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Robust Stability of a Class of Markovian Jump Nonlinear Systems

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Abstract— **This paper addresses the problem of robust stability analysis for a class of Markovian jump nonlinear systems subject to polytopic-type parameter uncertainty. A condition for robust local exponential mean square stability in terms of linear matrix inequalities is developed. An estimate of a robust domain of attraction of the origin is also provided. The approach is based on a stochastic Lyapunov function with polynomial dependence on the system state and uncertain parameters. A numerical example illustrates the proposed result.**

I. INTRODUCTION

Over the last decade, systems with Markovian jumps have been attracting an increasing attention. This class of systems is very appropriate to model plants whose structure is subject to random abrupt changes due to, for instance, changes of the operating point, random component failures, abrupt environment disturbance, etc. A number of control analysis and synthesis problems related to these systems has been analysed by several authors (see, e.g. [1], [2], [4]-[8], [11], [13]-[16], [18], [19], [21] and the references therein). In particular, robust mean square stability of continuous-time Markovian jump systems with uncertain parameters has been studied in, e.g. [1], [2] and [4]. A common feature of the existing methods of robust stability analysis is that they deal with either linear systems, or linear systems with unknown nonlinearities, such as Lipschitz-type and norm-bounded, which are treated as fictitious uncertainties. In addition, they are based on Lyapunov functions which are independent of the system uncertain parameters. Note that, since a common Lyapunov function is used to ensure stability for every admissible parameters value, these methods, referred to as *uncertainty-independent*, can be quite conservative. In spite of these developments, to the authors' knowledge, to-date the theory of robust stability analysis of Markovian jump nonlinear systems has not yet been fully addressed.

On the other hand, very recently a promising linear matrix inequality (LMI) approach of robust stability analysis and control for a class of nonlinear uncertain systems has been developed in [9] and [10]. This approach is based on a differential-algebraic representation of the system and employs a Lyapunov function, which is polynomial on the system state and uncertain parameters. In this paper, we extend the work of [10] to the context of Markovian jump nonlinear systems.

The purpose of this paper is to investigate the robust stability of a class of nonlinear systems with Markovian jumping parameters and subject to polytopic-type parameter uncertainty. An LMI method of robust local exponential mean square stability analysis is developed based on a stochastic Lyapunov function with polynomial dependence on the system state and uncertain parameters. Moreover, an estimate of a domain of attraction (DOA) inside a given polytopic region of the state-space containing the origin is also provided. The method can handle systems with rational functions of the state and uncertain parameters.

This paper is organized as follows. Section II deals with the class of systems considered in the paper and the problem to be addressed. Section III presents some preliminary results needed to obtain an LMI solution to the problem. Section IV develops the stability analysis method, which is then illustrated by a numerical example in Section V. Finally, concluding remarks are presented in Section VI.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices, $\|\cdot\|$ is the Euclidean vector norm, 0_n and $0_{m\times n}$ are the $n\times n$ and $m\times n$ matrices of zeros, I_n is the $n\times n$ identity matrix, diag$\{\cdots\}$ denotes a block-diagonal matrix, and Tr$[\cdots]$ stands for matrix trace. For a real matrix S, S' denotes its transpose, and $S>0$ ($S<0$) means that S is symmetric and positive-definite (negative-definite). For two polytopes $\mathscr{A}\subset\mathbb{R}^n$ and $\mathscr{B}\subset\mathbb{R}^m$, the notation $\mathscr{A}\times\mathscr{B}$ means that $(\mathscr{A}\times\mathscr{B})\subset\mathbb{R}^{(n+m)}$ is a meta-polytope obtained by the cartesian product and $\mathscr{V}(\mathscr{A}\times\mathscr{B})$ denotes the set of all vertices of $\mathscr{A}\times\mathscr{B}$. Mathematical expectation will be denoted by $\mathbf{E}[\cdot]$.

II. PROBLEM STATEMENT

Fix an underlying probability space $(\Omega,\mathscr{F},I\!P)$ and consider the stochastic system:

$$\dot{x}(t)=f(x(t),\delta,s_t),\ \ x(0)=x_0\in\mathscr{X} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $\delta\in\mathbb{R}^{n_\delta}$ is a vector of uncertain constant parameters, $\{s_t\}$ is a homogeneous Markov process with right continuous trajectories and taking values on the finite set $\Xi=\{1,2,\ldots,\sigma\}$, and $\mathscr{X}\subseteq\mathbb{R}^n$ is a given polytopic region that contains the origin and represents the domain of admissible initial states, x_0, to be considered in the stability analysis. $f(x,\delta,s_t)$ is a vector function that for each possible value of $s_t=i,\ i\in\Xi$ is given by

$$f_i(x,\delta):=f(x,\delta,i)$$

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where $f_i(x,\delta)$ is a continuous and bounded vector function of (x,δ) for all (x,δ) of interest. Moreover, the Markov process $\{s_t\}$ is assumed to have a stationary transition rate matrix $\Lambda = [\lambda_{ij}],\ i,j = 1,\ldots,\sigma$, such that

$$I\!P\{s_{t+h} = j \,|\, s_t = i\} = \begin{cases} \lambda_{ij}h + o(h), & i \neq j \\ 1 + \lambda_{ii}h + o(h), & i = j \end{cases}$$

where $h > 0$, $\lim_{h\downarrow 0} \frac{o(h)}{h} = 0$, $\lambda_{ij} \geq 0$ is the transition rate from the state i to j, $i \neq j$, and

$$\lambda_{ii} = -\sum_{\substack{j=1 \\ j\neq i}}^{\sigma} \lambda_{ij}. \tag{2}$$

It is supposed that system (1) satisfies the following assumptions:

A1 The uncertain parameter vector δ belongs to a given polytopic domain Δ.

