Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005



# $\mathcal{H}_\infty$ parameter-dependent state feedback control of linear time-varying systems in polytopic domains

Vinícius F. Montagner and Pedro L. D. Peres

*Abstract*— **The synthesis of $\mathcal{H}_\infty$ parameter-dependent state feedback controllers for linear time-varying systems in polytopic domains is addressed by means of linear matrix inequalities. Differently from other gain scheduled approaches in the literature, all the system matrices are supposed to be affected by time-varying parameters, which have bounded rates of variation and belong to a polytope. Moreover, there are no assumptions on the structure of the parameters and no gridding technique is required to determine the parameter-dependent controller. The scheduled state feedback gain is calculated through an analytical expression using the time-varying parameters and a set of matrices obtained from the feasibility of a linear matrix inequality problem. The proposed convex design conditions, based on parameter-dependent Lyapunov functions, allow to improve the $\mathcal{H}_\infty$ performance of the system when compared to other control strategies and also to cope with problems of time-varying actuator failures.**

## I. INTRODUCTION

The $\mathcal{H}_\infty$ norm is an important issue in system theory, with several interpretations in terms of allowable disturbances and robustness of uncertain linear and nonlinear systems, time-varying and time-invariant as well [1]. In the case of control design, the determination of a robust state feedback gain that assures to the closed-loop system an $\mathcal{H}_\infty$ norm (for precisely known systems – [2]) or an $\mathcal{H}_\infty$ guaranteed cost (for uncertain systems – [3]) is also a relevant task. Lyapunov-based techniques can be applied to linear systems with parametric uncertainty and the design of $\mathcal{H}_\infty$ controllers based on fixed state feedback gains can be accomplished through a convex optimization problem, expressed in terms of linear matrix inequalities (LMIs – see [4] for details). However, the results based on fixed gains are frequently conservative and gain scheduled controllers [5] can be used to improve the system performance.

Many gain scheduled strategies assume some structure for the time-varying parameters, as the linear-fractional transformation representation [6] or, in many cases, the problem is addressed through exhaustive grid techniques that usually demand high computational efforts [7]. The results that impose a special structure to the parameters are often restrictive and those based on gridding techniques can be unreliable when the parameters have fast time variations [8]. When the rates of variation of the time-varying parameters are not known *a priori* or even when these parameters can vary instantaneously as in the case of switched systems [9], the usual gain scheduled approaches cannot be applied. In this situation, an important and useful tool to assure the closed-loop stability is provided by the existence of a constant Lyapunov function (quadratic stability). As a matter of fact, quadratic stability has been largely used to cope with robust control and robust filtering synthesis, for both time-invariant and time-varying uncertain parameters, including $\mathcal{H}_\infty$ performance and other requirements such as pole-location (see [4] and references therein). In [10], [11] and [12], a class of $\mathcal{H}_\infty$ linear parameter-varying (LPV) controllers for linear time-varying systems in polytopic domains has been investigated through the quadratic stability approach, but only in special cases for which some of the system matrices are not allowed to be time-varying the design can be cast as a convex optimization problem with no need of interpolation. More recently, a systematic way to design LPV controllers for systems with time-varying linear fractional parameters using full block multipliers was proposed in [13].

Parameter-dependent Lyapunov functions have shown good results when addressing the problem of stability of linear time-invariant systems in polytopic domains by means of sufficient LMI conditions [14]. The stability analysis of linear systems with time-varying uncertainties with bounded rates of variation was addressed by means of the multiconvexity in [15] and [16], but less conservative evaluations can be obtained using affine parameter-dependent Lyapunov functions, as pointed out in [17]. The design of gain scheduled controllers for polytopic systems with bounded time derivatives on the parameters has been addressed by means of LMIs in [7] and [18]. However, the conditions must be solved upon a grid on the parameter space, and the stability cannot be assured for the overall domain. Moreover, the numerical complexity of the tests grows rapidly. When the plant and the controller admit a linear fractional transformation, the existence of a stabilizing control can be determined through the feasibility of a finite set of LMIs [6], [19], but in these cases the strategy is not suitable to cope with actuator failures [20].

