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$\mathcal{L}_2$ performance design problem for systems presenting nested saturations

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***Abstract*— This paper addresses the problem of stabilization for systems presenting nested saturations and $\mathcal{L}_2$ limited disturbance. The $\mathcal{L}_2$ performance design problem of the system are studied. LMI conditions and associated convex optimization problems are proposed in order to determine both an inner and an outer ellipsoids. Indeed, the closed-loop trajectories remain bounded in the outer ellipsoid despite the action of the disturbance provided that the initial states are starting from the inner ellipsoid. In addition, it is also derived a condition which allows to synthesize the saturating gains in order to minimize the $\mathcal{L}_2$ gain from the disturbance to the controlled output by the solution of an LMI optimization problem.**

I. INTRODUCTION

This paper focuses on the class of nonlinear systems resulting from nested saturations. This kind of systems can be used to represent the behavior of nonlinear actuators, and, therefore, to study the stability of control systems subject to both amplitude and dynamic actuator saturations (see, for example, [9], [7], [1]). On the other hand, analysis and design methodologies for systems presenting nested saturations can be useful to address stability issues of more general classes of nonlinear systems. For instance, the use of nested saturations becomes very interesting when one uses forwarding techniques for cascade systems with linear part [14], [17], [18].

Hence, the system under consideration in this paper has the following form:

$$\begin{array}{rcl} \dot{x} & = & A_p x + B_p sat_p\left(A_{p-1}x + B_{p-1}sat_{p-1}\left(A_{p-2}x\right.\right. \\ & & \left.\left. + \cdots (A_1 x + B_1 sat_1(Fx))\cdots\right) + B_w w \\ z & = & Cx + Dsat_p\left(A_{p-1}x + B_{p-1}sat_{p-1}\left(A_{p-2}x\right.\right. \\ & & \left.\left. + \cdots (A_1 x + B_1 sat_1(Fx))\cdots\right) + B_z w \end{array} \quad (1)$$

where $x \in \Re^n$ is the state of the system, $w \in \Re^q$ is the disturbance and $z \in \Re^l$ is an auxiliary performance vector. For all $j \in \{1,\dots,p\}$, A_j, B_j and F are matrices of appropriate dimensions (eventually having different dimensions depending on the index j). B_w, C, D and B_z are given matrices of appropriate dimensions. Furthermore, sat_j is a componentwise saturation map $\Re^{m_j} \to \Re^{m_j}$ defined as: $(sat_j(v))_{(i)} = sat_j(v_{(i)}) = sign(v_{(i)})\min(u_{j(i)}, |v_{(i)}|)$, $\forall i = 1,\dots,m_j$ where $u_{j(i)}$, $u_{j(i)} > 0$, denotes the ith bound of the saturation function.

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The disturbance vector w is assumed to be limited in energy, that is $w \in \mathcal{L}_2^q$ and for some scalar δ, $0 < \frac{1}{\delta} < \infty$, one gets:

$$\| w(t) \|_2^2 = \int_0^t w(\tau)'w(\tau)d\tau \leq \delta^{-1}, \forall t > 0 \quad (2)$$

One of the classical way to measure the disturbance tolerance should consist in computing the $\mathcal{L}_2$ gain, which corresponds to the largest ratio between the $\mathcal{L}_2$-norms of the controlled output z and the disturbance w. Note that due to the presence of nested saturations, a large disturbance can lead system (1) to have unbounded trajectories. In this case, the $\mathcal{L}_2$ gain will be not well defined. Hence, it is of major importance to ensure that the trajectories of the system are bounded for any w satisfying (2), provided that the initial condition belongs to a certain admissible set. Furthermore, in order to ensure the asymptotic stability when the disturbances are vanishing one has to guarantee that the state of the system does not leave the basin of attraction of the origin. Based on these considerations, in the current paper, the class of disturbance under consideration are those bounded in energy, as defined in (2).

The problem we intend to solve can therefore be summarized as follows.

Problem 1: Determine the gains A_j, $j = 1,...,p-1$ and F, two sets $\mathcal{E}_0$ and $\mathcal{E}_1$ such that the following properties are satisfied with respect to the resulting closed-loop system:

1) Internal stability: when $w = 0$, for any $x(0) \in \mathcal{E}_0$ the closed-loop trajectories asymptotically converge towards the origin;
2) Input-to-State stability: when $w \neq 0$, the closed-loop trajectories remain confined in $\mathcal{E}_1$ for any $x(0) \in \mathcal{E}(0)$ and any disturbance satisfying (2).

We want to address the problem above defined by exploiting some properties of the nested saturation functions. Hence, the current paper can be viewed as an extension of the results developed in [16], since here we consider additional disturbances and the $\mathcal{L}_2$-performance design problem. Differently from [1], considering the modeling of the saturated systems by a polytopic differential inclusion, it should be pointed out that we do not consider particular assumptions about the structure and the dimensions of matrices involved in the description of the system. Furthermore, our results allow also to address global stability issues, which is not possible with the approach proposed in [1].