A2 The origin $x = 0$ is an equilibrium point for all $s_t \in \Xi$ and $\delta \in \Delta$.

In this paper it is assumed that system (1) can be described by the following differential-algebraic representation (DAR):

$$\begin{cases} \dot{x}(t) = A_1(s_t)x(t) + A_2(s_t)\pi(x(t),\delta,s_t) \\ \ \ 0 = \Omega_1(x(t),\delta,s_t)x(t) + \Omega_2(x(t),\delta,s_t)\pi(x(t),\delta,s_t) \end{cases} \tag{3}$$

where $\pi(x,\delta,s_t) \in \mathbb{R}^{n_\pi}$ is an auxiliary nonlinear vector function of (x,δ) for each possible value of $s_t \in \Xi$ representing the nonlinear terms in $f(x,\delta,s_t)$, $A_1(s_t)$ and $A_2(s_t)$ are constant matrices for each possible value of $s_t = i,\ i \in \Xi$, denoted by A_{1i} and A_{2i}, respectively, and $\Omega_1(x,\delta,s_t) \in \mathbb{R}^{m\times n}$ and $\Omega_2(x,\delta,s_t) \in \mathbb{R}^{m\times n_\pi}$ are affine matrix functions of (x,δ) for each possible value of $s_t = i,\ i \in \Xi$, which are denoted by

$$\Omega_{1i}(x,\delta) := \Omega_1(x,\delta,i), \quad \Omega_{2i}(x,\delta) := \Omega_2(x,\delta,i)$$

where $\Omega_{1i}(x,\delta)$ and $\Omega_{2i}(x,\delta)$, $i = 1,\ldots,\sigma$ are affine matrix functions of (x,δ).

Note that a broad class of Markovian jump nonlinear systems can be represented in the form (3), such as systems with rational nonlinearities as well as some trigonometric nonlinearities. Indeed, it can be shown that (3) includes the linear fractional representation of [12], and as such it can model the whole class of systems with rational functions of the state and uncertain parameters without singularities at the origin; for further details see [9] and [10].

In addition to **A1** and **A2**, we shall adopt the following assumption to guarantee that the DAR (3) is well defined and thus, the uniqueness of the solution x is ensured.

A3 The matrix function $\Omega_{2i}(x,\delta)$, $i = 1,\ldots,\sigma$ has full column-rank for all (x,δ) belonging to $\mathscr{X} \times \Delta$.

To illustrate the DAR (3), consider a scalar system with

$$f_1(x) = a_0 x + a_1 x^3;\ \ f_2(x) = -\frac{c_0 x}{m(x)},\ \ m(x) = c_1 + x + c_2 x^2.$$

The above system can be written in the DAR (3) with:

$$A_{11} = a_0, \quad A_{21} = [0 \ \ a_1],$$

$$\pi(s_t = 1) = \begin{bmatrix} x^2 \\ x^3 \end{bmatrix}, \quad \Omega_{11} = \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad \Omega_{21} = \begin{bmatrix} -1 & 0 \\ x & -1 \end{bmatrix},$$

$$A_{12} = 0, \quad A_{22} = [-c_0 \ \ 0]$$

$$\pi(s_t = 2) = \begin{bmatrix} \frac{x}{m(x)} \\ \frac{x^2}{m(x)} \end{bmatrix}, \quad \Omega_{12} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \Omega_{22} = \begin{bmatrix} c_1 + x & c_2 x \\ -x & 1 \end{bmatrix}.$$

Note that the matrix Ω_{21} is nonsingular for all x and the Assumption **A3** for Ω_{22} is equivalent to $m(x) \neq 0$, which is a regularity condition for $f_2(x)$.

This paper addresses the problem of robust local stability analysis of the equilibrium point $x = 0$ of the system (3). The notion of stochastic stability used in this paper is in the mean square sense. In the sequel we introduce the following concepts of local *exponential mean square stability* for the system (3).

Definition 1: System (3) with a known parameter vector δ is *locally exponentially mean square stable*, if for any initial condition $x_0 \in \mathscr{X}$ and $s_0 \in \Xi$, there exist positive scalars α and β such that the solution $x(t)$ to (3) satisfies

$$\mathbf{E}[\,\|x(t)\|^2\,] \leq \beta \|x_0\| e^{-\alpha t},\ \forall t > 0.$$

Definition 2: System (3) with an uncertain $\delta \in \Delta$ is *robustly locally exponentially mean square stable*, if (3) is locally exponentially mean square stable for every $\delta \in \Delta$.

This paper is aimed at developing an LMI based condition for robust local exponential mean square stability of the system (3) as well as to provide an estimate of a robust domain of attraction of the origin, $\mathscr{D}_a$, defined as a subset of $\mathscr{X}$ such that for any $x_0 \in \mathscr{D}_a$ and $s_0 \in \Xi$, then $\mathbf{E}[x(t)] \in \mathscr{D}_a, \forall\ t \geq 0$ and $\lim_{t\to\infty} \mathbf{E}[x(t)] = 0$ for every $\delta \in \Delta$.

