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ZERO STRUCTURE ASSIGNMENT OF MATRIX PENCILS: THE CASE OF STRUCTURED ADDITIVE TRANSFORMATIONS

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Abstract— **Matrix Pencil Models are natural descriptions of linear networks and systems. Changing the values of elements of networks, that is redesigning them implies changes in the zero structure of the associated pencil by structured additive transformations. The paper examines the problem of zero assignment of regular matrix pencils by a special type of structured additive transformations. For a certain family of network redesign problems the additive perturbations may be described as diagonal perturbations and such modifications are considered here. This problem has certain common features with the pole assignment of linear systems by structured static compensators and thus the new powerful methodology of global linearisation [1, 2] can be used. For regular pencils with infinite zeros, families of structured degenerate additive transformations are defined and parameterised and this lead to the derivation of conditions for zero structure assignment, as well as methodology for computing such solutions. Finally the case of regular pencils with no infinite zeros is considered and conditions of zero assignment are developed. The results here provide the means for studying certain problems of linear network redesign by modification of the non-dynamic elements.**

I. INTRODUCTION

The general problem that is addressed is the redesign of networks and systems by either modifying the topology of interconnections and/or changing the type and values of the elements. Within this general family of problems that belong to this class, there exist a family of structure assignment problems formulated on matrix pencils [16] and one of these problems is considered here. Matrix Pencil models are natural descriptions of implicit descriptions of networks [17]. The structure assignment problems [6]-[8] which we consider are equivalent to a zero assignment of the regular matrix pencil $sF+G+H$, where $sF+G$ may express the internal dynamics matrix of a system (described in extended state space form) and $H=U\Lambda V$ may represent a static structural change; in fact, U,V are known graph incidence matrices (they may express a topology modification) and Λ is a diagonal matrix of continuous design parameters. In reality, the three matrices U, V, Λ are design parameters. Here we shall assume that the incidence matrices U,V are fixed and thus only the diagonal matrix Λ is free for the assignment of zeros $sF+G+U\Lambda V$. A large family of such problems can be reduced to the case of diagonal additive perturbations and this is the problem considered here in some detail. The paper deals with both the study of solvability conditions, as well as the method for computation of solutions, whenever such solutions exist.

The general properties of the frequency assignment map are considered first and the notion of degenerate transformations, i.e. those making the pencil $sF+G+H$ singular are defined. For the case of pencils with infinite zeros, a parametrisation of the set of degenerate transformations H is given based on the nature of the resulting singularity of the pencil. The significance of degenerate solutions is emphasised by establishing the property that if the differential of the frequency assignment map at a degenerate point H_0 is onto, then this implies assignability of zero structure of the pencil by some appropriate H. The explicit form of the differential at a degenerate point is computed and it is shown that for a generic pencil there exist degenerate points H_0 such that the corresponding differential is onto. Using as the starting point such degenerate solutions, it is shown that the non-degenerate transformations H, may be constructed to assign the zeros of $sF+G+H$ in the neighbourhood of any arbitrary symmetric set of complex numbers. The proposed methodology is a Quasi-Newton type numerical approach and its convergence properties are examined. Finally, the case of pencils with no infinite zeros is considered and conditions for the complex zero assignment are derived in terms of invariants associated with the pencil.

II. ZERO ASSIGNMENT OF MATRIX PENCILS: BACKGROUND RESULTS

Linear networks and systems may be described in a natural way by matrix pencil models [17]. Frequently, issues of redesign of the parameters and/or interconnection topology of the system arise [16]; such problems are not of the traditional control type, but they may be studied with control theoretic tools. There is a large number of state space redesign problems [16] and here we consider one of the most basics which is equivalent to zero structure assignment of matrix pencils by additive perturbations. The mathematical formulation of this abstract problem can be stated as follows:

Problem formulation: Given a square matrix pencil [4] $sA+B$ such that $A, B \in R^{n\times n}$, $rankA = n_1 < n$ the problem to be examined here is to investigate the solvability of the equation:

$$\det(sA + B + \Lambda) = \varphi(s) \qquad (1)$$

with respect to $\Lambda = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$when $\phi(s)$is a given polynomial of n_1 degree. □

Notation: $Q_{m,n}$ is the set of lexicographically ordered sequences of m integers from n set of integers and D_n is any sequence of n integers from (1,2,...,n) with possible repetition and any order.

