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# A Local Field of Extremals Near Boundary Arc - Interior Arc Junctions

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*Abstract*— **For optimal control problems with state space constraints given by control-invariant manifolds of relative degree 1, we describe a local field of extremals near junctions between boundary and interior arcs. These results continue earlier analysis by incorporating junctions of boundary arcs with singular arcs and bang arcs not considered in before. The problem of minimizing the base transit time in homojunction bipolar transistors is used to illustrate the results.**

## I. Introduction

Optimal control problems with state space constraints naturally arise in many practical problems of engineering or scientific interest. Examples come from various disciplines like the space shuttle re-entry problem considered in [2], the problem of minimizing the base transit time in bipolar transistors [15], or simple models for optimal control problems in cancer chemotherapy [4]. Because of the more complicated structure of necessary conditions for optimality (due to the presence of measures as multipliers when state constraints are active,) naturally the theory of sufficient conditions for optimality is less developed, still mostly at the stage of results which in practice are difficult to apply (such as the results in [17] or results related to the existence of viscosity solutions to the dynamic programming equation [3], [16]). In particular, no geometric theory of synthesis exists like for problems without state space constraints [1], [11]. Given the fact that state constraints often have strong geometric properties in practical applications, for example, are given by embedded submanifolds and intersections thereof, it does not seem unreasonable to formulate geometric synthesis type conditions for optimality for these cases. While these are typically more cumbersome to formulate, simply since geometric properties need to be described, once done, they are easily applicable. In this spirit, for boundary arcs of *order* or *relative degree* 1, (i.e. the state constraints are control-invariant submanifolds of relative degree 1), under specific generic assumptions in [7] a local field of extremals near a reference trajectory that consisted of finite concatenations of bang-bang and boundary arcs was constructed, and the resulting sufficient conditions for strong local optimality have been verified in [14]. In this paper we continue this construction of a local field of extremals for problems with state-space constraints near boundary arcs of order 1 by incorporating a type of boundary-bang arc junction different from the one considered in [7]. But mostly the emphasis here is on including junctions with singular arcs into a local synthesis. These constructions are motivated by the problem of minimizing the base transit time in bipolar transistors [15] and we include a discussion of local fields of extremals around reference trajectories for this problem.

## II. Mathematical Model

As in [7] we consider an optimal control problem for a *fixed time interval* $[0,T]$ and without terminal constraints: **(P)** minimize

$$J(u) = \int_0^T \left(L_0(t,x) + uL_1(t,x)\right)\,dt + \varphi(x(T)) \qquad (1)$$

over all Lebesgue measurable functions $u$ defined on $[0,T]$ with values in a compact interval $[a,b] \subset \mathbb{R}$, subject to the dynamics

$$\dot{x}(t) = f(t,x) + ug(t,x), \quad x(0) = x_0, \qquad (2)$$

and state space constraints

$$h_\alpha(t,x) \leq 0 \quad \text{for } \alpha = 1,\dots,r. \qquad (3)$$

The restriction to systems which are linear in the control (both dynamics and objective) is not essential, but it simplifies the presentation and this assumption is satisfied for many realistic problems. Having a single-input system implies that typically only one constraint will be active at a specific time, but we want to allow for the fact that different constraints may be active along a trajectory as, for example, it is the case in the semiconductor problem considered below (see also [15]).

We assume that the time-varying vector fields $f$ and $g$, $[0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, in the dynamics and the functions $L_0$ and $L_1$, $[0,T]\times\mathbb{R}^n \to \mathbb{R}$, in the objective are twice continuously differentiable in all variables. The state-space constraints are defined by twice continuously differentiable time-varying vector fields,

$$h_\alpha : [0,T]\times\mathbb{R}^n \to \mathbb{R},\ (t,x) \mapsto h_\alpha(t,x),$$

$\alpha = 1,\dots,r$, and we assume that the gradients $\nabla h_\alpha(t,x)$ do not vanish on the sets

$$M_\alpha = \{(t,x) : h_\alpha(t,x) = 0\}. \qquad (4)$$

In particular, each $M_\alpha$ is an embedded submanifold of codimension 1 of $(t,x)$-space.