We conclude this section by recalling a version of Finsler's lemma which will be used in this paper.

Lemma 1 (Finsler's lemma): ([3]) Given matrices $\Psi_i = \Psi_i' \in \mathbb{R}^{n\times n}$ and $H_i \in \mathbb{R}^{m\times n}$, $i = 1,\ldots,\nu$, then

$$x_i'\Psi_i x_i > 0, \forall x_i \in \mathbb{R}^n : H_i x_i = 0,\ x_i \neq 0;\ i = 1,\ldots,\nu$$

if and only if there exist matrices $L_i \in \mathbb{R}^{n\times m}$, $i = 1,\ldots,\nu$, such that

$$\Psi_i + L_i H_i + (L_i H_i)' > 0, \quad i = 1,\ldots,\nu.$$

III. Preliminary Results

This section presents some basic results needed to derive an LMI based method of robust exponential mean square stability analysis for the Markov jump nonlinear system (3). We shall introduce a parameter-dependent stochastic Lyapunov function candidate and some related properties.

Consider the following Lyapunov function candidate

$$V(x,\delta,s_t) = x'\mathscr{P}(x,\delta,s_t)x \tag{4}$$

where $\mathscr{P}(x,\delta,s_t)$ is a positive definite matrix function of (x,δ) for each possible value of $s_t = i,\ i \in \Xi$, given by $\mathscr{P}_i(x,\delta) = \mathscr{P}(x,\delta,i)$. In order to obtain an LMI based stability condition, the following structure is adopted for the matrix $\mathscr{P}(x,\delta,s_t)$:

$$\mathscr{P}(x,\delta,s_t) = \begin{bmatrix} \Theta(x,\delta,s_t) \\ I_n \end{bmatrix}' P(s_t) \begin{bmatrix} \Theta(x,\delta,s_t) \\ I_n \end{bmatrix} \tag{5}$$

where $\Theta(x,\delta,s_t)\in\mathbb{R}^{n_\theta\times n}$, $i=1,\ldots,\sigma$ are given polynomial matrix functions of (x,δ) for each possible value of $s_t=i,\ i\in\Xi$, denoted by $\Theta_i(x,\delta)$, and $P(s_t)=P_i$, when $s_t=i,\ i\in\Xi$, with $P_i=P_i'$ being constant matrices to be determined.

The matrices $\Theta_i(x,\delta)$ define the Lyapunov function complexity. In general, as complex as these matrices are, less conservative will be the results at the cost of extra computational effort. From the authors' experience, a good compromise between conservativeness and computational effort is achieved by choosing $\Theta_i(x,\delta)$ as an affine matrix function[1] of (x,δ), namely

$$\Theta_i(x,\delta)=\sum_{k=1}^{n}T_{i_k}x_k+\sum_{k=1}^{n_\delta}S_{i_k}\delta_k+U_i \tag{6}$$

where T_{i_k},S_{i_k} and U_i are constant matrices having the same dimensions as $\Theta_i(x,\delta)$, and x_k and δ_k are the components of x and δ, respectively.

In view of (5), $V(x,\delta,s_t)$ can be also written as

$$V(x,\delta,s_t)=\xi'P(s_t)\xi \tag{7}$$

where ξ is an auxiliary vector defined by

$$\xi=\begin{bmatrix}\Theta(x,\delta,s_t)\\ I_n\end{bmatrix}x\in\mathbb{R}^{\kappa},\ \ \kappa=n_\theta+n. \tag{8}$$

Note that in view of (6) and the DAR (3), it can be readily established that ξ satisfies:

$$\dot{\xi}=\hat{\Theta}(x,\delta,s_t)[A_1(s_t)x+A_2(s_t)]\pi \tag{9}$$

$$0=H_1(x,\delta,s_t)\xi \tag{10}$$

where

$$H_1(x,\delta,s_t)=\begin{bmatrix}I & -\Theta(x,\delta,s_t)\end{bmatrix} \tag{11}$$

$$\hat{\Theta}(x,\delta,s_t)=\begin{bmatrix}\Theta_a(x,\delta,s_t)\\ I_n\end{bmatrix} \tag{12}$$

$$\Theta_a(x,\delta,i)=\Theta(x,\delta,i)+\sum_{k=1}^{n}T_{i_k}xe_k \tag{13}$$

with e_k denoting the k-th row of I_n.

It should be remarked that in (7), ξ is not an arbitrary vector in $\mathbb{R}^{\kappa}$. Indeed, ξ is required to satisfy the algebraic constraints of (10).

Let $\mathscr{A}$ be the infinitesimal generator of the Markov process $\{(x(t),s_t),\ t\geq 0\}$. Considering (4), (5), (8) and (9), it can be ready obtained that (see, e.g. [17]):

$$\mathscr{A}\cdot V(x,\delta,s_t)=2\xi'P(s_t)\hat{\Theta}(x,\delta,s_t)[A_1(s_t)x+A_2(s_t)\pi]+\xi'\sum_{j=1}^{\sigma}\lambda_{s_tj}P_j\xi. \tag{14}$$

Introducing the vector

$$\zeta=\begin{bmatrix}(\Theta(x,\delta,s_t)x)' & x' & \pi'\end{bmatrix}', \tag{15}$$

it follows that (14) can be rewritten as

$$\mathscr{A}\cdot V(x,\delta,s_t)=\zeta'\Gamma(x,\delta,s_t)\zeta \tag{16}$$

[1]The reader can refer to [20] for the general case.