Definition (1): A sequence $\omega = (i_1, i_2, ..., i_n) \in Q_{n,2n}$ characterises a minor α_ω of $C_n[I_n, \Lambda]$. On such sequences we define the following:

(a) The operation π on $\omega \in \mathrm{Q_{n,2n}}$ is defined as:

$$\pi(\omega) \triangleq (\pi(i_1), \pi(i_2), ..., \pi(i_n) = (j_1, ..., j_n)$$
$$\pi(i_k) = \left\{ \begin{array}{ccc} i_k & if & i_k \le n \\ \hat{i}_k = i_k - n & if & i_k > n \end{array} \right.$$

b) A sequence $\omega = (i_1, i_2, ..., i_n) \in \mathrm{Q_{n,2n}}$ is called *degenerate*, if $\pi(\omega) = (j_1, j_2 ..., j_n)$ has at least two equal elements (i.e. $j_l = j_k$) and it is *nondegenerate*, if $\pi(\omega) = (j_1, j_2 ..., j_n)$ has distinct elements.

c) For a sequence $\omega \in \mathrm{Q_{n,2n}}$, which is nondegenerate we define as the *sign of* $\omega, sign(\omega) = \sigma(\omega) = sign(j_1, j_2 ..., j_n)$ and as the *trace of* ω, the subset of the elements of $\pi(\omega) = (j_1, j_2 ..., j_n)$ which correspond to $i_k > n$ and thus is the set $<\omega>= (\hat{i}_{k_1}, \hat{i}_{k_2}, ..., \hat{i}_{k_\mu})$, *ì≤n.* □

Proposition (1): Let $[I_n, \Lambda] \in \Re^{n\times 2n}$ and denote

$$C_n\left[I_n, \Lambda\right] = [........, a_\omega, ...] \in \Re^{1\times \binom{2n}{n}}, \quad \omega \in Q_{n,2n}$$

Then a_ω are defined as follows:

- $a_\omega=0$, if ω is degenerate
- $a_\omega \neq 0$, if ω is nondegenerate

Furthermore, if ω is nondegenerate, $\sigma(\omega)$ is the sign of ω and $<\omega>= \left\{\hat{i}_{k_1}, \hat{i}_{k_2}, ..., \hat{i}_{k_\mu}\right\}$ is the trace of ω, then $a_\omega = \sigma(\omega)\lambda_{\hat{i}_{K_1}}\lambda_{\hat{i}_{k_2}}...\lambda_{\hat{i}_{k_\mu}}$. □

The set of $Q_{n,2n}$ sequences may thus be divided into two disjoint sets, the set $Q^d_{n,2n}$ of degenerate sequences and the set $Q^{nD}_{n,2n}$ of nondegenerate sequences. Both subsets of sequences are assumed to be lexicographically ordered. Consider now the characteristic redesigned polynomial

$$\Phi(s) = \det(sA + B + \Lambda) \triangleq \Phi(A, B, \Lambda)$$

By the Binet-Cauchy theorem we have that:

$$det[sA + B + \Lambda] = det([I_n, \Lambda_n] \cdot [sA^T + B^T, I_n]^T) =$$
$$C_n([I_n, \Lambda_n])C_n([sA^T + B^T, I_n]^T) = \Phi(s). \qquad (2)$$