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Given a control $u : [0,T] \to [a,b]$, the initial value problem (2) has a unique solution defined on some maximal open interval of definition $I$. For a control to be admissible we require that $I \supset [0,T]$ and we call the solution the *trajectory* corresponding to the control $u$; the pair $(x,u)$ is a *controlled trajectory*. Given an open subset $P$ of $\mathbb{R}^n$, let $t_{in}$ and $t_f$ be two continuous functions satisfying $t_{in}(p) < t_f(p)$ for all $p \in P$ and let $D = \{(t,p) : t_{in}(p) \leq t \leq t_f(p), p \in P\}$. If $u = u(t,p)$ denotes some parameterized family of admissible controls defined on $D$, let $x = x(t,p)$ denote the corresponding trajectories. Then we define the corresponding flow as the map defined by the graphs of the trajectories, i.e.

$$\sigma : D \to \mathbb{R} \times \mathbb{R}^n,\ (t,p) \mapsto (t, x(t,p)). \tag{5}$$

If the time-varying equations are written as an autonomous system, this exactly is the standard flow of the trajectories. Similarly, if we set

$$F(t,x) = \begin{pmatrix} 1 \\ f(t,x) \end{pmatrix} \quad \text{and} \quad G(t,x) = \begin{pmatrix} 0 \\ g(t,x) \end{pmatrix}, \tag{6}$$

then for a continuously differentiable function $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, the functions $\mathcal{L}_F k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(t,x) \mapsto (\mathcal{L}_F k)(t,x)$, and $\mathcal{L}_G k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(t,x) \mapsto (\mathcal{L}_G k)(t,x)$ defined by

$$(\mathcal{L}_F k)(t,x) = \frac{\partial k}{\partial t}(t,x) + \frac{\partial k}{\partial x}(t,x) f(t,x) \tag{7}$$

and

$$(\mathcal{L}_G k)(t,x) = \frac{\partial k}{\partial x}(t,x) g(t,x) \tag{8}$$

are the Lie-derivatives of the function $k$ along the vector fields $F$ and $G$, respectively. In terms of this notation, the derivative of the function $h_\alpha$ (defining the manifold $M_\alpha$) along trajectories of the system is given by

$$\dot{h}_\alpha(t,x(t)) = \mathcal{L}_F h_\alpha(t,x(t)) + u(t)\mathcal{L}_G h_\alpha(t,x(t)).$$

If the function $\mathcal{L}_G h_\alpha$ does not vanish at a point $(\tilde{t},\tilde{x}) \in M_\alpha$, then there exists a neighborhood $V$ of $(\tilde{t},\tilde{x})$ such that there exists a unique control $u_\alpha = u_\alpha(t,x)$ which solves the equation $\dot{h}_\alpha(t,x) = 0$ on $V$ and $u_\alpha$ is given in feedback form as

$$u_\alpha(t,x) = -\frac{\mathcal{L}_F h_\alpha(t,x)}{\mathcal{L}_G h_\alpha(t,x)}. \tag{9}$$

We call the manifold $M_\alpha$ *control-invariant* of *relative degree* 1 for problem $(P)$ if the Lie derivative of $h_\alpha$ with respect to $G$ does not vanish anywhere on $M_\alpha$ and if the function $u_\alpha(t,x)$ defined by (9) is admissible, i.e. takes values in the control set $[a,b]$. Since we are dealing with a problem with control constraints, here we explicitly include the requirement that the control $u_\alpha$ satisfies these constraints in the definition. Thus, for a control-invariant submanifold of relative degree 1, the control that keeps the manifold invariant is unique and the corresponding dynamics (2) induces a unique flow on the constraint.

In this paper we assume throughout that:

**(A)** All constraint manifolds $M_\alpha$ are control-invariant of relative degree 1.

This assumption corresponds to the least degenerate, or, equivalently, most common scenario and is satisfied for many practical problems like for example [2], [4], [15].