This paper addresses the problem of designing parameter-dependent state feedback $\mathcal{H}_\infty$ controllers for linear time-varying systems in polytopic domains whose parameters have bounded rates of variation. All the system matrices are supposed time-varying and no restrictive assumptions are made on the structure of the time-varying parameters. Extending the results from [21] to encompass $\mathcal{H}_\infty$ performance specifications, sufficient LMI conditions are given to compute parameter-dependent gains as a function of the system parameter vector (gain scheduled strategy), supposed to be on-line available, and of a set of fixed matrices. The

V. F. Montagner and P. L. D. Peres are with School of Electrical and Computer Engineering, University of Campinas, CP 6101, 13081-970, Campinas - SP - Brazil {montagne,peres}@dt.fee.unicamp.br

controller is obtained from the solution of a convex problem, with no need of gridding on the space of parameters. The conditions proposed here can provide better results than the ones based on quadratic stability or than gain scheduled methods that use interpolation, being also useful to cope with the problem of time-varying actuator failures.

## II. PRELIMINARIES

Consider the linear time-varying system

$$\dot{x}(t) = A(\alpha(t))x(t) + B_1(\alpha(t)))w(t) + B_2(\alpha(t))u(t) \quad (1)$$

$$z(t) = C(\alpha(t))x(t) + D_1(\alpha(t))w(t) + D_2(\alpha(t))u(t) \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^r$ is an exogenous input, $u(t) \in \mathbb{R}^m$ is the control input, $z(t) \in \mathbb{R}^p$ is the controlled output, $A(\alpha(t)) \in \mathbb{R}^{n\times n}$, $B_1(\alpha(t)) \in \mathbb{R}^{n\times r}$, $B_2(\alpha(t)) \in \mathbb{R}^{n\times m}$, $C(\alpha(t)) \in \mathbb{R}^{p\times n}$, $D_1(\alpha(t)) \in \mathbb{R}^{p\times r}$ and $D_2(\alpha(t)) \in \mathbb{R}^{p\times m}$ are time-varying matrices that belong to the polytope $\mathcal{P}$ given by

$$\begin{aligned}\mathcal{P} = \Big\{&(A, B_1, B_2, C, D_1, D_2)(\alpha(t)) \;:\\ &(A, B_1, B_2, C, D_1, D_2)(\alpha(t)) =\\ &\sum_{j=1}^{N} \alpha_j(t)(A, B_1, B_2, C, D_1, D_2)_j \;,\\ &\sum_{j=1}^{N} \alpha_j(t) = 1 \;,\; \alpha_j(t) \geq 0, \quad j = 1, \ldots, N\Big\}\end{aligned} \quad (3)$$

The system matrices are written as a convex combination of the vertices of the polytope $\mathcal{P}$ in terms of the time-varying parameters $\alpha(t)$. The time derivatives of the parameters are subject to the bounds

$$|\dot{\alpha}_i(t)| \leq \rho_i \;,\; i = 1, \ldots, N-1 \quad (4)$$

Notice that the constraint $\sum_{j=1}^{N} \alpha_j(t) = 1$ implies, without loss of generality, $\dot{\alpha}_N(t) = \sum_{j=1}^{N-1} \dot{\alpha}_j(t)$ and the bound on this parameter can be expressed by $|\dot{\alpha}_N(t)| \leq \sum_{i=1}^{N-1} \rho_i$.

The aim of this paper is to investigate the existence of a parameter-dependent state feedback control law

$$u(t) = K(\alpha(t))x(t) \quad , \quad K(\alpha(t)) \in \mathbb{R}^{m\times n} \quad (5)$$

such that with

$$\begin{aligned}A_{cl}(\alpha(t)) &\triangleq A(\alpha(t)) + B_2(\alpha(t))K(\alpha(t))\\ C_{cl}(\alpha(t)) &\triangleq C(\alpha(t)) + D_2(\alpha(t))K(\alpha(t))\end{aligned} \quad (6)$$

the closed-loop system given by

$$\dot{x}(t) = A_{cl}(\alpha(t))x(t) + B_1(\alpha(t))w(t) \quad (7)$$

$$z(t) = C_{cl}(\alpha(t))x(t) + D_1(\alpha(t))w(t) \quad (8)$$

has the following properties:

i) $A_{cl}(\alpha(t))$ is asymptotically stable;
ii) with $x(0) = \mathbf{0}$, for any input $w(t) \in \mathcal{L}_2$ it is possible to determine a bound $\gamma > 0$ such that $z(t) \in \mathcal{L}_2$ verifies