Definition (2): Let $Q^D_{n,2n}Q^{nD}_{n,2n}$ be the ordered subjects of degenerate and nondegenerate of $Q_{n,2n}$ associated with the $[I_n,I_n]$ structure. We shall denote by $\tilde{C}_n([I_n,\Lambda])$ the subvector of $C_n([I_n,\Lambda])$ obtained by omitting all zero co-ordinates corresponding to $Q^d_{n,2n}$ sequences and similarly by $\tilde{C}_n([sA^t + B^t, I_n])$ the reduced subvector of $C_n([sA^t + B^t, I_n])$ derived by deleting the $Q^D_{n,2n}$ set of coordinates. The subvectors $\tilde{C}_n([I_n,\Lambda])$, $\tilde{C}_n([sA^t + B^t, I_n])$ will be referred to as *$[I_n, I_n]$-structured projections*. Note that

$$C_n[I_n, \Lambda]C_n\left(\left[\begin{array}{c} sA + B \\ I_n \end{array}\right]\right) =$$
$$\tilde{C}_n\left([I_n, \Lambda]\right)\tilde{C}_n\left(\left[\begin{array}{c} sA + B \\ I_n \end{array}\right]\right) = \quad \Phi(s) \qquad (3)$$

and given that

$$\tilde{C}_n\left([I_n, \Lambda]\right) = [....., a_\omega, ...]$$
$$= \left[...., \sigma(\omega)\lambda_{\hat{i}_{k_1}}....\lambda_{i_{k_\mu}}, ...\right]$$
$$= \left[....., \lambda_{\hat{i}_{\kappa_1}}....\lambda_{\hat{i}_{\kappa_\mu}}, ...\right] diag\left\{...., \sigma(\omega), ...\right\} \qquad (4)$$
$$= \hat{C}_n\left([I_n, \Lambda]\right) D\left(\sigma\left(\omega\right)\right) \quad \omega \in Q^{nD}_{n,2n}$$

then

$$\Phi\left(s\right) = \tilde{C}_n\left([I_n, \Lambda]\right) D\left\{\sigma(\omega)\right\}\tilde{C}_n\left(\left[\begin{array}{c} sA + B \\ I_n \end{array}\right]\right)$$
$$= \tilde{C}_n\left([I_n, \Lambda]\right)\tilde{C}_n\left(\left[\begin{array}{c} sA + B \\ I_n \end{array}\right]\right) \qquad (5)$$

The vectors

$$\hat{C}_n\left([I_n, \Lambda]\right) \triangleq \tilde{C}_n\left[I_n, \Lambda\right] D\left\{\sigma(\omega)\right\} \in \Re^{1\times 2^n} \qquad (6)$$

$$\hat{C}_n\left(\left[\begin{array}{c} sA + B \\ I_n \end{array}\right]\right) \triangleq D\left\{\sigma\left(\omega\right)\right\}\tilde{C}_n\left[\begin{array}{c} sA + B \\ I_n \end{array}\right]$$
$$= \underline{\hat{P}}\left(s\right) \in \Re^2[s]$$

will be referred to as *normalised* $[I_n, I_n]-$*structured projections* of $C_n\left([I_n, \Lambda]\right)$, $C_n\left([sA^t + B^t, I_n]\right)^t$ respectively. In particular, $\hat{C}_n\left([sA^t + B^t, I_n]\right)^t = \underline{\hat{P}}(s)$, will be called the $[I_n, I_n]-$*Grassmann representative*. □

Proposition (2): The normalised $[I_n,I_n]$-structured projection of $\hat{C}_n\left([I_n, \Lambda]\right)$ may be expressed as:

$$\tilde{C}_n\left([I_n, \Lambda]\right) = (1, \lambda_1) \otimes (1, \lambda_2) \otimes ..... \otimes (1, \lambda_n) \qquad (7)$$

where $\otimes$ denotes the standard tensor product. □

The above result follows by inspection of the expression of $\hat{C}_n([I_n,\Lambda])$. The characteristic polynomial is expressed as in (5) and it is generated by the $[I_n, I_n]$-Grassmann representative of the system i.e.

$$\underline{\hat{P}}(s) = \hat{C}_n\left(\begin{bmatrix} sA+B \\ I_n \end{bmatrix}\right) \quad (8)$$

Remark (1) : For any sA+B, $C_n([sA^t + B^t, I_n])$ is a polynomial vector; however, $\underline{\hat{P}}(s)$ is not necessarily coprime.