## III. Necessary Conditions for Optimality

First-order necessary conditions for optimality are given by the Pontryagin maximum principle [12]. Mathematically the presence of the state-space constraints complicates matters in that it brings in additional multipliers which a priori are only known to be non-negative Radon measures. However, under condition $(A)$ it can be shown that these measures are absolutely continuous with respect to Lebesgue measure on intervals where the state lies in one of the constraint manifolds [14, Prop. 3.1]. Since admissible controls are only Lebesgue-measurable, in principle the sets where a constraint is active can be arbitrarily complicated sets. Nevertheless, in most practical situations this set often is a union of intervals and in this case more stringent necessary conditions for optimality are valid which we will use to formulate sufficient conditions for local optimality. Following the notation introduced by Maurer [9] we call a piece $\Gamma$ of the graph of a trajectory defined over an open interval $I$ which does not intersect the boundary an *interior arc* and call $\Gamma$ a *boundary arc* if at least one constraint is active on all of $I$. More specifically we call $\Gamma$ an $M_\alpha$-boundary arc over $I$ if only the constraint $h_\alpha \leq 0$ is active on $I$. The times $\tau$ when interior arcs and boundary arcs meet are called *junction times* and the corresponding pairs $(\tau, x(\tau))$ junction points. The necessary conditions for optimality for problem $(P)$ can then be summarized as below:

*Theorem 3.1:* Suppose the state space constraints in problem $(P)$ are given by control-invariant submanifolds of relative degree 1. Let $u_* : [0,T] \to [a,b]$ be an optimal control for problem $(P)$ with corresponding trajectory $x_*$ and assume $x_*$ is a finite concatenation of interior and boundary arcs with junction times $t_i^*$, $i = 1,\dots,m$, $0 = t_0^* < t_1^* < \dots < t_m^* < t_{m+1}^* = T$, such that on each interval $(t_i^*, t_{i+1}^*)$, $i = 0,\dots,m$, at most one constraint is active and that the boundary controls take values in the interior of the control set. Then there exist a constant $\lambda_0 \geq 0$ and piecewise continuous functions $\lambda_*$, $\lambda_* : [0,T] \to (\mathbb{R}^n)^*$, $\lambda_*(T) = \lambda_0 \frac{\partial \varphi}{\partial x}(x_*(T))$, and $\nu_\alpha$, $\nu_\alpha : [0,T] \to \mathbb{R}$, $\alpha = 1,\dots,r$, which do not all vanish identically, such that the following conditions (a)-(d) hold:

(a) $\lambda_*$ is continuously differentiable on each interval $(t_i^*, t_{i+1}^*)$, $i = 0,\dots,m$, and satisfies the adjoint equation in the form

$$\begin{aligned}\dot{\lambda}_*(t) = &- \lambda_0 \left( \frac{\partial L_0}{\partial x}(t,x_*) + u_* \frac{\partial L_1}{\partial x}(t,x_*) \right) \\ &- \lambda(t)_* \left( \frac{\partial f}{\partial x}(t,x_*) + u_* \frac{\partial g}{\partial x}(t,x_*) \right) \\ &- \sum_{\alpha=1}^{r} \nu_\alpha \frac{\partial h_\alpha}{\partial x}(t,x_*).\end{aligned} \tag{10}$$

(b) If the constraint $M_\alpha$ is not active, then $\nu_\alpha(t) = 0$; if $(t_i^*, t_{i+1}^*)$ is the domain of an $M_\alpha$-boundary arc, then $\nu_\alpha(t)$

is given by

$$\frac{1}{\mathcal{L}_G h_\alpha(t,x_*)}\left\{\lambda_0\left(\frac{\partial L_1}{\partial t}(t,x_*)\right.\right. \quad (11)$$
$$\left. +\frac{\partial L_1}{\partial x}(t,x_*)f(t,x_*)-\frac{\partial L_0}{\partial x}(t,x_*)g(t,x_*)\right)$$
$$\left. +\lambda_*\left(\frac{\partial g}{\partial t}(t,x_*)+[f,g](t,x_*)\right)\right\}.$$

(c) The multiplier $\lambda_*$ is continuous at interior junctions; it is continuous at a junction between an interior arc and an $M_\alpha$-boundary arc if the graph $t \mapsto (t, x_*(t))$ is transversal to $M_\alpha$ at the junction point $(\tau, x_*(\tau))$. This is the case if and only if the control $u_*$ is discontinuous at $\tau$.
(d) With $\Phi_*(t) = \lambda_0 L_1(t, x_*(t)) + \lambda_*(t)g(t, x_*(t))$ the control satisfies

$$u_*(t) = \begin{cases} b & \text{if } \Phi_*(t) < 0 \\ a & \text{if } \Phi_*(t) > 0 \end{cases}. \quad (12)$$

Along an $M_\alpha$-boundary arc the control $u_*$ is given by

$$u_*(t) = u_\alpha(t, x_*(t)) = -\frac{\mathcal{L}_F h_\alpha(t, x_*(t))}{\mathcal{L}_G h_\alpha(t, x_*(t))}. \quad (13)$$