$$\|z(t)\|_2 < \gamma\|w(t)\|_2 \quad (9)$$

Any value of $\gamma$ that satisfies (9) is called an $\mathcal{H}_\infty$ guaranteed cost of the closed-loop system (7)-(8) and it is of great interest to determine the gain $K(\alpha(t))$ which provides the smallest $\gamma$ (best attenuation of disturbances $w(t)$). Although the choice of a robust state feedback gain (through quadratic stability as, for instance, in [3]) simplifies the problem to be solved and does not demand the on-line availability of the time-varying parameters $\alpha(t)$, there are some systems that do not admit a fixed quadratically stabilizing state feedback gain, or, which occurs quite frequently, the system admits a fixed quadratically stabilizing feedback control but this fixed gain does not provide an adequate $\mathcal{H}_\infty$ attenuation level for the closed-loop system.

In the sequel, a sufficient condition for the existence of a parameter-dependent state feedback gain is given. The condition is formulated as a set of LMIs involving only the vertices of the polytope $\mathcal{P}$ and the bounds on the time derivatives of the parameters (4), encompassing the results from quadratic stability in the sense that a feasible solution is obtained whenever the system is quadratically stabilizable by fixed gains.

## III. PROPOSED CONDITIONS

Next theorem provides a systematic way to determine an $\mathcal{H}_\infty$ state feedback gain scheduled controller based on the parameter-dependent Lyapunov function

$$v(x) = x'P(\alpha(t))x \quad (10)$$

with

$$\begin{aligned}&P(\alpha(t)) = \sum_{j=1}^{N} \alpha_j(t)P_j \quad , \quad P_j = P_j' > 0 \quad ,\\ &\sum_{j=1}^{N} \alpha_j(t) = 1 \quad , \quad \alpha_j(t) \geq 0 \;,\; j = 1, \ldots, N\end{aligned} \quad (11)$$

*Theorem 1:* For given real scalars parameters $\rho_i \geq 0$, $i = 1, \ldots, N-1$, if there exist symmetric positive definite matrices $W_j \in \mathbb{R}^{n\times n}$ and matrices $Z_j \in \mathbb{R}^{m\times n}$, $j = 1, \ldots, N$, such that[1]

$$H_j \triangleq \begin{bmatrix} T_1 & B_{1j} & W_jC_j' + Z_j'D_{2j}' \\ \star & -\mathbf{I} & D_{1j}' \\ \star & \star & -\mu\mathbf{I} \end{bmatrix} < 0 \;, \quad j = 1, \ldots, N \quad (12)$$

$$T_1 \triangleq A_jW_j + W_jA_j' + B_jZ_j + Z_j'B_j' + \sum_{i=1}^{N-1} \pm\rho_i(W_i - W_N) \quad (13)$$

$$H_{jk} \triangleq \begin{bmatrix} T_2 & B_{1j} + B_{1k} & T_3 \\ \star & -2\mathbf{I} & D_{1j}' + D_{1k}' \\ \star & \star & -2\mu\mathbf{I} \end{bmatrix} < 0 \;, \quad j = 1, \ldots, N-1 \;,\; k = j+1, \ldots, N \quad (14)$$

[1]The LMIs must be implemented with all the combinations $\pm$.

with

$$T_2 = A_j W_k + W_k A_j' + A_k W_j + W_j A_k' + B_j Z_k + Z_k' B_j' + B_k Z_j + Z_j' B_k' + 2 \sum_{i=1}^{N-1} \pm \rho_i (W_i - W_N) \quad (15)$$

$$T_3 \triangleq W_j C_k' + W_k C_j' + Z_j' D_{2k}' + Z_k' D_{2j}' \quad (16)$$

then the parameter-dependent gain

$$K(\alpha(t)) = Z(\alpha(t)) W(\alpha(t))^{-1} \quad (17)$$

with

$$(Z,W)(\alpha(t)) = \sum_{j=1}^{N} \alpha_j(t) (Z,W)_j \ , \quad \sum_{j=1}^{N} \alpha_j(t) = 1 \ , \ \alpha_j(t) \geq 0 \ , \ j = 1, \ldots, N \quad (18)$$

assures the stability of the closed-loop system with an $\mathcal{H}_\infty$ guaranteed cost given by $\gamma = \sqrt{\mu}$ and the Lyapunov function (10)-(11).