Definition (3) : The greatest common divisor of the entries of $\underline{\hat{P}}(s)$ will be denoted by $\Phi_{A,B}(s)$ and this will be referred to as the *$[I_n, I_n]$-fixed polynomial* of the system. A system for which $\Phi_{A,B}(s)$=1 will be called *$[I_n, I_n]$-irreducible*; otherwise, it will be called *$[I_n, I_n]$-reducible*.

The following result can be readily established:

Theorem (1): The fixed zeros of the redesigned polynomial $\Phi(A, B, \Lambda)$ for all possible Λ are only the roots of $\Phi_{A,B}(s)$polynomial. □

We can now easily establish that:

$$det[sA + B + \Lambda] =$$
$$(1, \lambda_1) \otimes (1, \lambda_2) \otimes \ldots\ldots \otimes (1, \lambda_n) = \underline{\hat{P}}(s)$$

By equating the coefficients of the powers of s we get:

$$(1, \lambda_1) \otimes (1, \lambda_2) \otimes \ldots \otimes (1, \lambda_n) \cdot P = \phi$$

where ϕ is the coefficient vector of $\phi(s)$ and P is called the *Plucker matrix* for the problem [3].

Example (1) : Let a system matrix of a circuit be:

$$sA + B = \begin{bmatrix} s+5 & s-1 & s \\ 2s & s & s+3 \\ 1 & 2 & -1 \end{bmatrix}$$

In this case the $C_3([I_3,\Lambda_3])$ matrix is

$$C_3[I_3, \Lambda_3] \equiv C_3 \begin{bmatrix} 1 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \lambda_3 \end{bmatrix}$$

And can be calculated to be:

$$(1, 0, 0, \lambda_3, 0, -\lambda_2, 0, 0, 0, \lambda_2\lambda_3, \lambda_1, 0, 0, 0,$$
$$-\lambda_1\lambda_3, 0, \lambda_1\lambda_2, 0, 0, \lambda_1\lambda_2\lambda_3)$$

The [sA+B, $I_n]^T$ matrix is expressed as:

$$\begin{bmatrix} sA+B \\ I_3 \end{bmatrix} \equiv \begin{bmatrix} s+5 & s-1 & s \\ 2s & s & s+3 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The nonzero elements of $C_3([I_3,\Lambda_3])$ are (1, λ_3,- λ_2, $\lambda_2\lambda_3$, λ_1, - $\lambda_1\lambda_3$, $\lambda_1\lambda_2$, $\lambda_1\lambda_2\lambda_3$) and the corresponding elements of $C_3([sA+B,I_3]^T)$ are ($3s^2$-21s-33, $-s^2$+7s, 2s+5, s+5, -3s-6, -s, -1, 1). Therefore:

$$det(sA+B+\Lambda) = [1, \lambda_3, \lambda_2, \lambda_2\lambda_3, \lambda_1, \lambda_1\lambda_3, \lambda_1\lambda_2, \lambda_1\lambda_2\lambda_3].$$
$$[3s^2 - 21s - 33, -s^2 + 7s, 2s+5, s+5, -3s-6, -s, -1, 1]^T$$

□

The problem described involves the solution of a set of nonlinear algebraic equations. When the number of solutions is finite, this number is combinatorially large (one can prove that the degree is $n!$) and this makes the problem difficult to be investigated via the standard Groebner basis tools [9] especially when n is large. To construct a solution of the problem we will follow the methodology in [2] by studying the local properties of degenerate solutions.

The *Frequency Assignment Map* associated with the problem is the map assigning Λ to the coefficient vector ϕ i.e.

$$F : R^n \rightarrow R^n : F(\Lambda) = \varphi$$

A diagonal matrix Λ_0 is *degenerate* iff:

$$F(\Lambda_0) = 0 \text{ or equivalently}$$

$$\det(sA + B + \Lambda_0) = 0$$

In other words, Λ_0 is degenerate if the pencil $sA + B + \Lambda_0$ becomes singular. The following theorem shows the great importance of degenerate matrices.