Condition (c) is a reformulation of Maurer's well-known junction conditions [9] and condition (d) is equivalent to the more standard formulation that the optimal control minimizes the Hamiltonian,

$$H = \lambda_0 (L_0(t,x) + uL_1(t,x)) + \lambda f(t,x) + ug(t,x)), \quad (14)$$

over the control set $[a, b]$ along $(\lambda_0, \lambda_*(t), x_*(t))$, i.e.

$$H(\lambda_0, \lambda_*(t), x_*(t), u_*(t)) = \min_{a\le w\le b} H(\lambda_0, \lambda_*(t), x_*(t), w). \quad (15)$$

We refer to the function $\Phi_*$ as the switching function for the problem.

We call control-trajectory pairs $(x, u)$ for which there exist multipliers such that these conditions are satisfied *extremals*. For general control problems it cannot be excluded that $\lambda_0$ vanishes and extremals with $\lambda_0 = 0$ are called abnormal, while those with $\lambda_0 > 0$ are called normal. In this case we can normalize $\lambda_0 = 1$. There exist several results which can be used to establish the normality of extremals, (for example, see [13], [8]) and in our construction below we will also need to assume that:

**(B)** The reference trajectory is normal.

## IV. Construction of a Local Field of Extremals Near a Boundary-arc Singular-arc Junction

Our aim is to formulate sufficient conditions for strong local optimality of an arc of a reference extremal $(x_*, u_*)$ which is a finite concatenation of interior arcs and $M_\alpha$-boundary arcs. In [7] we analyzed one generic type of junctions between boundary arcs and interior arcs corresponding to constant controls. Here we now consider the case of junctions with interior singular arcs. While the local synthesis around the boundary arc is identical to the one in [7], (and thus is valid under the same conditions), the synthesis is qualitatively different at the junction.

An interior arc $\Gamma$ corresponding to a trajectory $x_*$ defined over an open interval $I$ is called *singular* if $\Phi_*$ vanishes identically on $I$. In this case all the derivatives of $\Phi_*$ must vanish as well. The first derivative $\dot\Phi_*$ does not depend on the control and the second derivative $\ddot\Phi_*$ is of the form

$$\ddot\Phi(t) = \Psi(t) + u_*(t)\Xi(t) \quad (16)$$

where the functions $\Psi$ and $\Xi$ only depend on the multiplier $\lambda_*$ and the trajectory $x_*$, but not on the control. It is a necessary condition for minimality of the singular control, the so-called Legendre-Clebsch condition [5], that

$$\Xi(t) = \frac{\partial}{\partial u}\frac{d^2}{dt^2}\frac{\partial H}{\partial u}(t, \lambda_*(t), x_*(t), u) \le 0 \quad (17)$$

and if this inequality is strict we say that the strengthened Legendre-Clebsch condition is satisfied. In this case the singular control is called of order 1 and the equation $\ddot\Phi_* \equiv 0$ can formally be solved to compute the singular control as a function of $t$, $x$ and $\lambda$. In order to be admissible, the values need to lie in the control set $[a, b]$.

Singular controls share many features with boundary controls and, as will be seen below, at junctions between boundary and singular controls in the non-degenerate scenario the structure of the local synthesis does not change, unlike for the more complicated type of exit junctions between boundary arcs and interior bang arcs considered in [7]. We now consider the following scenario:

**(E-ref)** *Let $\Gamma_\alpha$ be an $M_\alpha$-boundary arc of an extremal input-trajectory pair $\Gamma = (x_*, u_*)$ defined over an interval $[\tau_1, \tau_2] \subset (0, T)$ with corresponding multipliers $\lambda_*$ and $\nu_\alpha$. Suppose $\tau_1$ and $\tau_2$ are the entry- and exit-times, respectively, and assume there exists an $\varepsilon > 0$ such that the control $u_*$ is constant on the interval $(\tau_1 - \varepsilon, \tau_1)$ and is singular on $(\tau_2, \tau_2 + \varepsilon)$. For sake of argument suppose $u_*(t) \equiv a$ on $(\tau_1 - \varepsilon, \tau_1)$.*

