementation of controllers in the Introduction.

In order to write equations (2) and (3) for the Chaplygin sleigh with a moving mass explicitly, let

$$\xi^1 = \frac{(M + m)(p_1 - m\dot{a}b) + mb(p_2 + M\dot{a})}{(M + m)(I + ma^2) + Mmb^2},$$

$$\xi^2 = \frac{m[b(p_1 - m\dot{a}b) - (I + ma^2)\dot{a}] + [I + m(a^2 + b^2)]p_2}{(M + m)(I + ma^2) + Mmb^2},$$

and define η by

$$\frac{[Mmb^2 + I(M + m)]\dot{b} + a[(M + m)p_1 + mb(p_2 + M\dot{a})]}{(M + m)(I + ma^2) + Mmb^2}.$$

The momentum dynamics (2) for the Chaplygin sleigh with a moving mass is computed to be

$$\dot{p}_1 = -m\eta\xi^2, \quad \dot{p}_2 = m\eta\xi^1. \quad (9)$$

Observe that for $(a, b) = \text{const}$, equations (9) reduce to (7).

After solving equations (9), the group configuration variables (θ, x, y) are obtained from the reconstruction equations

$$\dot{\theta} = \xi^1, \quad \dot{x} = \xi^2 \cos \theta, \quad \dot{y} = \xi^2 \sin \theta.$$

B. Controllability of Asymptotic Directions

Recall that if (p_1, p_2) is constant and $p_2 \neq 0$, there are three types of motions for the uncontrolled ($\dot{a}^2 + \dot{b}^2 = 0$) dynamics:

1. If $a \neq 0$ and $(M + m)p_1 + mbp_2 \neq 0$, then the system's trajectory is a curve that approaches straight-line motions as $t \rightarrow \pm\infty$ (see Figure 3).
2. If $a = 0$ and $(M + m)p_1 + mbp_2 \neq 0$, then the system moves along a circle in the xy -plane at a constant rate.
3. If $(M + m)p_1 + mbp_2 = 0$, the system moves along a straight line in the xy -plane at a constant speed.

We remark that the first type is *generic* (i.e., observed with probability one when the initial conditions are randomly generated) whereas the second and the third types are not.

The objective of this paper is: Assuming that the sleigh is sliding (that is, $\xi^2 \neq 0$), find the control inputs that put the system on a trajectory which asymptotically approaches a straight line with the desired direction in the xy -plane. In the theorems below we prove that it is possible to change the trajectory type by controlling parameters a and b . The existence of the desired steering control algorithm follows immediately from these theorems.

Theorem 1: Assume that the initial motion of the system is circular, i.e., $a = 0$, $b = \text{const}$, and $(M + m)p_1 + mbp_2 \neq 0$.

Then there exist a continuously-differentiable function $a(t)$ and constants A , T_1 , and T_2 with properties

- $a(t) = 0$ when $t \leq T_1$ and $a(t) = A$ when $t \geq T_2$,
- $a(t)$ is increasing when $T_1 < t < T_2$,

such that the trajectory of the system with $a = a(t)$, $b = \text{const}$ asymptotically approaches a straight line motion with a given direction in the xy -plane.

Proof: Without loss of generality assume that $\theta = 0$ at $t = 0$. Choose positive constants A and T and consider a continuously-differentiable function $f(t)$ such that $f(t) = 0$ for $t \leq 0$, $f(t) = A$ for $t \geq T$ and $f(t)$ is increasing on $0 < t < T$. The actual shape of $f(t)$ on the interval $0 < t < T$ is not important. For instance, one can define $f(t)$ as

$$F_{A,T}(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{2A}{T^2} t^2 & \text{if } 0 < t \leq T/2 \\ A - \frac{2A}{T^2} (t - T)^2 & \text{if } T/2 < t \leq T \\ A & \text{if } t > T \end{cases}$$

(this will be our default choice).

Set $a = f(t)$. At the end of transition interval $0 < t < T$ the value of a becomes A . According to the classification of motions given above, the trajectory of the system for $t > T$ is either of type 1 or type 3. Let ϕ be the angle between the asymptotic direction of this trajectory as $t \rightarrow \infty$ (or the trajectory itself if it is a straight line) and the positive direction of the x -axis. Let ψ be the angle between the desired (asymptotic) direction of motion and the positive direction of the x -axis.

For the initial circular trajectory, let $T_1 \in \mathbb{R}$ be such that $\theta(T_1) = \psi - \phi$. Set $a(t)$ equal to $f(t - T_1)$. Then the trajectory of the system with $a = a(t)$ and $b = \text{const}$ satisfies the statement of the theorem. Indeed, this trajectory is obtained from the one corresponding to $a = f(t)$ by rotation about the center of the initial circular trajectory by the angle $\psi - \phi$. Therefore, the asymptotic direction of the trajectory forms the angle ψ with the positive direction of the x -axis. ■

Theorem 2: Assume that the system is moving along a trajectory of type 1, i.e., $a = A \neq 0$ and $(M + m)p_1 + mbp_2 \neq 0$.³ Then there exist a continuously-differentiable function $a(t)$ and constants T_1 and T_2 with properties

- $a(t) = A$ when $t \leq T_1$ and $a(t) = 0$ when $t \geq T_2$,
- $a(t)$ is decreasing when $T_1 < t < T_2$,

such that the trajectory of the system with $a = a(t)$, $b = \text{const}$ becomes a circle for $t > T_2$.