Theorem (2): If there exists a degenerate matrix $\Lambda_0$such that the differential $DF_{\Lambda o}$ is onto then any set of n frequencies can be assigned via some diagonal perturbation. □

For a generic $n \times n$ pencil when n is small the set of all degenerate matrices may be constructed by use of Groebner Basis algorithm [9].

Example (2) Consider the Pencil

$$sA + B = \begin{bmatrix} -3s & 2+4s & -1-s \\ -3+4s & 5+s & -1-2s \\ -4+s & 6+5s & -1-3s \end{bmatrix}$$

then the set of equations defining all the degenerate matrices *diag*{*x,y,z*} is given by:

x - 4 y - x y + 6 z + 5 x z + x y z=0
-5 + x - 3 x y - 11 z + x z - 3 y z=0
-2 + 7 x + 10 y - 19 z=0

a Groebner Basis for the above set of equations is:

480 + 5312*x + 16433*x^2 + 21474*x^3 + 15452*x^4 + 5726*x^5 + 147*x^6=0
1579680 - 10392988*x - 18923271*x^2 - 12885549*x^3 - 3302425*x^4 - 81879*x^5 + 2714400*y=0
2122560 - 12293068*x - 18923271*x^2 - 12885549*x^3

-3302425*x^4 - 81879*x^5 + 5157360*z=0

which gives 3! solutions 4 real and 2 complex.

One can calculate the number of degenerate matrices for a generic pencil as follows:

Theorem (3) For a generic $n \times n$ pencil $sA + B$ such that $rank(A) = n - 1$ the number of degenerate diagonal matrices is finite and equal to $n!$. □

III. CLASSIFICATION OF THE SET OF DEGENERATE COMPENSATORS

We may classify the degenerate matrices Λ of a Pencil *sA+B* according to the sizes of row or column minimal indices of $sA + B - \Lambda$.

Definition: A degenerate matrix Λ of a Pencil $sA + B$ is of degree k if the polynomial module that spans the right Kernel of $sA + B - \Lambda$ has Forney dynamical order k.

Theorem (4): For a generic nxn pencil sA+B with rank(A)=n-1 the number B_d of degenerate diagonal matrices of degree d, (0≤d≤n-1) is finite

$$B_d = \begin{cases} \begin{pmatrix} n \\ d+1 \end{pmatrix} A_{d+1} & \text{if} \quad d > 0 \\ 1 & \text{if} \quad d = 0 \end{cases}$$

where $A_{d+1}$is the number of permutations of d+1 objects with no fixed points. □

Although the construction of degenerate matrices looks as though it has the same complexity as that of the problem we have started, there is a certain degenerate matrix that can be easily constructed via linear equations. These are the degenerate diagonal matrices of degree 0 and n-1.

Proposition (3): Let v^t, w vectors such that:

$$v^t A = 0, Aw = 0$$

then the diagonal matrices

$$\Lambda_0 = -diag\{\frac{v^t b_1}{v_1}, ..., \frac{v^t b_n}{v_n}\}, \Lambda_0 = -diag\{\frac{v^t b_1}{v_1}, ..., \frac{v^t b_n}{v_n}\},$$

$$\Lambda_{n-1} = -diag\{\frac{b_1^t w}{w_1}, ..., \frac{b_n^t w}{w_n}\}$$

where $b_i, (b_i^t)$ are the columns (rows) of B and $v_i(w_i)$ are the coordinates of $v(w)$, is degenerate. □

Another classification of the degenerate matrices are into infinite and finite. Infinite are those solutions that are taken as limits of sequences Λ_n whose one or more elements tend to infinity. The degenerate matrices constructed in proposition 4 are finite iff $v_i \neq 0$. If V is the basis matrix of the left kernel of A next theorem characterises V so that there exists at least one finite degenerate matrix.

Theorem (5): If $V = \begin{bmatrix} \underline{v}_1 & ... & \underline{v}_n \end{bmatrix}$is a basis matrix of the left kernel of A then there exists a $v \in V$ such that the corresponding degenerate matrix produced by v is finite iff $\underline{v}_i \neq 0$. □

Note that if the above defined V has not the desired properties if there exists a $k \times n$ submatrix of A, say A', such that $rank(A) = rank(A')$.