**(E-boundary)** *For all times in the closed interval $[\tau_1, \tau_2]$ the boundary control $u_*$ which keeps $M_\alpha$ invariant, $u_*(t) = u_\alpha(t, x_*(t))$, takes values in the interior of the control set and the multiplier $\nu_*(t)$ is positive.*

**(E-singular)** *There exists a codimension 1 submanifold $\mathcal{S}$ transversal to $M_\alpha$ at $(\tau_2, x_*(\tau_2))$ consisting of graphs of extremal trajectories corresponding to singular controls of order 1. The values of the singular controls lie in the interior of the control set and the strengthened Legendre-Clebsch condition is satisfied.*

This last condition describes the typical scenario for singular arcs in low dimensions. Under generic conditions, in $n = 2$ there exists a single singular arc and for $n = 3$ there is a surface on which a singular feedback control can be defined. But in dimensions $n \ge 4$ there are more degrees of freedom in defining the singular arcs. Also note that by assuming the "singular surface" $\mathcal{S}$ is transversal to $M_\alpha$ it follows that the controls are discontinuous at the exit junctions and this holds by assumption *(E-boundary)* also for the entry-junction. In particular, the multiplier $\lambda_*$ in the maximum principle remains continuous at entry and exit junctions [9], [14, Prop. 32]. We now construct a local synthesis around the reference trajectory.

*Lemma 4.1:* If $p_* = (\tau, x_*(\tau))$ is a point on the singular surface $\mathcal{S}$, the trajectories $x_\pm(t, p_*)$ corresponding to the constant controls $u_+(t, p_*) \equiv b$ and $u_-(t, p_*) \equiv a$, defined for $t < \tau$, $t$ sufficiently close to $\tau$, are extremals.

Proof. This is a consequence of the strengthened Legendre-Clebsch condition. Recall that $\lambda_*$ denotes the adjoint variable along the singular arc and let $\lambda_\pm$ denote the solutions to the adjoint equations corresponding to $u_\pm$ with terminal value $\lambda_\pm(\tau) = \lambda_*(\tau)$. Also define switching functions $\Phi_\pm$ using the multipliers $\lambda_\pm$. It then follows from the fact that the reference arc is singular that $\dot{\Phi}_\pm(\tau) = 0$. For the second derivatives we have that

$$\ddot{\Phi}_+(\tau) = \Psi(\tau) + b\Xi(\tau), \quad \ddot{\Phi}_-(\tau) = \Psi(\tau) + a\Xi(\tau), \quad (18)$$

while for the value $u_*(\tau) \in (a, b)$ we have

$$\ddot{\Phi}_*(\tau) = \Psi(\tau) + u_*(\tau)\Xi(\tau) = 0. \quad (19)$$

Since the strengthened Legendre-Clebsch condition holds, we have $\Xi(\tau) < 0$, and thus it follows that $\ddot{\Phi}_+(\tau) < 0$ and $\ddot{\Phi}_-(\tau) > 0$. Hence $\Phi_+$ is negative and $\Phi_-$ is positive for $t < \tau$, $t$ sufficiently close to $\tau$. Thus these trajectories satisfy the minimum condition (12) over these intervals. $\square$

Since the singular control takes values in the open interval $(a, b)$, the graphs of the trajectories $x_\pm(t, \cdot)$ leave $\mathcal{S}$ to opposite sides and thus define a local field of extremals near the singular surface. As the points on the singular surface come close to the constraint $M_\alpha$, in line with our choice of $u_- = a$ as the entry control in *(E-ref)*, the trajectories $x_+$ will need to be terminated as they hit $M_\alpha$. As in the synthesis described in [7] these trajectories will not be propagated further backward, but they are needed to have a field near the singular surface. The trajectories $x_-$ on the other hand move away from the constraint when run backwards and can be kept as long as $\Phi_-$ is positive (and thus no additional switchings become necessary).