Moreover, the problem to be treated consists in taking into account the disturbance, which enters the system independently from the nested inputs, in order to ensure a certain disturbance tolerance of the closed-loop system. Such a study is in the same line as the works [8], [12], [3], [13], [6] (which consider only systems presenting single saturation terms) but for more complex systems as those described in (1). Note that in the present paper, the additive disturbance is not input additive as studied, for example, in [10].

Notations. For any vector $x \in \Re^n$, $x \succeq 0$ means that all the components of x, denoted $x_{(i)}$, are nonnegative. For two vectors x, y of $\Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0$, $\forall i = 1, \ldots, n$. $\mathbf{1}$ and $\mathbf{0}$ denote respectively the identity matrix and the null matrix of appropriate dimensions. The elements of a matrix $A \in \Re^{m\times n}$ are denoted by $A_{(i,j)}$, $i = 1, ..., m$, $j = 1, ..., n$. $A_{(i)}$ denotes the ith row of matrix A. $|A|$ is the matrix constituted from the absolute value of each element of A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A. $1_m \stackrel{\triangle}{=} [1 \ldots 1]' \in \Re^m$.

II. THEORETICAL CONDITIONS

A. Preliminaries

For $j = 1, \ldots p$, define the following nonlinearities:

$$\begin{aligned}
\phi_1(x) &= sat_1(Fx) - Fx \\
\phi_2(x) &= sat_2\left((A_1 + B_1F)x + B_1\phi_1(x)\right) \\
&\quad - [(A_1 + B_1F)x + B_1\phi_1(x)] \\
&\ \ \vdots \\
\phi_p(x) &= sat_p\left((A_{p-1} + B_{p-1}\left(A_{p-2} + B_{p-2}\left(A_{p-3}\right.\right.\right. \\
&\quad \left.\left.\left. + \cdots + B_2(A_1 + B_1F)\right)\right)\right)x \\
&\quad + B_{p-1}\phi_{p-1}(x) + B_{p-1}B_{p-2}\phi_{p-2}(x) + \cdots \\
&\quad + B_{p-1}B_{p-2}\cdots B_1\phi_1(x)) \\
&\quad - [(A_{p-1} + B_{p-1}\left(A_{p-2} + B_{p-2}\left(A_{p-3}\right.\right. \\
&\quad \left.\left. + \cdots + B_2(A_1 + B_1F)\right)\right))x \\
&\quad + B_{p-1}\phi_{p-1}(x) + B_{p-1}B_{p-2}\phi_{p-2}(x) + \cdots \\
&\quad + B_{p-1}B_{p-2}\cdots B_1\phi_1(x)]
\end{aligned} \qquad (3)$$

Define now the following p matrices:

$$\begin{aligned}
\mathbb{A}_1 &= A_1 + B_1F \\
\mathbb{A}_2 &= A_2 + B_2(A_1 + B_1F) \\
&\ \ \vdots \\
\mathbb{A}_p &= A_p + B_p\left(A_{p-1} + B_{p-1}\left(A_{p-2} + \cdots\right.\right. \\
&\quad \left.\left. + B_2(A_1 + B_1F)\right)\right) \\
\mathbb{C} &= C + D\left(A_{p-1} + B_{p-1}\left(A_{p-2} + \cdots\right.\right. \\
&\quad \left.\left. + B_2(A_1 + B_1F)\right)\right)
\end{aligned} \qquad (4)$$

From (3) and (4), system (1) can be re-written as

$$\begin{aligned}
\dot{x} &= \mathbb{A}_p x + B_p\phi_p(x) + B_pB_{p-1}\phi_{p-1}(x) + \cdots \\
&\quad + B_pB_{p-1}\cdots B_1\phi_1(x) + B_w w \\
z &= \mathbb{C}x + D\phi_p(x) + DB_{p-1}\phi_{p-1}(x) + \cdots \\
&\quad + DB_{p-1}\cdots B_1\phi_1(x) + B_z w
\end{aligned} \qquad (5)$$

Note that in the absence of saturation one gets $\phi_j(x) = \mathbf{0}$, $j = 1, ..., p$.

Lemma 1: [16] If v and ω are elements of $S(v_0)$:

$$S(v_0) = \{v \in \Re^m, \omega \in \Re^m; -v_0 \preceq v - \omega \preceq v_0\} \qquad (6)$$

then the generic nonlinearity $\varphi(v)sat_{v_0}(v) - v$ satisfies the following inequality:

$$\varphi(v)'T(\varphi(v) + \omega) \leq 0 \qquad (7)$$

for any diagonal positive definite matrix $T \in \Re^{m\times m}$.