where

$$\Gamma(x,\delta,s_t)=N_1'P(s_t)\hat{\Theta}(x,\delta,s_t)\hat{A}(s_t)N_2+[N_1'P(s_t)\hat{\Theta}(x,\delta,s_t)\hat{A}(s_t)N_2]'+\sum_{j=1}^{\sigma}\lambda_{s_tj}N_1'P_jN_1 \tag{17}$$

$$\hat{A}(s_t)=\begin{bmatrix}A_1(s_t) & A_2(s_t)\end{bmatrix} \tag{18}$$

$$N_1=\begin{bmatrix}I_\kappa & 0_{\kappa\times n_\pi}\end{bmatrix},\ \ N_2=\begin{bmatrix}0_{n\times n_\theta} & I_n & 0_{n\times n_\pi}\\ 0_{n_\pi\times n_\theta} & 0_{n_\pi\times n} & I_{n_\pi}\end{bmatrix}. \tag{19}$$

Note that $N_1\zeta=\xi$ and $N_2\zeta=[x'\ \ \pi']'$.

Finally, observe that the vector ζ in (16) is not an arbitrary vector in $\mathbb{R}^{(\kappa+n_\pi)}$. In fact, ζ is coupled with $\Gamma(x,\delta,s_t)$ via the vectors x and δ and is such that

$$H_2(x,\delta,s_t)\zeta=0,$$

$$H_2(x,\delta,s_t)=\begin{bmatrix}I & -\Theta(x,\delta,s_t) & 0\\ 0 & \Omega_1(x,\delta,s_t) & \Omega_2(x,\delta,s_t)\end{bmatrix}.$$

IV. Main Results

First, we shall develop LMI conditions to ensure that $V(x,\delta,s_t)$ is positive definite and $\mathscr{A}\cdot V(x,\delta,s_t)$ is negative definite for each possible value of $s_t\in\Xi$. In view of (7) and (16), the latter requirements lead to inequalities of the form

$$w(x,\delta)'T(x,\delta)w(x,\delta)>0,\ \forall(x,\delta)\in\mathscr{X}\times\Delta \tag{20}$$

where $w(x,\delta)\in\mathbb{R}^q$ is affine in δ and quadratic in x, whereas $T(x,\delta)\in\mathbb{R}^{q\times q}$ depends affinely on x and δ. Notice that (20) could be tested via the following LMIs:

$$T(x,\delta)>0,\ \forall(x,\delta)\in\mathscr{V}(\mathscr{X}\times\Delta).$$

However, the above is conservative because: (a) w is not an arbitrary vector in $\mathbb{R}^q$; (b) w and T are coupled.

A way to reduce the above conservatism is to use Finsler's lemma together with a *linear annihilator* of x, namely, a matrix $\mathscr{N}(x)$ which is a linear function of x and such that $\mathscr{N}(x)x=0$. More specifically, if $\mathscr{N}_w(x,\delta)$ is a full row-rank matrix with affine dependence on x and δ and such that

$$\mathscr{N}_w(x,\delta)w(x,\delta)=0,\ \forall(x,\delta)\in\mathscr{X}\times\Delta$$

then by Lemma 1, (20) holds if

$$T(x,\delta)+L\mathscr{N}_w(x,\delta)+\mathscr{N}_w(x,\delta)'L'>0,\ \forall(x,\delta)\in\mathscr{V}(\mathscr{X}\times\Delta)$$

where L is a multiplier matrix to be determined.

It should be noted that the matrix representation of a linear annihilator of x, $\mathscr{N}(x)$, is not unique. For instance, given $x=[x_1\ \cdots\ x_n]'\in\mathbb{R}^n$, a natural choice of $\mathscr{N}(x)\in\mathbb{R}^{(n-1)\times n}$ is as below:

$$\mathscr{N}(x)=\begin{bmatrix}x_2 & -x_1 & 0 & 0 & \cdots & 0 & 0\\ 0 & x_3 & -x_2 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & x_n & -x_{n-1}\end{bmatrix}.$$

The above procedure will be used to obtain a solution to the robust local stability analysis problem in terms of state- and parameter-dependent LMIs as presented in the next theorem. To this end, first introduce the following matrices:

$$\mathcal{N}_{1i}(x,\delta) = \begin{bmatrix} I_\kappa & -\Theta_i(x,\delta) \\ 0 & \mathcal{N}(x) \end{bmatrix}, \ i=1,\ldots,\sigma \tag{21}$$

$$\mathcal{N}_{2i}(x,\delta) = \begin{bmatrix} 0 & \mathcal{N}(x) & 0 \\ I_\kappa & -\Theta_i(x,\delta) & 0 \\ 0 & \Omega_{1i}(x,\delta) & \Omega_{2i}(x,\delta) \end{bmatrix}, \ i=1,\ldots,\sigma \tag{22}$$

where $\mathcal{N}(x)$ is a linear annihilator of x. Note that $\mathcal{N}_{1i}\xi = 0$ and $\mathcal{N}_{2i}\zeta = 0$ whenever $s_t = i$. Moreover, the role of the matrix $\mathcal{N}(x)$ is to decrease the conservatism of testing state-dependent LMIs, and it was first used in [20].