The synthesis around the boundary arc is described in detail in [7] and [14] and we therefore only briefly summarize the construction: The synthesis on $M_\alpha$ is given by the flow corresponding to the unique control $u_\alpha$ which makes $M_\alpha$ invariant. Since $M_\alpha$ and $\mathcal{S}$ are transversal, they intersect in a codimension 1 submanifold $\Theta = M_\alpha \cap \mathcal{S}$ of $M_\alpha$. Integrating the system and adjoint equations backward from $\Theta$ using the boundary control $u_\alpha$ generates a parameter dependent multiplier $\nu_\alpha(t, \tilde{p})$, $\tilde{p} \in \Theta$, which is positive and bounded away from zero over a compact subset. The positivity of $\nu_\alpha$ implies that the control $u_- = a$ satisfies the minimality condition when integrating $u \equiv a$ backward from $(\tau, \tilde{p}) \in \tilde{D}$ and the corresponding trajectories will remain extremal for some positive duration $\delta > 0$ independent of the initial condition. Thus the positivity of the multiplier $\nu$ guarantees that we can integrate the system backward from points of the constraint manifold $M_\alpha$ using the control $u = a$ and get extremals. Furthermore, the flow $\tilde{\sigma}_-$ consisting of the graphs of solutions to the differential equation

$$\dot{x} = f(t, x) + ag(t, x), \quad (20)$$

with initial conditions $q = (\tilde{t}, x(\tilde{t}, \tilde{p}))$, $\tilde{p} \in \Theta$, on the manifold $M_\alpha$, is everywhere transversal to $M_\alpha$ and thus it follows from the uniqueness of solutions that the corresponding graphs cannot intersect. This gives the required local embedding of the arc of $\Gamma$ defined over the interval $[\tau_1 - \varepsilon, \tau_1]$. The qualitative structure of the synthesis is summarized in Fig. 1.

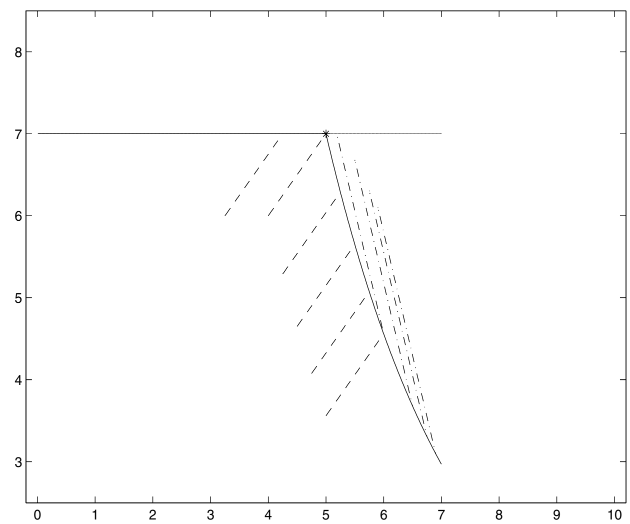


Fig. 1. Local synthesis at a boundary-arc singular-arc junction

## V. A Qualitatively Equivalent Local Field of Extremals Near a Boundary-arc Bang-arc Junction

In [7] and [14] we considered the case when a bang arc $\Gamma$ that concatenates with a boundary arc at an exit junction was embedded into a local field of extremals corresponding to the same control [7, assumption $(E3)$]. While this is a non-degenerate situation (for example, it is the typical situation for problems that arise in mathematical models for cancer chemotherapy like those considered in [4]), in general this property need not hold and a situation exactly as above is another typical scenario. (For example, consider the standard time-optimal control problem to the origin for the double integrator, but with constraints on the velocity). In this simpler case, as in Lemma 4.1 above, a local field of extremals is constructed which is transversal to the bang arc $\Gamma$ by integrating the other constant control backward from the reference arc $\Gamma$. If $\Gamma$ corresponds to the control $u \equiv b$, then, as shown in [7], there exists a bang-bang trajectory which switches from $u \equiv a$ to $u \equiv b$ at the exit-junction $\tau$ and thus only has a contact point with the constraint. If $\dot{\Phi}(\tau) < 0$, then for a small time a family of bang-bang extremals can be constructed by integrating $u \equiv a$ backward from $\Gamma$ by adjusting the multiplier. The corresponding local synthesis is qualitatively the same as for a junction with a singular arc shown in Fig. 1.

Combining the local syntheses described here with the results in [7] a local field of extremals can be constructed around a finite concatenation of interior bang-bang or singular and boundary arcs. The local optimality of the corresponding trajectories can then be shown with a regular synthesis type argument similar as in [14], but the details still need to be carried out. Instead, here we illustrate the immediate applicability of these conditions for a realistic example. In fact, this example has motivated the constructions both here and in [7].