B. Main results

Let us define the following matrices

$$\begin{aligned}
\mathbb{G}_{j-1} &= G_{j-1} + B_{j-1}\left(G_{j-2} + B_{j-2}\left(G_{j-3} + \cdots\right.\right. \\
&\quad \left.\left. + B_2(G_1 + B_1G_0)\right)\right),\ j = 2, ..., p
\end{aligned} \qquad (8)$$

The following proposition provides theoretical sufficient conditions to address Problem 1.

Proposition 1: If there exist a symmetric positive definite matrix W, matrices Z_{jj}, $j = 1, ..., p$, Y_{jl}, $j = 2, ..., p$, $l = 1, ..., p-1$, $j \neq l$, $j > l$, G_k, $k = 0, ..., p-1$, diagonal positive matrices S_j, $j = 1, ..., p$, of appropriate dimensions and positive scalars μ and δ satisfying[1]:

$$\begin{bmatrix}
L + L' & B_pB_{p-1}...B_1S_1 - Z'_{11} & B_pB_{p-1}...B_2S_2 - Z'_{22} & \ldots & B_pS_p - Z'_{pp} & B_w \\
\star & -2S_1 & -Y'_{21} & \ldots & -Y'_{p1} & \mathbf{0} \\
\star & \star & -2S_2 & \ldots & -Y'_{p2} & \mathbf{0} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\star & \star & \star & \star & -2S_p & \mathbf{0} \\
\star & \star & \star & \star & \star & -\mathbf{1}
\end{bmatrix} < 0 \qquad (9)$$

$$\begin{bmatrix}
W & G'_{0(i)} - Z'_{11(i)} \\
\star & \mu u^2_{1(i)}
\end{bmatrix} \geq 0,\ i = 1, ..., m_1 \qquad (10)$$

$$\begin{bmatrix}
W & Z'_{11} & Z'_{22} & \ldots & Z'_{j-1\,j-1} & \mathbb{G}'_{j-1(i)} - Z'_{jj(i)} \\
\star & 2S_1 & Y'_{21} & \ldots & Y'_{j-1\,1} & S_1B'_1...B'_{j-2}B'_{j-1(i)} - Y'_{j1(i)} \\
\star & \star & 2S_2 & \ldots & Y'_{j-1\,2} & S_2B'_2...B'_{j-2}B'_{j-1(i)} - Y'_{j2(i)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\star & \star & \star & \star & 2S_{j-1} & S_{j-1}B'_{j-1(i)} - Y'_{j\,j-1(i)} \\
\star & \star & \star & \star & \star & \mu u^2_{j(i)}
\end{bmatrix} \geq 0$$
$$i = 1, ..., m_j,\ j = 2, ..., p \qquad (11)$$

$$\delta - \mu \geq 0 \qquad (12)$$

with

$$L = A_pW + B_pG_{p-1} + ... + B_pB_{p-1}...B_2(G_1 + B_1G_0) \qquad (13)$$

then the gains $A_{j-1} = G_{j-1}W^{-1}$, $j = 2, ..., p$ and $F = G_0W^{-1}$, are such that:

- when $w \neq 0$, the closed-loop trajectories remain bounded in the set $\mathcal{E}_1 = \{x \in \Re^n; x'W^{-1}x \leq \mu^{-1}\}$ for any

[1]The symbol $\star$ stands for symmetric blocks.

$x(0) \in \mathcal{E}_0 = \{x \in \Re^n; x'W^{-1}x \leq \beta\}$ with $\beta = \mu^{-1} - \delta^{-1}$, and any disturbance satisfying (2).

- when $w = 0$, the set $\mathcal{E}_0 = \mathcal{E}_1 = \{x \in \Re^n; x'W^{-1}x \leq \mu^{-1}\}$ is included in the basin of attraction of the closed-loop system (1) and is contractive.
- matrix $\mathbb{A}_p$ is Hurwitz.

Proof. Consider matrices E_{11}, E_{22}, E_{21}, … E_{pp}, E_{pp-1}, …, E_{p1}, matrices of appropriate dimensions to be determined. According to the definition of nonlinearities ϕ_j, $j = 1, ..., p$, by applying Lemma 1 p-times, it follows:

- Firstly, in the case $\varphi = \phi_1$, relation (7) applies with

$$T = T_1;\ v = Fx;\ \omega = E_{11}x;\ v_0 = u_1$$

- Secondly, in the case $\varphi = \phi_2$, relation (7) applies with

$$T = T_2;\ v = (A_1 + B_1F)x + B_1\phi_1(x);$$
$$\omega = E_{22}x + E_{21}\phi_1(x);\ v_0 = u_2$$