Theorem 1: Consider the system (3) satisfying **A1-A3** and let $\mathcal{X}$ and Δ be given polytopic regions. Let $\Theta_i(x,\delta)$, $i=1,\ldots,\sigma$ be given affine matrix functions of (x,δ). Suppose that there exist matrices $P_i = P_i'$, L_i and M_i, $i=1,\ldots,\sigma$ satisfying the following LMIs:

$$P_i + L_i\mathcal{N}_{1i}(x,\delta) + \mathcal{N}_{1i}(x,\delta)'L_i' > 0, \ \forall(x,\delta) \in \mathcal{V}(\mathcal{X}\times\Delta), \quad i=1,\ldots,\sigma \tag{23}$$

$$\Gamma_i(x,\delta) + M_i\mathcal{N}_{2i}(x,\delta) + \mathcal{N}_{2i}(x,\delta)'M_i' < 0, \quad \forall(x,\delta) \in \mathcal{V}(\mathcal{X}\times\Delta), \ i=1,\ldots,\sigma \tag{24}$$

where

$$\Gamma_i(x,\delta) = N_1'P_i\hat{\Theta}(x,\delta,i)\hat{A}_iN_2 + [N_1'P_i\hat{\Theta}(x,\delta,i)\hat{A}_iN_2]' + \sum_{j=1}^{\sigma}\lambda_{ij}N_1'P_jN_1 \tag{25}$$

and $\hat{A}_i = \hat{A}(s_t)$ when $s_t = i$. Then, the system (3) is robustly locally exponentially mean square stable.

Proof. First, note that if the LMIs (23) and (24) are feasible, then, by convexity, they are also satisfied for all $(x,\delta) \in \mathcal{X}\times\Delta$.

Since the inequality (23) is strict, there exists a sufficiently small scalar $\beta > 0$ such that

$$P_i + L_i\mathcal{N}_{1i}(x,\delta) + \mathcal{N}_{1i}(x,\delta)'L_i' > \beta N_3'N_3, \ \forall(x,\delta) \in \mathcal{X}\times\Delta, \quad i=1,\ldots,\sigma$$

where

$$N_3 = \begin{bmatrix} 0_{n\times n_\theta} & I_n \end{bmatrix}.$$

Note that for ξ as in (8), $N_3\xi = x$. Hence, pre- and post-multiplying the above inequality by ξ' and ξ, respectively, and considering (7), leads to

$$V(x,\delta,i) > \beta\|x\|^2, \ \forall(x,\delta) \in \mathcal{X}\times\Delta, \ i=1,\ldots,\sigma.$$

On the other hand, since (x,δ) belongs to a polytope, there exist scalars $\gamma_i > 0$, $i=1,\ldots,\sigma$ such that

$$V(x,\delta,i) \leq \gamma_i\|x\|^2, \ \forall(x,\delta) \in \mathcal{X}\times\Delta, \ i=1,\ldots,\sigma. \tag{26}$$

Next, (24) implies that there exist sufficiently small scalars $\varepsilon_i > 0$, $i=1,\ldots,\sigma$ such that

$$\Gamma_i(x,\delta) + M_i\mathcal{N}_{2i}(x,\delta) + \mathcal{N}_{2i}(x,\delta)'M_i' < -\varepsilon_iN_4'N_4, \quad \forall(x,\delta) \in \mathcal{X}\times\Delta, \ i=1,\ldots,\sigma$$

where

$$N_4 = \begin{bmatrix} 0_{n\times n_\theta} & I_n & 0_{n\times n_\pi} \end{bmatrix}.$$

Since for ζ as in (15), $N_4\zeta = x$ and $\mathcal{N}_{2i}\zeta = 0$ when $s_t = i$, pre- and post-multiplying the latter inequality by ζ' and ζ, respectively, and considering (16), implies that

$$\mathscr{A}\cdot V(x,\delta,i) < -\varepsilon_i\|x\|^2, \ \forall(x,\delta) \in \mathcal{X}\times\Delta, \ i=1,\ldots,\sigma. \tag{27}$$

Therefore, considering (26) and (27), we have

$$\frac{\mathscr{A}\cdot V(x,\delta,i)}{V(x,\delta,i)} < -\alpha, \ \forall(x,\delta) \in \mathcal{X}\times\Delta, \ x\neq 0, \ i=1,\ldots,\sigma$$

where $\alpha := \min_{i\in\Xi}\{\varepsilon_i/\gamma_i\}$.

Applying Dynkin's formula [17] and the Gronwall-Bellman's lemma to the latter inequality, similarly as in the proof of Theorem 1 in [16], it follows that for $t \geq 0$:

$$\mathbf{E}[V(x,\delta,s_t)|x_0,s_0] \leq V(x_0,\delta,s_0)e^{-\alpha t}, \ \forall x_0 \in \mathcal{X}, \ \forall s_0 \in \Xi \tag{28}$$

and for every $\delta \in \Delta$, which implies that the system (3) is robustly locally exponentially mean square stable. ∇∇∇

Theorem 1 provides an LMI condition for robust local exponential mean square stability of the origin of uncertain Markovian jump systems with rational nonlinearities, over a given polytopic region of the state-space. The proposed condition is based on a parametric stochastic Lyapunov function which is a 4th-order polynomial in the state variables and depends quadratically on the system uncertain parameters.