## VI. EXAMPLE: MINIMIZATION OF THE BASE TRANSIT TIME IN SEMICONDUCTOR DEVICES [15]

This construction of a local field of extremals was motivated by the structure of extremals for the problem of minimizing the base transit time in bipolar transistors and we briefly recall this particular application[1] [15]. The active area in bipolar transistors is called *base region* and the time needed by the electric charges to cross the base region, the *base transit time* $\tau_B$, is one of the most important parameters related to the speed of bipolar transistors. The electric field which moves the charges is induced by tailoring the distribution of dopants in the base region and this so-called doping profile becomes a design parameter. The resulting problem of optimizing the base doping profile is one of the most well-studied problems in the electronics literature (see, for example, [18], [19], [20].) It can naturally be cast in the framework of optimal control theory. Assuming *low-level injunction* and *neglecting carrier recombination at the base*, two commonly made simplifying assumptions in modelling, the base transit time $\tau_B$ in bipolar transistors can be expressed as

$$\tau_B = \int_0^{W_B} \frac{1}{D_n(x)n_0(x)} \left( \int_0^x n_0(y)\ dy \right)\ dx + \frac{1}{n_0(W_B)v_{sat}} \int_0^{W_B} n_0(x)\ dx. \qquad (21)$$

In this formula the variable $x$ represents the coordinate for the base with baselength $W_B$, $0 \leq x \leq W_B$, $n_0(x)$ denotes the *minority carrier concentration* at $x$ in thermodynamic equilibrium and $D_n(x)$ denotes the *carrier diffusion coefficient*; $v_{sat}$ is a constant, the saturation velocity of the electrons. This formula is based on the path-breaking work of Kroemer [6] on his double-integral relation. The specific expression (21) given here, which includes a saturation velocity on the electrons, follows by integration by parts from a formula which has been derived by Suzuki [18].

In homojunction transistors, the minority carrier concentration $n_0$ is related to the doping concentration $N_A$ by the relation

$$n_0 = \frac{n_i^2}{N_A} \qquad (22)$$

where $n_i$, called the *intrinsic carrier concentration*, is a function generally also depending on $N_A$, which can be determined experimentally. The doping concentration $N_A(x)$ is constrained to lie between minimum and maximum levels, $0 < N_{A,\min} \leq N_A(x) \leq N_{A,\max}$. Different models exist for modelling the dependence of the diffusion coefficient $D_n$ and the intrinsic carrier concentration $n_i^2$ with respect to the doping concentration $N_A$. In [7] we considered the case when $D_n$ and $n_i^2$ do not depend explicitly on the independent variable $x$, but only through their dependence on the doping concentration $N_A$, i.e. $D_n(x) = D_n(N_A(x))$ and $n_i^2(x) = n_i^2(N_A(x))$. Except for this and some basic smoothness conditions, the forms for $D_n$ and $n_i^2$ are *arbitrary*. Furthermore, a fixed *base-resistance*, expressed as an integral constraint of the form

$$\int_0^{W_B} C(N_A(x))dx = G, \qquad (23)$$

where $G$ denotes the inverse of the fixed base resistance to be achieved, is added as extra constraint. The function $C$, which only depends on the doping profile, generally needs to be determined experimentally.

Formulating the problem as an optimal control problem, we take as the control in the problem, $u$, the space derivative of the doping profile,

$$u(x) = \frac{d}{dx}N_A(x) = N_A'(x). \qquad (24)$$

Henceforth we use a prime to denote $x$ derivatives. We assume that $u$ takes values in a compact interval $[U_{\min}, U_{\max}]$ where $-\infty < U_{\min} < 0 < U_{\max} < \infty$ and define the states $\xi_1$, $\xi_2$ and $\xi_3$ as

$$\xi_1(x) = \int_0^x \frac{n_i^2(N_A(y))}{N_A(y)}\, dy, \qquad (25)$$

$$\xi_2(x) = N_A(x), \qquad (26)$$

$$\xi_3(x) = \int_0^x C(N_A(y))dy. \qquad (27)$$

With this notation the problem of minimizing the base transit time $\tau_B$ for a fixed base resistance can be reformulated as to: **(T)** minimize

$$\tau(u) = \int_0^{W_B} \frac{\xi_1\xi_2}{D_n(\xi_2)n_i^2(\xi_2)}\, dx + \frac{\xi_1\xi_2}{v_{sat}n_i^2(\xi_2)}_{|(W_B)}$$

over all locally bounded Lebesgue measurable functions $u : [0, W_B] \to [U_{\min}, U_{\max}]$ subject to the differential equations