- …
- Finally, in the case $\varphi = \phi_p$, relation (7) applies with

$$\begin{array}{l} T = T_p; \\ v = \\ (A_{p-1} + B_{p-1}(A_{p-2} + B_{p-2}(A_{p-3} + \cdots + B_2(A_1 + B_1F))))x \\ +B_{p-1}\phi_{p-1}(x) + B_{p-1}B_{p-2}\phi_{p-2}(x) + \cdots \\ +B_{p-1}B_{p-2}\cdots B_1\phi_1(x); \\ \omega = E_{pp}x + E_{pp-1}\phi_{p-1}(x) + \cdots + E_{p1}\phi_1(x); \\ v_0 = u_p \end{array}$$

where matrices T_1, T_2, …, T_p are p diagonal and positive definite matrices.

Consider now

$$E_{jj} = Z_{jj}W^{-1},\ j = 1, ..., p$$
$$E_{jl} = Y_{jl}S_j^{-1},\ j = 2, ..., p,\ l \neq j,\ j > l$$

The satisfaction of relations (10) and (11) implies that the set $\mathcal{E}_1$ is included in $\cap_{j=1}^{p} S(u_j)$ [16]. Hence, the nonlinearities ϕ_j, $j = 1, ..., p$, associated to the appropriate v and ω defined above, satisfy the sector conditions (7) for all $x \in \mathcal{E}_1$.

Consider now the quadratic Lyapunov function $V(x) = x'Px$, with $P = P' > 0$. The time-derivative of $V(x)$ along the trajectories of closed-loop system (5) reads:

$$\begin{array}{rcl} \dot{V} & = & x'(\mathbb{A}_p'P + P\mathbb{A}_p)x + 2x'PB_p\phi_p \\ & & +2x'PB_pB_{p-1}\phi_{p-1} \\ & & + \cdots + 2x'PB_pB_{p-1}\cdots B_1\phi_1 + 2x'PB_ww \end{array}$$

Define $\mathcal{L} = \dot{V} - w'w$. Since (10) and (11) are satisfied, sector conditions (7) hold $\forall \phi_j, j = 1, \ldots, p$, $\forall x \in \mathcal{E}_1$, considering the vectors v and ω defined from the matrices E_{ji} and T_j. Hence, $\forall x \in \mathcal{E}_1$ it follows that

$$\begin{array}{rcl} \mathcal{L} & \leq & x'(\mathbb{A}_p'P + P\mathbb{A}_p)x + 2x'PB_p\phi_p \\ & & +2x'PB_pB_{p-1}\phi_{p-1} \\ & & + \cdots + 2x'PB_pB_{p-1}\cdots B_1\phi_1 \\ & & -2\phi_1'T_1(\phi_1 + E_{11}x) \\ & & -2\phi_2'T_2(\phi_2 + E_{22}x + E_{21}\phi_1) - \cdots \\ & & -2\phi_p'T_p(\phi_p + E_{pp}x + E_{pp-1}\phi_{p-1} + E_{pp-2}\phi_{p-2} + \\ & & ... + E_{p1}\phi_1) \\ & & +2x'PB_ww - w'w \end{array}$$

Considering $\xi = \begin{bmatrix} x' & \phi_1' & \phi_2' & \ldots & \phi_p' & w' \end{bmatrix}'$, the inequality above can be re-written as $\dot{V} - w'w \leq \xi'\mathcal{M}\xi$ with

$$\mathcal{M} = \begin{bmatrix} \mathbb{A}_p'P + P\mathbb{A}_p & \star & \star & \cdots & \star & \star \\ B_1'\cdots B_{p-1}'B_p'P - T_1E_{11} & -2T_1 & \star & \cdots & \star & \star \\ B_2'\cdots B_{p-1}'B_p'P - T_2E_{22} & -T_2E_{21} & -2T_2 & \cdots & \star & \star \\ B_3'\cdots B_{p-1}'B_p'P - T_3E_{33} & -T_3E_{31} & -T_3E_{32} & \cdots & \star & \star \\ \vdots & \vdots & \vdots & \cdots & \star & \star \\ B_p'P - T_pE_{pp} & -T_pE_{p1} & -T_pE_{p2} & \cdots & -2T_p & \star \\ B_w'P & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{1} \end{bmatrix}$$