**Proof** Define[2]

$$\dot{v} + \mu^{-1} z' z - w' w \triangleq \beta' M(\alpha) \beta \quad (19)$$

with $\beta' = [x' \ \ w']$,

$$M(\alpha) = \begin{bmatrix} M_1 & P(\alpha) B_1(\alpha) + \mu^{-1} C_{cl}(\alpha)' D_1(\alpha) \\ \star & \mu^{-1} D_1(\alpha)' D_1(\alpha) - \mathbf{I} \end{bmatrix} \quad (20)$$

and

$$M_1 \triangleq A_{cl}(\alpha)' P(\alpha) + P(\alpha) A_{cl}(\alpha) + \dot{P}(\alpha) + \mu^{-1} C_{cl}(\alpha)' C_{cl}(\alpha) \quad (21)$$

Using Schur complement, expression (20) can be rewritten as

$$M(\alpha) = \begin{bmatrix} M_2 & P(\alpha) B_1(\alpha) & C_{cl}(\alpha)' \\ \star & -\mathbf{I} & D_1(\alpha)' \\ \star & \star & -\mu \mathbf{I} \end{bmatrix} \quad (22)$$

with

$$M_2 \triangleq A_{cl}(\alpha)' P(\alpha) + P(\alpha) A_{cl}(\alpha) + \dot{P}(\alpha) \quad (23)$$

Multiplying $M(\alpha)$ at left by $T(\alpha)$, at right by $T(\alpha)'$, with

$$T(\alpha) \triangleq \begin{bmatrix} P(\alpha)^{-1} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \quad (24)$$

using $A_{cl}(\alpha)$ and $C_{cl}(\alpha)$ given by (6), and taking into account the variable transformations $P(\alpha)^{-1} = W(\alpha)$, $Z(\alpha) = K(\alpha) W(\alpha)$ one has

$$H(\alpha) = \begin{bmatrix} H_1 & B_1(\alpha) & W(\alpha) C(\alpha)' + Z(\alpha)' D_2(\alpha)' \\ \star & -\mathbf{I} & D_1(\alpha)' \\ \star & \star & -\mu \mathbf{I} \end{bmatrix} \quad (25)$$

[2]The time dependency of the variables is omitted until the end of the proof, for sake of simplicity.

with

$$H_1 \triangleq A(\alpha) W(\alpha) + W(\alpha) A(\alpha)' + B_2(\alpha) Z(\alpha) + Z(\alpha)' B_2(\alpha)' + W(\alpha) \dot{P}(\alpha) W(\alpha) \quad (26)$$

Since $P(\alpha)^{-1} = W(\alpha)$ one has

$$P(\alpha) W(\alpha) = I \ , \ \dot{P}(\alpha) = -W(\alpha)^{-1} \dot{W}(\alpha) W(\alpha)^{-1} \quad (27)$$

which leads to

$$H(\alpha) = \begin{bmatrix} H_2 & B_1(\alpha) & W(\alpha) C(\alpha)' + Z(\alpha)' D_2(\alpha)' \\ \star & -\mathbf{I} & D_1(\alpha)' \\ \star & \star & -\mu \mathbf{I} \end{bmatrix} \quad (28)$$

with

$$H_2 \triangleq A(\alpha) W(\alpha) + W(\alpha) A(\alpha)' + B_2(\alpha) Z(\alpha) + Z(\alpha)' B_2(\alpha)' - \dot{W}(\alpha) \quad (29)$$

Notice that

$$\dot{W}(\alpha) = \sum_{j=1}^{N} \dot{\alpha}_j W_j = \sum_{i=1}^{N-1} \dot{\alpha}_i (W_i - W_N) = \Big( \sum_{j=1}^{N} \alpha_j^2 + 2 \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \alpha_j \alpha_k \Big) \sum_{i=1}^{N-1} \dot{\alpha}_i (W_i - W_N) \quad (30)$$

Using expression (3) for $(A, B_2)(\alpha)$, and (18) for $Z(\alpha)$, $W(\alpha)$, and taking (30) into account, one can write

$$H(\alpha) = \sum_{j=1}^{N} \alpha_j^2 H_j + \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \alpha_j \alpha_k H_{jk} \quad (31)$$

with $H_j$ given by (12) and $H_{jk}$ given by (14)-(15). Theorem 1 imposes $H_j < 0$ and $H_{jk} < 0$, which is sufficient to assure $H(\alpha) < 0$ for all $\alpha$ defined in (3), thus guaranteeing the closed-loop stability with $\gamma = \sqrt{\mu}$ attenuation level. □