IV. GENERICITY RESULTS AND CONSTRUCTION OF SOLUTIONS

The differential of the frequency assignment map F related to our problem, plays a very important role in the determination of the onto properties of the map and therefore in the solvability of the problem. This can be calculated in many ways and for a general square rank deficient polynomial matrix A(s) it can be proved that:

Lemma (1): The following holds true:

$$\det(A(s) + xB(s)) = x.trace(Adj(A(s))B(s)) + O(x^2)$$

this shows that if $adj(sA + B - \Lambda_0) = g(s) \cdot v^t(s)$ then $DF_{\Lambda 0}$ can be represented by the coefficient matrix of the polynomial vector $(g_1(s)v_1(s), ..., g_n(s)v_n(s))$. Next we will prove the following result:

Proposition (4): For a generic Pencil the degenerate diagonal matrix Λ_0 of the zero assignment map of the problem, satisfies $rankDF_{\Lambda_0} = n$ □

Next we will prove that a Quasi-Newton type of numerical method starting from a regular degenerate matrix can produce diagonal matrices which assign the desired frequencies and it is within an r distance from the degenerate matrix.

Theorem (6): Let $M = n \left\| DF_{\Lambda_0}^{-1} \right\| \|T\|$, $a = \left\| DF_{\Lambda_0}^{-1} \underline{\phi} \right\|$ and ε,r be such that:

$$\varepsilon \leq \frac{r}{a} < \frac{(M^{-1} + \|\Lambda_0\|^{n-1})^{\frac{1}{n-1}} - \|\Lambda_0\|}{a} = \varepsilon_0$$

then a sequence Λ_k produced by the iteration:

$$\Lambda_{k+1} = \Lambda_k - DF_{\Lambda_0}^{-1}(F(\Lambda_k) - \varepsilon \underline{\phi})$$

converges to a Λ that satisfies:

$$F(\Lambda) = \varepsilon \underline{\phi}, \quad \|\Lambda - \Lambda_0\| \leq r$$

□

The above suggest the following methodology for the solution of the problem:

1) Construct the degenerate matrix Λ_0 as above
2) Use the iteration:

$$\Lambda_{k+1} = \Lambda_k - DF_{\Lambda_0}^{-1}(F(\Lambda_k) - \varepsilon \underline{\phi})$$

with the parameters as in the Theorem (6) and starting from Λ_0, until convergence is reached.

Example (3): Consider a network whose system matrix $T(s)$ is defined by:

$$\begin{bmatrix} G1+G2+sC & -G2 \\ -G2 & G2+G3+1/[(sL+(1/G4)] \end{bmatrix}$$

Letting X_3=*1/[(sL+(1/G4)]*X_2 , T(s) is transformed to:

$$\begin{bmatrix} G1+G2+sC & -G2 & 0 \\ -G2 & G2+G3 & 1 \\ 0 & 1 & -sL-(1/G4) \end{bmatrix}$$

when the values are: C=1,L=1,G1=4,G2=1,G3=0,G4=∞ the system matrix becomes:

$$T_1(s) = \begin{bmatrix} s+5 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -s \end{bmatrix}$$

assuming that we would like to change the natural frequencies of the above system by tuning the values of G2,G3,G4, we get the following perturbation:

$$\begin{bmatrix} G2 & -G2 & 0 \\ -G2 & G2+G3 & 0 \\ 0 & 0 & G4 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G2 & 0 & 0 \\ 0 & G3 & 0 \\ 0 & 0 & G4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U\Lambda U^T$$

Which is equivalent to applying a diagonal perturbation $\Lambda = diag(G2, G3, G4)$ to the system

$$U^{-1}T_1(s)(U^T)^{-1} = \begin{bmatrix} s+5 & s+4 & 0 \\ s+4 & s+4 & 1 \\ 0 & 1 & -s \end{bmatrix}$$