By recalling that $W = P^{-1}$, $S_j = T_j^{-1}$, $j = 1, ..., p$, by using the change of variables $A_{j-1} = G_{j-1}W^{-1}$, $j = 2, ..., p$ and $F = G_0W^{-1}$ and by pre- and post-multiplying the matrix $\mathcal{M}$ above defined by $\begin{bmatrix} W & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & S_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & S_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \star & \star & \star & \star & S_p & \mathbf{0} \\ \star & \star & \star & \star & \star & \mathbf{1} \end{bmatrix}$, it follows that, if relation (9) is satisfied, one has

$$\dot{V}(x) - w'w < 0, \tag{14}$$

Therefore, integrating (14), it follows

$$V(x(t)) < V(x(0)) + \| w(t) \|_2^2 \leq V(x(0)) + \delta^{-1}$$

Hence, provided that $x(0) \in \mathcal{E}_0$, since the satisfaction of relation (12) means that the scalar $\beta = \mu^{-1} - \delta^{-1}$ is non-negative, it follows that $V(x(t)) \leq \mu^{-1}$, $\forall t \geq 0$ and for any disturbance satisfying (2). Hence, the satisfaction of relations (9), (10), (11) and (12) means that the trajectories of the closed-loop system (1) remain bounded in $\mathcal{E}_1$, $\forall x(0) \in \mathcal{E}_0$ and any disturbance satisfying (2). This completes the proof of the first point.

By considering $w = 0$, it follows directly that $\dot{V}(x) < 0$, for all $x(0) \in \mathcal{E}_1$ and the second point is proven. The third point is then a direct consequence. □

Remark 1: The reachable set of the closed-loop system (1), given by [2] $\mathcal{W}_0 = \{x(t) \in \Re^n\ ;\ x(0) = 0$ and $\int_0^t w(\tau)'w(\tau)\ dt \leq \frac{1}{\delta}\}$, is included in the ellipsoid $\mathcal{E}_1$.

Remark 2: The results in [12] appears as a particular case of Proposition 1 since just a single saturation is considered (that is, $p = 1$ and $m_1 = m$). Moreover in [12], $\phi_1(x)$ satisfies the "classical" sector condition: $\phi_1(x)'T(\phi_1(x) + \Lambda Fx) \leq 0$, $\forall x \in \{x \in \Re^n; -u_1 \preceq (\mathbf{1} - \Lambda)Fx \preceq u_1\}$, where Λ is a positive diagonal matrix. This "classical" sector condition can be deribed from relation (7) by choosing $v = Fx$ and $\omega = \Lambda Fx$ (or equivalently $E_{11} = \Lambda F$). Following a similar procedure to the one applied in the proof of Proposition 1, the following conditions, in plus of (12), are obtained

$$\begin{bmatrix} WA_1' + A_1W + B_1G_0 + G_0'B_1' & B_1S_1 - G_0'\Lambda & B_w \\ \star & -2S_1 & \mathbf{0} \\ \star & \star & -\mathbf{1} \end{bmatrix} < 0 \tag{15}$$

$$\begin{bmatrix} W & (1-\Lambda_{(i,i)})G'_{0(i)} \\ \star & \mu u^2_{1(i)} \end{bmatrix} \geq 0,\ i=1,...,m \qquad (16)$$

Note that these matrix inequalities are bilinear in variables G_0 and Λ.

Proposition 1 presents a local stabilization condition for the nested saturated system (1). On the other hand, in the case where A_p is Hurwitz the global asymptotic stabilization of the system can be obtained using the following result.

Corollary 1: If there exist a symmetric positive definite matrix W, G_k, $k=0,...,p-1$, and diagonal positive matrices S_j, $j=1,...,p$, of appropriate dimensions satisfying:

$$\begin{bmatrix} L+L' & B_pB_{p-1}...B_1S_1-G'_0 & B_pB_{p-1}...B_2S_2-(G_1+B_1G_0)' & ... & B_pS_p-\mathbb{G}'_{p-1} & B_w \\ \star & -2S_1 & -S_1B'_1 & ... & -S_1(B_{p-1}...B_1)' & \mathbf{0} \\ \star & \star & -2S_2 & ... & -S_2(B_{p-1}...B_2)' & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \star & \star & \star & \star & -2S_p & \mathbf{0} \\ \star & \star & \star & \star & \star & -\mathbf{1} \end{bmatrix} < 0 \qquad (17)$$

where $\mathbb{G}_{p-1}$ and L are defined in (8) and (13), respectively, then the system (1) with $A_{j-1}=G_{j-1}W^{-1}$, $j=2,...,p$ and $F=G_0W^{-1}$, is such that

- $\forall w \in \mathcal{L}_2^q$ and $\forall x(0) \in \Re^n$, the closed-loop state trajectories are bounded. An estimate to the reachable region in the state space is given by the set $\mathcal{E}_1$ where $\mu^{-1}=\beta+\delta^{-1}$, $\beta=x(0)'W^{-1}x(0)$ and $\delta^{-1}=||w(t)||_2^2$.
- when $w=0$, the closed-loop system (1) is globally asymptotically stable.