In the case of a completely known Markov jump linear system, namely for a system described by

$$\dot{x}(t) = A(s_t)x(t) \tag{29}$$

where $\{s_t\}$ is a Markov process as in system (1) and $A(s_t)$ is a known constant matrix for each possible value of $s_t = i$, $i \in \Xi$, denoted by A_i, Theorem 1 turns out to be equivalent to a well known necessary and sufficient condition for exponential mean square stability as below:

Lemma 2: ([16]) System (29) is exponentially mean square stable if and only if there exist matrices $X_i > 0$, $i=1,\ldots,\sigma$ satisfying the following LMIs:

$$X_iA_i + A_i'X_i + \sum_{j=1}^{\sigma}\lambda_{ij}X_j < 0, \ i=1,\ldots,\sigma. \tag{30}$$

□

Let the system (29) be rewritten in the DAR form (3) with:

$$\pi = x, \ A_{1i} = A_i, \ A_{2i} = 0_n, \ \Omega_{1i} = I_n, \ \Omega_{2i} = -I_n, \ i=1,\ldots,\sigma. \tag{31}$$

Then, we have the following result.

Lemma 3: The system (29) together with the DAR form (3) with (31) is exponentially mean square stable if and only if there exist matrices $P_i = P_i'$, L_i and M_i, $i=1,\ldots,\sigma$ satisfying the LMIs (23) and (24) with $\Theta_i = 0_{1\times n}$ and

$$L_i = \begin{bmatrix} \varepsilon_i & 0 \\ 0 & 0_n \end{bmatrix}, \ M_i = \begin{bmatrix} 0 & -\alpha_i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_iI_n \end{bmatrix}, \ P_i = \begin{bmatrix} 0 & 0 \\ 0 & Q_i \end{bmatrix}, \quad i=1,\ldots,\sigma \tag{32}$$

where $Q_i \in \mathbb{R}^{n\times n}$, ε_i and α_i are arbitrary positive scalars, and β_i is a sufficiently small positive scalar.

Proof. With the matrices L_i, M_i, P_i and Θ_i as above and the DAR of (31), it can be readily verified that the left-hand side of (23) and (24), denoted by Φ_i and Ψ_i, respectively, become:

$$\Phi_i = \begin{bmatrix} 2\varepsilon_i & 0 \\ 0 & Q_i \end{bmatrix}, \quad \Psi_i = J' \begin{bmatrix} -2\alpha_i & 0 & 0 \\ 0 & -2\beta_i I_n & \beta_i I_n \\ 0 & \beta_i I_n & \Upsilon_i \end{bmatrix} J \quad (33)$$

where

$$\Upsilon_i = Q_i A_i + A_i' Q_i + \sum_{j=1}^{\sigma} \lambda_{ij} Q_j, \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix}.$$

Finally, applying Schur's complement and considering that β_i is arbitrary small, it follows from (33) that the conditions $\Phi_i > 0$ and $\Psi_i < 0$, $i = 1, \ldots, \sigma$ are equivalent to

$$Q_i > 0, \quad \Upsilon_i < 0, \ i = 1, \ldots, \sigma$$

which concludes the proof. $\nabla\nabla\nabla$

A. DOA Estimate

In the sequel, we shall introduce an estimate of the domain of attraction $\mathscr{D}_a$, which can be obtained by solving an LMI problem. First, under the assumption that $V(x, \delta, s_t)$ proves the local stability of system (3) over $\mathscr{X} \times \Delta$, let the regions

$$\mathscr{R}_i = \{x \in \mathbb{R}^n : V(x, \delta, i) \leq 1, \ \forall \delta \in \Delta\}, \ i = 1, \ldots, \sigma \quad (34)$$

subject to the constraint $\mathscr{R}_i \subset \mathscr{X}, \forall \delta \in \Delta$. Note that in view of (28), to obtain an estimate of the DOA, one could consider the intersection $\mathscr{R}$ of all $\mathscr{R}_i$, namely

$$\mathscr{R} := \bigcap_{i \in \Xi} \mathscr{R}_i.$$

The motivation for this estimate is that $\mathscr{R}$ is a positively invariant set, in the sense that if $x_0 \in \mathscr{R}$, then $\mathbf{E}[V(x(t), \delta, i) | x_0, s_0] \leq 1$, for all $t > 0$, $\delta \in \Delta$ and $s_0, i \in \Xi$.

As the set $\mathscr{R}$ is, in general, non-convex, its computation may be intractable. To overcome this problem, we consider the following DOA estimate

$$\tilde{\mathscr{R}} := \{x \in \mathbb{R}^n : \tilde{V}(x, \delta) \leq 1, \ \forall \ \delta \in \Delta\} \quad (35)$$

where $\tilde{V}(x, \delta)$ is chosen such that $\tilde{\mathscr{R}} \subseteq \mathscr{R}_i, \forall i \in \Xi$ and $\tilde{\mathscr{R}}$ can be obtained in a convex way. It this paper, we shall adopt

$$\tilde{V}(x, \delta) = x' \tilde{\mathscr{P}}(x, \delta) x = \tilde{\xi}' \tilde{P} \tilde{\xi} \quad (36)$$

with $\tilde{P} = \tilde{P}' \in \mathbb{R}^{(\kappa\sigma) \times (\kappa\sigma)}$ and

$$\tilde{\mathscr{P}}(x, \delta) = \tilde{\Theta}(x, \delta)' \tilde{P} \tilde{\Theta}(x, \delta), \ \tilde{\xi} = \begin{bmatrix} \xi_1' & \ldots & \xi_\sigma' \end{bmatrix}'$$