Proof: Choose the values T_1 , T_2 and set $T = T_2 - T_1$, $a(t) = A - f(t - T_1)$, where $f(t)$ is the function introduced in Theorem 1. Then at the end of the transition period the value of a equals 0. Therefore, the trajectory of the system for $t > T_2$ is either a circle, or a straight line. Adjusting the initial and terminal moments T_1 and T_2 of the transition period if

necessary, it is possible to have $(M + m)p_1 + mbp_2 \neq 0$. Therefore, the trajectory becomes circular for $t > T_2$. ■

Theorem 3: Assume that the system is moving along a straight line, i.e., $b = B_1 = \text{const}$ and $(M + m)p_1 + mbp_2 = 0$. Assume that $a = A > 0$. Then there exist a continuously-differentiable function $b(t)$ and constants $B_2 \neq B_1$, T_1 , and T_2 with properties

- $b(t) = B_1$ when $t \leq T_1$ and $b(t) = B_2$ when $t \geq T_2$,
- $b(t)$ is monotonic when $T_1 < t < T_2$,

such that the trajectory of the system with $a = A$, $b = b(t)$ becomes type 1 for $t > T_2$.

Proof: Define $b(t)$ by the formula

$$B_1 + F_{B_2 - B_1, T_2 - T_1}(t).$$

By adjusting the values of T_1 , T_2 , and B_2 , it is possible to satisfy the condition

$$(M + m)p_1(T_2) + mb(T_2)p_2(T_2) \neq 0.$$

Since the value of a has not changed, the trajectory of the system becomes type 1 for $t > T_2$. ■

Remark: The statement of the last theorem can be extended to the case of an initial straight line motion with $a = 0$. One just needs to change the value of a from 0 to A and then apply the algorithm of Theorem 3.

The reorientation algorithm can now be stated in the following steps:

1. Check if the trajectory of the sleigh is a straight line. If no, go to step 2. If yes, use Theorem 3 to transfer the sleigh to a generic trajectory and then go to step 2.
2. Check if the trajectory is circular. If yes, go to step 3. If no, use the control from Theorem 2 to transfer the sleigh to a circular trajectory and then go to step 3.
3. Using Theorem 1, exit the circular trajectory at an appropriate moment.

Remark: By Theorem 1, any point outside a circular trajectory in the plane belongs to an “exit” trajectory. It is now evident that the above three reorientation algorithm steps can be used to steer the Chaplygin sleigh through any point in the plane.

IV. SIMULATIONS

In this section we illustrate the control primitives obtained in Theorems 1–3. We assume that the numerical values of the parameters of the system are $I = 10$, $M = 2$, and $m = 1$. In all simulations the initial value of b is set to 0.

Figure 4 illustrates the steering algorithm of Theorem 1. The value of a on the circular trajectory equals 0, and $f(t)$ is chosen to be $F_{1,4}(t)$. If $a = f(t)$, the system’s trajectory leaves the circle along the dashed curve in Figure 4, right. The trajectory corresponding to $a = a(t) = f(t - 30)$ is the solid curve in Figure 4, right.

Figure 5 illustrates the control input that steers the system from a generic trajectory to a circular one. The initial value of a is 1, and $a(t)$ is set to $1 - F_{1,4}(t - 2)$.

Figure 6 illustrates the transfer from a straight line to a generic trajectory. The initial value of a is 0.1 and $b(t)$ equals $F_{2,4}(t)$.

³We assume, without loss of generality, that $A > 0$.

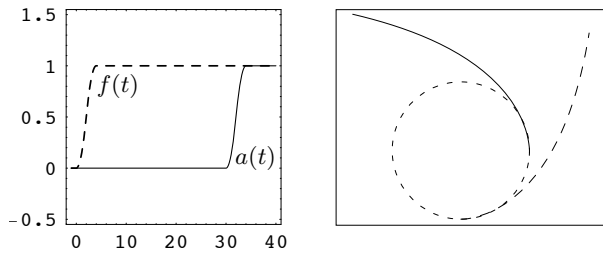


Fig. 4. Transition from a circle to the trajectory with desired direction.

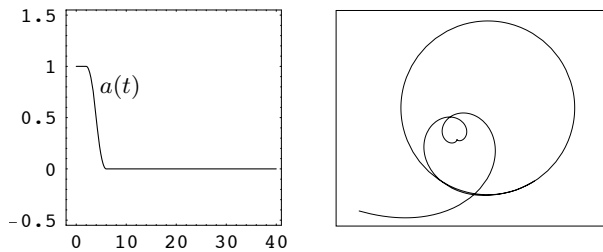


Fig. 5. Transition from a generic trajectory to a circle.

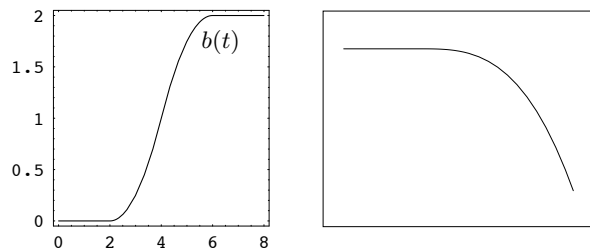


Fig. 6. Transition from a straight line to a generic trajectory.

V. CONCLUSIONS

In this paper we have developed a dynamical system approach to controlling the asymptotic dynamics of the Chaplygin sleigh. The key feature of our algorithm is the use of the controlled dynamics only for switching to and from circular trajectories. As a consequence, the controller remains unpowered most of the time.

While our control algorithm design is problem specific (the uncontrolled dynamics change for each mechanical system chosen) as a philosophy it is a general principle. Whether it

is applicable to a given situation depends of course on the nature of both the uncontrolled dynamics and controllers. We intend to extend this approach to a wider class of systems in a future publication.

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