$$\xi_1'(x) = \frac{n_i^2(\xi_2)}{\xi_2},\ \xi_2'(x) = u,\ \xi_3'(x) = C(\xi_2) \qquad (28)$$

with initial conditions $\xi_1(0) = 0$, $\xi_2(0) = N_A(0)$, and $\xi_3(0) = 0$, prescribed terminal conditions $\xi_2(W_B)$ and $\xi_3(W_B) = G$, and state-space constraints

$$0 < \xi_2^{min} = N_{A,\min} \leq \xi_2 \leq N_{A,\max} = \xi_2^{max}. \qquad (29)$$

It is clear that the constraints are control-invariant submanifolds of relative degree 1 and the invariant controls are given by $u_\alpha \equiv 0$. In [15] normal extremals given by concatenations of bang, boundary and singular arcs have been computed that have the following general structure: (1) a bang arc corresponding to $u = U_{\max}$ which steers the system from its initial condition to the upper saturation limit

[1]The material on semiconductor devices is based on joint work with Paolo Rinaldi.

$N_{A,\max}$ followed by (2) a boundary arc, (3) a singular arc that steers the system from its upper limit $N_{A,\max}$ to the lower limit $N_{A,\min}$, (4) another boundary arc and (5) a final portion corresponding to $u = U_{\max}$ which steers the system to its desired terminal condition. Some of the pieces in this description may not be present. For example, in practical situations often $\xi_2(0) = N_{A,\max}$ and $\xi_2(W_B) = N_A(W_B)$ are taken so that the first and last piece are not present. On the other hand, it is possible that the singular arc in (3) saturates at its lower value $U_{\min}$ (this is only possible at the upper limit $N_{A,\max}$) and then the singular arc needs to be replaced by a concatenation of a trajectory for $u = U_{\min}$ followed by the singular arc. In [15] explicit analytical formulas are given for the singular arc and a system of linear equations is formulated for the junction times allowing the explicit determination of extremal trajectories.

Our results above and in [7] provide a complete geometric analysis of a local field of extremals near such a trajectory implying its strong local optimality. In this case, it can be shown that this analysis is globally valid and thus the extremals computed in [15] are indeed the globally optimal solutions. We illustrate the synthesis along a typical reference extremal with saturating singular arc (but only for the base doping profile $N_A$) in Fig. 2 below. The saturation point is indicated in the graph with a '*'.

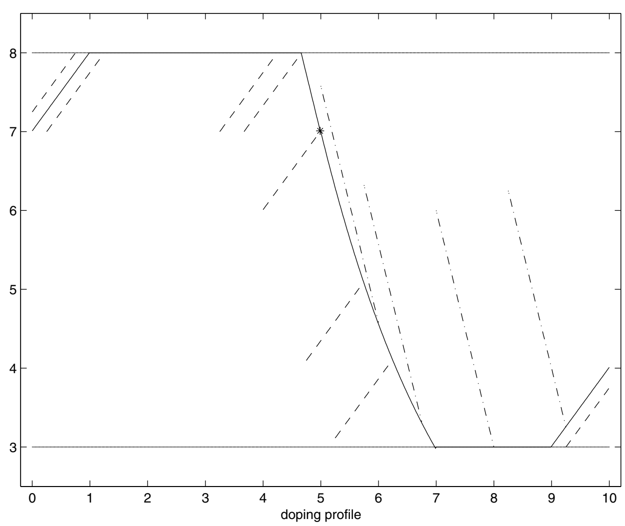


Fig. 2. Example of local synthesis for (T)

## VII. CONCLUSION

In many practical problems of engineering or scientific interest problems with state space constraints arise where the constraints have strong geometric properties and typically have boundaries which are manifolds or intersections of manifolds. Examples include the space shuttle re-entry problem considered in [2], the problem of minimizing the base transit time in bipolar transistors [15], or simple models for optimal control problems in cancer chemotherapy [4]. If the constraints are control-invariant manifolds of relative degree 1 - in our view a very natural condition - the results formulated here in conjunction with [7] allow to *imbed a wide class of reference trajectories that have boundary segments into a local field of extremals*. Our assumptions on the singular surface, however, make this realistic only in small dimensions. Based on these embeddings the optimality of the corresponding solutions in the strong sense, i.e. in $C([0,T])$, can in fact be proven.

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