The degenerate perturbations are defined by:

$$f2(G2, G3, G4) = -1 - G2 - G3 = 0$$
$$f1(G2, G3, G4) = -5 - 4G2 - 5G3 - G2G3 + G4 + \\ +G2G4 + G3G4 = 0$$
$$f0(G2, G3, G4) = -5 - G2 + 4G4 + 4G2G4 + 5G3G4 + \\ +G2G3G4 = 0$$

and the finite solutions are given by:

a) G2=-2, G3=1, G4=-3
b) G2=0, G3=-1, G4=-5

both of them are full (or regular), so both can be used as staring points for a numerical Quasi-Newton method to place the characteristic polynomial at any given second order one, $p(s)$:

$$x_{n+1} = x_n - (Jf)^{-1}_{x_0}(f - ep)$$

where

$$x = (G2, G3, G4)^T, p = [1, 8, 15]^T, f = [f2, f1, f0]^T$$

and $x_0 = (-2, 1, -3)^T$. Starting with e=0.5 the method converges after about 60 iterations to $x_{60} = (-2,5507, 1,050697, -2,74137)^T$. Taking now this as a starting point we repeat the method for e=1.2 and so on. The following table displays the various solutions we obtain through this algorithm the last column being the Euclidean distance of the solution from the degenerate one:

Iterations	E	G2	G3	G4	Dist from deg perturbation
0	0	-2	1	-3	0
60	0,5	-2,55	1,050	-2,741	0,610
50	1,2	-3,325	1,125	-2,652	1,375
85	2,5	-4,706	1,206	-2,611	2,741
135	5	-7,278	1,278	-2,594	5,301
250	10	-12,33	1,333	-2,588	10,34
80	18	-20,36	1,365	-2,586	18,37

V. ARBITRARY ASSIGNMENT IN TERMS OF THE PLUCKER MATRIX: THE CASE N = N$_i$

The onto properties of a polynomial map such as F can be examined in terms of its differential. The rank of the differential of a complex algebraic map although it is a local invariant may determine its global properties [14].

Proposition (5): If F is a algebraic map between two complex varieties X,Y such that dimX$\leq$Y then: there exists x $\in$ X: rankDF$_X$ =dim Y iff F is (almost) onto. □

This shows that the invariant that characterises the onto property of the map F is the n-th exterior product of its differential DF$_X$ and in the case we examine, this invariant is the determinant of the Jacobian of F , i.e. det(J(F)$_X$). Due to the property that F(x) = f(x).P, where f(x)=[1,x$_l$,x$_2$,...,x$_1$x$_2$..x$_n$] the Jacobian of the pole placement map, that can be calculated in terms of the Jacobian of f and the Plucker matrix P, i.e. det(J(F)$_x$)=C$_n$(J(f)).C$_n$(P). Thus, the calculation of det(J(F)$_x$) is reduced to calculating C$_n$(J(f)). The calculation of J(f) is easily achieved by the following result:

Theorem (7): The partial derivative of f with respect to x$_i$, is given by:

$$(1, x_1) \otimes \ldots \otimes (1, x_{I-1}) \otimes (0, 1) \otimes (1, x_{I+1}) \otimes \ldots \otimes (1, x_n)$$

□

Using the above, we select n entries of the vector f(x) say $a = [a_l, a_2, ..., a_n]$, and call the Jacobian of the function a, J(a); then this is a square $n \times n$ matrix whose determinant is one of the coordinates of the vector C$_n$(J(f)), conversely, all the coordinates C$_n$(J(f)) are of the form det (J(a)) for some a. The following result provides a the description of the compound C$_n$(J(f)).