Some remarks about Theorem 1 are now in order. First, if a feasible solution exists for a given set of bounds (4), then the gain $K(\alpha(t))$, analytically determined through (17)-(18), assures the system stability with a $\gamma$ disturbance attenuation. Notice that there is no need of grids on the parameter space neither restrictive assumptions on the parameter structure to determine $K(\alpha(t))$. The number of LMIs to be solved in this case is $N + N2^{N-1} + N(N-1)2^{N-2}$ (including $W_j > 0 \ , \ j = 1, \ldots, N$) and, although this number increases rapidly with $N$, there are polynomial time based algorithms available to solve the problem [4], [22]. Second, if the conditions of Theorem 1 are feasible for arbitrarily high values of $\rho_i$, $i = 1, \ldots, N-1$, they lead to the solution $W_1 \simeq W_2 \simeq \ldots W_N$ and the system is quadratically stable. This can be viewed from the fact that, since the LMIs (12)-(16) must be fulfilled for $+\rho_i(W_i - W_N)$ as well as for $-\rho_i(W_i - W_N)$, the only possible feasible solution in this case will be such that $W_1 \simeq \cdots \simeq W_N$ in order to annihilate the influence of the terms $\pm\rho_i(W_i - W_N)$ for high values of $\rho_i$ (reducing their contribution as much as possible by making $W_1 \simeq \cdots \simeq W_N$). Notice that the resulting gain

(17) will be almost LPV in this case. Third remark, when the bounds on the time derivatives $\rho_i$, $i = 1, \ldots, N-1$ are not known *a priori*, it is possible to use line searches to find bounds $\rho_i$ for which Theorem 1 ensures the closed-loop stability with an $\mathcal{H}_\infty$ guaranteed performance. Finally, note that time-varying systems are frequently represented with an affine dependence on the parameters. In this case, it is possible to rewrite the system into the polytopic form since there exists a linear relationship between the parameters in each representation.

### A. Actuator failures

Actuators of physical systems suffer from deterioration that comes from aging, malfunction etc. Actuator failures can be modeled as control inputs given by $F(t)u(t)$, $F(t) \in \mathbb{R}^{m\times m}$, with $F(t) = \text{diag}[f_1(t) \quad f_2(t) \quad \ldots \quad f_m(t)]$. The parameters are such that $0 \leq f_i(t) \leq 1$, $i = 1, \ldots, m$, $|\dot{f}_i(t)| \leq \beta_i$, $i = 1, \ldots, m$ and they describe the degree of failure of each actuator $u_i(t)$, $i = 1, \ldots, m$. For instance, if $f_1(t) = 1$, actuator $u_1(t)$ works with no failure and if $f_1(t) = 0$, this actuator fails completely. The bounds $\beta_i$, $i = 1, \ldots, m$, give information about how fast an actuator loses its strength during operation. One can model reliable actuators as having slow rates of variation (small values for $\beta_i$) and unreliable or critical actuators as having high rates of variation (high values for $\beta_i$). It is also possible to assume arbitrarily high values for $\beta_i$ in order to model instantaneous failures. The conditions from Theorem 1 can be directly applied to this problem, dealing with the matrix products $B_2(\alpha(t))F(t)$ and $D_2(\alpha(t))F(t)$, and thus representing the original system (1)-(2) with $N$ vertices by means of a polytopic system with $2^m N$ vertices, as shown in the third example.

## IV. NUMERICAL EXAMPLES

Consider system (1)-(2) with randomly generated vertices given by

$$A_1 = \begin{bmatrix} 10.9 & 1.0 \\ 10.0 & 18.0 \end{bmatrix},\ B_{11} = \begin{bmatrix} -0.1 \\ -0.5 \end{bmatrix},\ B_{21} = \begin{bmatrix} 2.1 \\ 1.1 \end{bmatrix} \quad (32)$$

$$C_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix},\ D_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ D_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (33)$$

$$A_2 = \begin{bmatrix} -19.1 & 1.0 \\ -10.0 & -14.0 \end{bmatrix} \quad (34)$$

$$B_{12} = B_{11},\ B_{22} = B_{21},\ C_2 = C_1,\ D_{12} = D_{11},\ D_{22} = D_{21} \quad (35)$$

This system is quadratically stabilizable through the fixed gain

$$K = \begin{bmatrix} -8.5616 & -48.4999 \end{bmatrix} \quad (36)$$

with an $\mathcal{H}_\infty$ guaranteed cost given by $\gamma_Q = 0.9427$ (for arbitrary rates of parametric variations).