$$\tilde{\Theta}(x, \delta) = \begin{bmatrix} (\tilde{\Theta}_1(x, \delta))' & \ldots & (\tilde{\Theta}_\sigma(x, \delta))' \end{bmatrix}'$$

$$\tilde{\Theta}_i(x, \delta) = \begin{bmatrix} \Theta_i(x, \delta)' & I \end{bmatrix}', \ \xi_i = \tilde{\Theta}_i(x, \delta) x, \ i = 1, \ldots, \sigma$$

and subject to the constraints:

$$\tilde{\xi}' \tilde{P} \tilde{\xi} - \xi_i' P_i \xi_i > 0, \forall (x, \delta) \in \mathscr{X} \times \Delta, \ i = 1, \ldots, \sigma. \quad (37)$$

Note that (37) ensures that $\tilde{\mathscr{R}} \subseteq \mathscr{R}_i$, $i = 1, \ldots, \sigma$.

Next, the inclusion $\mathscr{R}_i \subset \mathscr{X}$ will be expressed in terms of matrix inequalities (see [3] for further details). To this end, first the polytopic region $\mathscr{X}$ is recast as a set of LMIs:

$$\mathscr{X} = \left\{x \in \mathbb{R}^n : c_k' x \leq 1, \ i = 1, \ldots, n_e\right\} \quad (38)$$

where c_k are given constant vectors defining the n_e edges of $\mathscr{X}$. Hence, $\mathscr{R}_i \subset \mathscr{X}$ can be described by the conditions:

$$2 - 2c_k' x \geq 0, \ \forall \ x \in \mathbb{R}^n : x' \mathscr{P}_i(x, \delta) x - 1 \leq 0, \ \forall \delta \in \Delta. \quad (39)$$

Since $x' \mathscr{P}_i(x, \delta) x = \xi' P_i \xi$, by applying the $\mathscr{S}$-procedure [3] to (39) leads to the following sufficient conditions for (39) to hold, or equivalently, $\mathscr{R}_i \subset \mathscr{X}$:

$$\xi_a' \Pi_{ik} \xi_a \geq 0, \ k = 1, \ldots, n_e, \ i = 1, \ldots, \sigma \quad (40)$$

where

$$\xi_a = \begin{bmatrix} 1 \\ \xi \end{bmatrix}, \ \Pi_{ik} = \begin{bmatrix} 1 & c_{ak}' \\ c_{ak} & P_i \end{bmatrix}, \ c_{ak} = \begin{bmatrix} 0_{n_\theta \times 1} \\ -c_k \end{bmatrix}. \quad (41)$$

The next theorem deals with the DOA estimate $\tilde{\mathscr{R}}$. A way to maximize the size of $\tilde{\mathscr{R}}$ is to minimize $\mathrm{Tr}\{\tilde{\mathscr{P}}(x, \delta)\}$, which is a non-convex problem. Alternatively, by considering (36) and Lemma 1, we shall minimize $\mathrm{Tr}\{\tilde{P} + F\tilde{\mathscr{N}} + \tilde{\mathscr{N}}' F'\}$, where F is a multiplier matrix to be determined and $\tilde{\mathscr{N}}$ is such that $\tilde{\mathscr{N}} \tilde{\xi} = 0$ and given by:

$$\tilde{\mathscr{N}}(x, \delta) = \mathrm{diag}\{\mathscr{N}_{11}(x, \delta), \ldots, \mathscr{N}_{1\sigma}(x, \delta)\}. \quad (42)$$

Theorem 2: Consider the system (3) satisfying **A1-A3** and let $\mathscr{X}$ and Δ be given polytopic regions. Let $\Theta_i(x, \delta)$ be given affine matrix functions of (x, δ). Suppose that there exist matrices $\tilde{P} = \tilde{P}'$, F, $P_i = P_i'$, L_i, M_i, G_i and K_{ij}, $i = 1, \ldots, \sigma$, $j = 1, \ldots, n_e$, and a scalar η solving the following LMI problem:

minimize η, subject to (23), (24) and

$$\eta - \mathrm{Tr}\{\tilde{P} + F\tilde{\mathscr{N}} + \tilde{\mathscr{N}}' F'\} > 0 \quad (43)$$

$$\tilde{P} + F\tilde{\mathscr{N}} + \tilde{\mathscr{N}}' F' > 0 \quad (44)$$

$$\tilde{P} - \tilde{N}_i' P_i \tilde{N}_i + G_i \tilde{\mathscr{N}} + \tilde{\mathscr{N}}' G_i' > 0, \forall i \in \Xi \quad (45)$$

$$\Pi_{ik} + K_{ij} \mathscr{N}_{3i} + \mathscr{N}_{3i}' K_{ij}' > 0, \forall i \in \Xi, \ j = 1, \ldots, n_e \quad (46)$$

for all $(x, \delta) \in \mathscr{V}(\mathscr{X} \times \Delta)$, where $\tilde{N}_i$ is such that $\tilde{N}_i \tilde{\xi} = \xi_i$ and

$$\mathscr{N}_{3i}(x, \delta) = \begin{bmatrix} x & \mathscr{I} \\ 0 & \mathscr{N}_{1i}(x, \delta) \end{bmatrix}, \ \mathscr{I} = \begin{bmatrix} 0_{n \times n_\theta} & -I_n \end{bmatrix}. \quad (47)$$

Then, the set $\tilde{\mathscr{R}}$ defined by (35)-(37) is such that for any initial state $x_0 \in \tilde{\mathscr{R}}$ and $s_0 \in \Xi$, we have that $\mathbf{E}[x(t)] \in \tilde{\mathscr{R}}$, $\forall t > 0$ and $\lim_{t \to \infty} \mathbf{E}[x(t)] = 0$, for every $\delta \in \Delta$.