Proof. Associated to the sets $S(u_j)$ as defined in proof of Proposition 1, it suffices to consider $E_{11}=F$, $E_{22}=A_1+B_1F$, ..., $E_{pp}=A_{p-1}+...+B_{p-1}...B_2(A_1+B_1F)=\mathbb{A}_{p-1}$, $E_{21}=B_1$, $E_{pp-1}=B_{p-1}$, ..., $E_{p1}=B_{p-1}B_{p-2}...B_2B_1$. In this case, $S(u_j)=\Re^n$, $j=1,\ldots,p$ and it follows that the sector conditions (7), applied p-times to the nonlinearities defined in (3), are globally satisfied, i.e. they are satisfied $\forall x \in \Re^n$. □

Remark 3: The application of Proposition 1 in the case where the disturbance satisfies (2) and is vanishing (that is, after some $t=t_f$ it is equal to zero), allows to conclude that the corresponding system trajectory will converge asymptotically to the origin after time t_f, provided that $x(0) \in \mathcal{E}_0$. On the other hand, if the condition of Corollary 1 is verified, the asymptotic convergence of the trajectories to the origin, after some time t_f, is ensured for any initial condition $x(0) \in \Re^n$ and any vanishing disturbance $w(t) \in \mathcal{L}_2^q$.

Similar conditions to those of Proposition 1 and Corollary 1 could be obtained in a context of analysis, that is when all the gains A_j, $j=1,...,p-1$, and F are a priori given.

C. $\mathcal{L}_2$ gain from w to z

In addition to the guarantee of the disturbance tolerance, which can be achieved by using the conditions stated in Proposition 1, it is also interesting to compute the gains in order to ensure an upper bound on the $\mathcal{L}_2$ gain from w to z. The following proposition provides theoretical sufficient conditions to address this last case when $x(0)=\mathbf{0}$.

Proposition 2: If there exist a symmetric positive definite matrix W, matrices Z_{jj}, $j=1,...,p$, Y_{jl}, $j=2,...,p$, $l=1,...,p-1$, $j\neq l$, $j>l$, G_k, $k=0,...,p-1$, diagonal positive matrices S_j, $j=1,...,p$, of appropriate dimensions, and positive scalars η, μ and δ satisfying relations (10), (11), (12) and:

$$\begin{bmatrix} M & \mathcal{C}' \\ \star & -\eta\mathbf{1} \end{bmatrix} < 0 \qquad (18)$$

where matrix M corresponds to the matrix of relation (9) and $\mathcal{C}$ is defined by

$$\mathcal{C}=\begin{bmatrix} \mathbb{C}W & DB_{p-1}...B_1S_1 & DB_{p-1}...B_2S_2 & ... & DB_{p-1}S_{p-1} & DS_p & B_z \end{bmatrix} \qquad (19)$$

then the gains $A_{j-1}=G_{j-1}W^{-1}$, $j=2,...,p$ and $F=G_0W^{-1}$, are such that the $\mathcal{L}_2$ gain from w to z is less than or equal to $\sqrt{\eta}$.

Proof. The proof follows the same lines than those of the proof of Proposition 1. By considering $\xi=\begin{bmatrix} x' & \phi'_1 & \phi'_2 & ... & \phi'_p & w' \end{bmatrix}'$, one can write z as:

$$z=\begin{bmatrix} \mathbb{C} & DB_{p-1}...B_1 & DB_{p-1}...B_2 & ... & DB_{p-1} & D & B_z \end{bmatrix}\xi$$

Thus, one proves that the satisfaction of relation (18) implies, $\forall w$ satisfying (2): $\dot{V}+\frac{1}{\eta}z'z-w'w<0$ with $V(x)=x'Px$, $P=P'>0$. Thus, by noting that one considers $x(0)=\mathbf{0}$ and that $V(x)\geq 0$ it follows that $\| z \|_2 \leq \sqrt{\eta} \| w \|_2$. Any positive scalar satisfying the two above inequalities is called an $\mathcal{H}_\infty$ guaranteed cost for the closed-loop system (1). □

Remark 4: In the case $x(0)\neq\mathbf{0}$, one considers

$$\int_0^t z'(\tau)z(\tau)d\tau < \eta V(x(0)) + \eta\int_0^t w'(\tau)w(\tau)d\tau$$

or equivalently

$$\| z \|_2 \leq \bar{\beta}+\sqrt{\eta} \| w \|_2$$

for any $x(0)\in\mathcal{E}_0$. The scalar $\bar{\beta}$ is a bias depending on the initial condition.

D. Disturbance Rejection

In a similar context to that one of Problem 1, it is natural to investigate the disturbance rejection. The problem of disturbance rejection is considered in [1], but in the case of amplitude limited disturbances.