Theorem (8): The Jacobian J(a) is given by:

$$J(a) = diag(x_l^{-1}, x_2^{-1}, ..., x_n^{-1})I(a)diag(a_l, a_2, ..., a_n)$$

where the ij entry of I(a) is 1 if a_j contains x_i and 0 otherwise. Therefore the determinant of J(a) is :

$$det(J(a)) = I(a)a_1, a_2...a_n/x_1x_2...x_n \quad \square$$

Every selection of p monomials $a = [a_1, a_2...a_n]$ corresponds to a minor M_a of P. For a given monomial m consider the sum $P_m = \Sigma \det(I(a))M_a$ where the sum is taken when $a_1,a_2...a_n/x_1x_2...x_n$ = m and $\det(I(a) \neq 0$. The collection of all P_m constitutes a system invariant characterising the onto properties of the pole placement map. In fact:

Theorem (9): The complex pole placement map is onto if there exists m such that $P_m \neq 0$. □

VI. CONCLUSIONS

A special problem of structure assignment formulated on matrix pencils under structured additive perturbations has been considered and conditions for its solvability (as well as computation of solutions) have been derived. The abstract problem that has been considered belongs to the general class of redesign of networks [16], by either modifying the topology of interconnections and/or changing the type and values of the elements. The case considered here corresponds to the diagonal perturbations; the results can also be extended to the structured perturbations case, since such cases can also be handled within the current exterior algebra framework. Structure assignment problems may be formulated on pencils [7], [8], [10], but they may be also defined on general polynomial models [5], [6] and are related to zeros, or other types of invariants. The current framework is suitable for zero assignment problems.

REFERENCES

[1] J. Leventides and N. Karcanias, (1995) Sufficient conditions for arbitrary Pole assignment by constant decentralised output feedback,. Mathematics of Control for Signals and Systems, 1995,8,222-240.

[2] J. Leventides and N. Karcanias. (1995) Global asymptotic linearisation of the pole placement map: A closed form solution for the constant output feedback problem, Automatica, Vol. 31, No. 9, pp 1303-1309.

[3] Karcanias N. and Giannakopoulos C. (1984). Grassmann invariants, Almost zeros and the Determinantal zero, pole assignment problems of linear systems. Int. J. Control, Vol. 40, 673698.

[4] F.R. Gantmacher(1959), The Theory of Matrices, Vol.2, Chelsea, New York.

[5] H.H. Rosenbrock and B.A. Rowe, 1970, "Allocation of poles and zeros", Proc. IEE, Vol. 117,pp 1079.

[6] N. Karcanias and C. Giannacopoulos, 1989, "Necessary and sufficient conditions for Zero assignment by Constant Squaring Down", Linear algebra and Its Applic, Vol. 122/123/124, pp. 415-446.

[7] N. Karcanias, 1992, "The model projection problems in the global instrumentation of a process", IFAC Workshop on Interactions between Process Design and Process Control, Imperial College, London, Sept. 6-8, 1992, Pergamon Press, pp. 167-170.

[8] J.J. Loiseau, S. Mondie, I. Zaballa and P. Zagalak, 1997, "The problem of assigning the Kronecker Invariants of a matrix pencil by row or column completion", Linear Algebra and Its Applications.

[9] T. Becker, V. Weispfenning (1991), "Groebner bases: A computational approach to Commutative Algebra", Springer Verlag.

[10] W. Helton, J. Rosenthal and X. Wang (1997), Matrix Extensions and eigenvalue completions, the generic case, AMS Trans, Vol. 349, No 8, pp 3401-3408.

[11] S. Friedland (1977), Inverse eigenvalue problems, Linear Algebra and Applications 17,15-51.

[12] Leventides J. and Karcanias N. (1995), Sufficient conditions for arbitrary pole assignment by constant decentralised output feedback, MCSS, Vol 8, 222-240.

[13] Wang X. (1994), Decentralised pole assignment and product Grassmannians, SIAM J. Cont. and Opt., pp 855-875.

[14] Humphreys J.E. (1975), Linear Algebraic Groups, Graduate Texts in Mathematics, Springer Verlag, NY.

[15] Fulton W. (1984), " Intersection Theory", Springer-Verlag, New York.

[16] Karcanias N. and Leventides J. (2000), System Redesign and Control Theory, Control Engineering Centre Research Report, May 2000, City University.

[17] Karcanias N. and Hayton G.E. (1981), Generalised Autonomous Dynamical Systems, Algebraic Duality and Geometric Theory. Proceedings of 8^{th} IFAC World Congress, Kyoto, Japan, Pergamon Press (1982), 289-294.