Proof. First, (23) and (24) imply that $V(x, \delta, s_t)$ is a Lyapunov function for the system (3) inside $\mathscr{X} \times \Delta$.

Since $\mathscr{N}_{3i} \xi_a = 0$ when $s_t = i$, pre- and post-multiplying (46) by ξ_a' and ξ_a, respectively, leads to (40), which implies $\mathscr{R}_i \subset \mathscr{X}, \forall i \in \Xi$. Next, as $\tilde{\mathscr{N}} \tilde{\xi} = 0$, pre- and post-multiplying (45) by $\tilde{\xi}'$ and $\tilde{\xi}$, respectively, leads to (37) and thus $\tilde{\mathscr{R}} \subseteq \mathscr{R}_i, \forall i \in \Xi$. Therefore, $\tilde{\mathscr{R}}$ is a positively invariant set. Finally, the above optimization minimizes $\mathrm{Tr}\{\tilde{P} + F\tilde{\mathscr{N}} + \tilde{\mathscr{N}}' F'\}$.

V. An Illustrative Example

Consider a Markovian jump system as in (1) with two operating modes described by:

$$f_1(x) = \begin{bmatrix} -x_1 + x_1^3 + 0.5x_2 \\ -0.5x_1 - x_2 + x_2^3 \end{bmatrix}, \ f_2(x) = \begin{bmatrix} x_1 + 0.5x_1^3 \\ x_1 - 2.3x_2 + 0.5x_2^3 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} -3 & 3 \\ 6 & -6 \end{bmatrix}. \quad (48)$$

Notice that the system (48) becomes unstable as the magnitude of the initial state $x_1(0)$ or/and $x_2(0)$ increases. Thus, the equilibrium point $x = 0$ is not globally exponentially mean square stable. Hence, we shall bound the state-space by the following parameterized polytope:

$$\mathscr{X}(\rho) = \{x : |x_i| \le \rho,\ i = 1,2\}.$$

System (48) can be rewritten in the differential-algebraic form (3) with $\pi = [x_1^2\ \ x_2^2\ \ x_1^3\ \ x_2^3]'$ and

$$A_{11} = \begin{bmatrix} -1 & 0.5 \\ -0.5 & -1 \end{bmatrix},\ A_{12} = \begin{bmatrix} 1 & 0 \\ 1 & -2.3 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ A_{21} = \begin{bmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix},$$

$$\Omega_{1i}(x) = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ \Omega_{2i}(x) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ x_1 & 0 & -1 & 0 \\ 0 & x_2 & 0 & -1 \end{bmatrix},\ i = 1,2$$

Theorem 1 is applied to the above system with

$$\mathscr{N}(x) = \begin{bmatrix} x_2 & -x_1 \end{bmatrix},\ \Theta_i(x) = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_1 & x_2 \end{bmatrix}',\ i = 1,2.$$

By a linear search on ρ, we get from Theorem 1 that the maximum ρ that ensures the local exponential mean square stability of the origin over $\mathscr{X}$ is $\rho = 1.10$. For the same ρ, Theorem 2 is then applied leading to the DOA estimate shown in Figure 1.

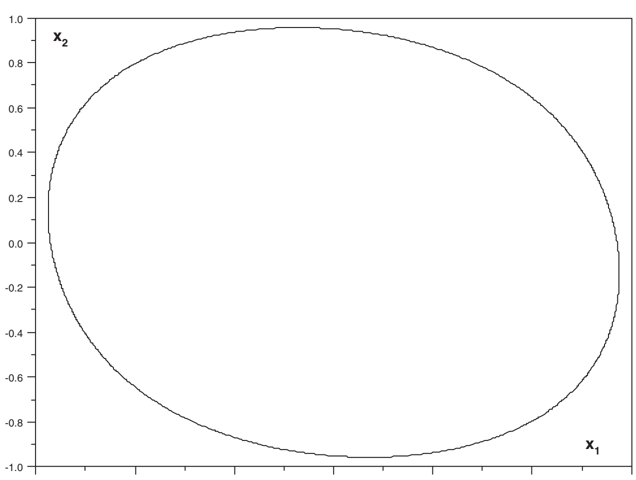

Fig. 1. Domain of Attraction Estimate.

VI. CONCLUSIONS

This paper has investigated the robust stability of a class Markovian jump nonlinear systems subject to polytopic-type parametric uncertainty. The system is assumed to be described by a differential-algebraic representation, which can model a broad class of Markovian jump nonlinear systems, such as the whole class of systems with rational functions of the state and uncertain parameters as well as some trigonometric nonlinearities. An LMI method has been developed for assessing the robust local exponential mean square stability of the origin over a given polytopic region $\mathscr{X}$ of the state-space and to obtain an estimate of a domain of attraction of the origin inside $\mathscr{X}$. The proposed method is based on a stochastic Lyapunov function which is a 4th-order polynomial in the state variables and depends quadratically on the system uncertain parameters.

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