Let us denote by α the disturbance rejection level. Thus, mimicking Proposition 1, the disturbance rejection problem is tackled by satisfying:

$$\dot{V}(x)-\alpha w'w<0$$

Integrating this above inequality it follows:

$$V(x(t)) < V(x(0)) + \alpha \int_0^t w'(\tau)w(\tau)d\tau \leq V(x(0)) + \alpha\delta^{-1}$$

In this case, the $(p+2, p+2)$-block (namely $-\mathbf{1}$) of the matrix in relation (9) is changed by $-\alpha\mathbf{1}$, whereas relations (10), (11) and (12) are kept unchanged. Thus, the inner set $\mathcal{E}_0$ is defined with $\beta = \mu^{-1} - \alpha\delta^{-1}$, whereas the definition of the outer set is unchanged.

Note that the difference of the radius of the inner ellipsoid $\mathcal{E}_0$ and the outer ellipsoid $\mathcal{E}_1$ is equal to $\alpha\delta^{-1}$. A good way to state the disturbance rejection consists in minimizing the distance between the outer and the inner sets, in other words in minimizing the level α.

E. Possible Extension

In the control design, the designer is often faced to model uncertainty. In order to capture better the system behavior, it is well-known that, in addition to a good nominal model, we need also to describe, in a convenient way, the involved uncertainties.

Thus, the results presented in the previous sections could be extended by considering that system (1) is subject to uncertainties of norm-bounded type. Hence, nominal matrices A_p, B_p, C and D are replaced by the following ones: $(A_p + N_1 I M_1)$, $(B_p + N_1 I M_2)$, $(C + N_2 I M_1)$ and $(D + N_2 I M_2)$ where matrices N_1, N_2, M_1 and M_2 are constant matrices of appropriate dimensions defining the structure of the uncertainty. Matrix $I \in \mathcal{S}_I$ with

$$\mathcal{S}_I = \{I : \Re^+ \to \Re^{l\times q}; I'I \leq I_q, \forall t \geq 0\} \quad (20)$$

is the uncertainty parameter (which can depend on the time), frequently considered in the robust control literature (see, for example, [5], [15] and references therein).

The following proposition provides theoretical sufficient conditions to address Problem 1.

Proposition 3: If there exist symmetric positive definite matrices W, R_1, matrices Z_{jj}, $j = 1,...,p$, Y_{jl}, $j = 2,...,p$, $l = 1,...,p-1$, $j \neq l$, $j > l$, G_k, $k = 0,...,p-1$, diagonal positive matrices S_j, $j = 1,...,p$, of appropriate dimensions, and positive scalars μ and δ satisfying relations (10), (11), (12) and:

$$\begin{bmatrix} \tilde{M} & \tilde{C}' \\ \star & -R_1 \end{bmatrix} < 0 \quad (21)$$

where matrix $\tilde{M}$ is defined by

$$\tilde{M} = M + \begin{bmatrix} N_1 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} R_1 \begin{bmatrix} N_1 & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \quad (22)$$

with M matrix of relation (9), and $\tilde{C}$ is defined by

$$\begin{array}{c} \tilde{C} = [\ \tilde{L} \quad M_2B_{p-1}...B_1S_1 \quad M_2B_{p-1}...B_2S_2 \\ \dots \quad M_2B_{p-1}S_{p-1} \quad M_2S_p \quad \mathbf{0}\ ] \\ \tilde{L} = M_1W + M_2G_{p-1} + \dots + M_2B_{p-1}...B_2(G_1 + B_1G_0) \end{array} \quad (23)$$

then the gains $A_{j-1} = G_{j-1}W^{-1}$, $j = 2,...,p$ and $F = G_0W^{-1}$ are such that:

- when $w \neq 0$, the closed-loop trajectories remain bounded in the set $\mathcal{E}_1$ for any $x(0) \in \mathcal{E}_0$ and any disturbance satisfying (2).
- when $w = 0$, the set $\mathcal{E}_0 = \mathcal{E}_1$ is included in the basin of attraction of the closed-loop system (1) and is contractive.
- matrix $\mathbb{A}_p + N_1 I(M_1 + M_2(A_{p-1} + \dots + B_2(A_1 + B_1F)))$ is Hurwitz, $\forall I \in \mathcal{S}_I$.

Proof. The proof mimics that one of Proposition 1 and is omitted for reasons of place. □

Results for computing an upper bound on the $\mathcal{L}_2$ gain from w to z (section II-C) or a disturbance rejection level could be addressed similarly (section II-D).

III. NUMERICAL ISSUES

A. Optimization Issues

Let us stress that the conditions of Propositions 1 or 2 are LMIs in the decision variables. Note that by using the modeling of the saturation terms given in [1] to solve Problem 1, LMI conditions could be also derived. Nevertheless, as soon as the numbers n and $m_1 = m_2 = \cdots = m_p$ are sufficiently large, it leads to solve an LMI condition with more lines and more variables than the conditions of Propositions 1 or 2 (see the discussion in [16] in an analogous context). Therefore, by invoking [4], the LMI conditions of Propositions 1 and 2 lead to an optimization problem which is numerically easier to solve than the optimization problem obtained with the method of [1].

Depending on the energy bound on the disturbance, δ, is given by the designer or not, the following optimization problems can be considered:

- given δ, we want to optimize the size of the set $\mathcal{E}_0$. This case can be addressed if we consider a set Ξ_0 with a given shape and a scaling factor τ. For example, let Ξ_0 be defined as a polyhedral set described by its vertices: $\Xi_0 = Co\{v_r;\ r = 1,...,n_r, v_r \in \Re^n\}$. We want then to satisfy $\tau\ \Xi_0 \subset \mathcal{E}_0$. The goal consists in maximizing τ, which corresponds to define, through Ξ_0, the directions in which we want to maximize $\mathcal{E}_0$.
- δ being a decision variable, we want to minimize it. Such a problem can be reinterpreted as the problem to find the largest disturbance tolerance.

1) Proposition 1 with given δ:

$$\begin{array}{c} \min \zeta_1\mu - \zeta_2\gamma \\ \text{subject to relations } (9),(10),(11),(12) \\ \begin{bmatrix} 1 - \frac{\mu}{\delta} & \gamma v_r' \\ \star & W \end{bmatrix} \geq 0, r = 1,...,n_r \end{array} \quad (24)$$

where ζ_i, $i = 1,2$ are tuning positive parameters. Note that the satisfaction of the last constraint in (24) allows to guarantee the inclusion $\tau\ \Xi_0 \subset \mathcal{E}_0$ with $\tau = \frac{\gamma}{\sqrt{\mu}}$.

2) Proposition 1 with unknown δ:

$$\begin{array}{c} \min \delta \\ \text{subject to relations } (9),(10),(11),(12) \end{array} \quad (25)$$

3) $\mathcal{L}_2$ gain from w to z:

$$\begin{array}{c} \min \eta \\ \text{subject to relations } (18),(10),(11),(12) \end{array} \quad (26)$$

4) Disturbance Rejection: Let us consider the value δ_{min} obtained from (25).

$$\begin{array}{c} \min \alpha \\ \text{subject to relations } (9-\alpha),(10),(11),(12) \\ \delta \geq \delta_{min} \end{array} \quad (27)$$

where (9-α) corresponds to (9), in which $-\alpha\mathbf{1}$ replaces $-\mathbf{1}$.

B. Numerical Example

Consider the longitudinal dynamics of the F-8 aircraft borrowed from [19]. System (1), in the case $p = 3$, $m_3 = m_2 = 2$, $m_1 = 1$ and $n = 4$, is described by the following data:

$$A_3 = \begin{bmatrix} -0.8 & -0.006 & -12 & 0 \\ 0 & -0.014 & -16.64 & -32.2 \\ 1 & -0.0001 & -1.5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} -19 & -3 \\ -0.66 & -0.5 \\ -0.16 & -0.5 \\ 0 & 0 \end{bmatrix}; B_w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; B_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}; u_3 = u_2 = \begin{bmatrix} 15 \\ 15 \end{bmatrix}; u_1 = 15$$

Note that, since $m_3, m_2 \neq m_1$, the approach of [1] cannot be applied. By applying Proposition 1 and by considering the optimisation issue of III-A.2, one obtains:

$$\begin{array}{l} A_2 = \begin{bmatrix} 0.1050 & -0.0096 & -0.5484 & 0.6687 \\ 0.0119 & -0.0048 & 0.0507 & -0.0238 \end{bmatrix} \\ A_1 = \begin{bmatrix} -0.0112 & -0.0013 & 0.1303 & -0.1347 \\ 0.0329 & -0.0048 & -0.1179 & 0.1619 \end{bmatrix} \\ F = \begin{bmatrix} 0.0144 & -0.0016 & -0.0666 & 0.0841 \end{bmatrix} \\ eig(A_3 + B_3(A_2 + B_2(A_1 + B_1 F))) \\ \quad = \{-1.1595 \pm j3.4520; -0.5983 \pm j0.2754\} \\ \delta = 0.0434 \end{array}$$

IV. CONCLUSION

This paper addressed the problem of stabilizing gains design for systems presenting nested saturations and $\mathcal{L}_2$ limited disturbance. Given δ an energy bound on the disturbance, LMI conditions were given to compute the inner ellipsoid (the set of initial conditions) and the outer ellipsoid (the set bounding the closed-loop trajectories). With the same approach, some connected problems was also considered.

Many issues remain open. In particular we conjecture that it can be a great help to combine our approach with [11], where the problem of tracking trajectories of feedforward systems is solved by the construction of a strict Lyapunov function. Moreover, some other class of nested nonlinearities, as slope restricted nonlinearities, could be studied in order to deal with various nonlinear actuators.

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