Assuming bounded rates of parametric variations given by $|\dot{\alpha}_1(t)| \leq \rho_1$ (recall $\alpha_2(t) = 1 - \alpha_1(t)$), Theorem 1 allows to obtain gain scheduled controllers that reduce the values of $\mathcal{H}_\infty$ guaranteed cost, as shown in Fig. 1. Observe that for each value of $\rho_1$ in Fig. 1, the conditions from Theorem 1 provide a gain scheduled controller that ensure the closed-loop stability with values of $\mathcal{H}_\infty$ guaranteed costs that are always smaller or equal to the value from quadratic stability with the gains given in (36). The lowest value obtained through (12)-(16) is $\gamma = 0.7397$, for $\rho_1 = 0$ (time-invariant case) and as $\rho_1$ increases, the values of $\gamma$ obtained from Theorem 1 tend to $\gamma_Q = 0.9427$ from quadratic stability with a fixed gain.

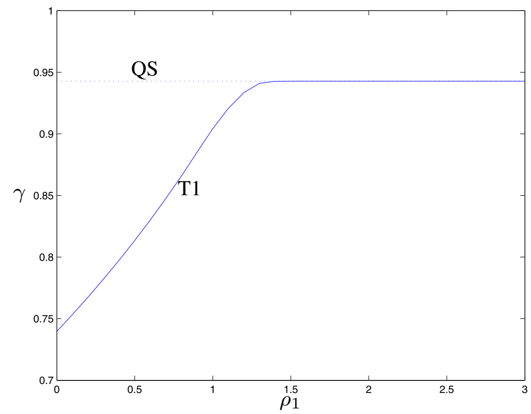


Fig. 1. $\mathcal{H}_\infty$ guaranteed costs $\gamma$ for system (1)-(2) with vertices (32)-(35) as a function of the bound on the rates of parametric variations $\rho_1$ using a robust control gain obtained from quadratic stability (QS) and scheduled gains from Theorem 1 (T1).

The entries of the parameter-dependent gain

$$K(\alpha_1(t)) = \begin{bmatrix} k_{11}(\alpha_1(t)) & k_{12}(\alpha_2(t)) \end{bmatrix}$$

provided by Theorem 1 for $\rho_1 = 0$ and $\rho_1 = 3$ are shown in Fig. 2. Observe the nonlinear behavior of $k_{11}(\alpha_1(t))$ and $k_{12}(\alpha_1(t))$ for $\rho_1 = 0$ and how these entries tend to the fixed values obtained through the quadratic condition, given by (36), when $\rho_1 = 3$. The better performances (lower values of $\gamma$, better rejection of disturbances) are assured by Theorem 1 at the price of implementing a gain scheduled strategy, as shown in Fig. 2, at left.

As a second example, consider system (1)-(2) with randomly generated vertices

$$A_1 = \begin{bmatrix} 0.7 & 0.5 \\ 0.9 & 0.5 \end{bmatrix},\ B_{11} = \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix},\ B_{21} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \quad (37)$$

$$C_1 = \begin{bmatrix} 3 & 2 \end{bmatrix},\ D_{11} = \begin{bmatrix} 0.6 \end{bmatrix},\ D_{21} = \begin{bmatrix} 0.8 \end{bmatrix} \quad (38)$$

$$A_2 = \begin{bmatrix} 0.3 & 0.3 \\ 0.1 & 0.2 \end{bmatrix},\ B_{12} = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix},\ B_{22} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \quad (39)$$

$$C_2 = \begin{bmatrix} 0 & 4 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0.1 \end{bmatrix},\ D_{22} = \begin{bmatrix} 1 \end{bmatrix} \quad (40)$$

This system is not quadratically stabilizable through fixed gains [4] but the conditions of Theorem 1 allow stabilization

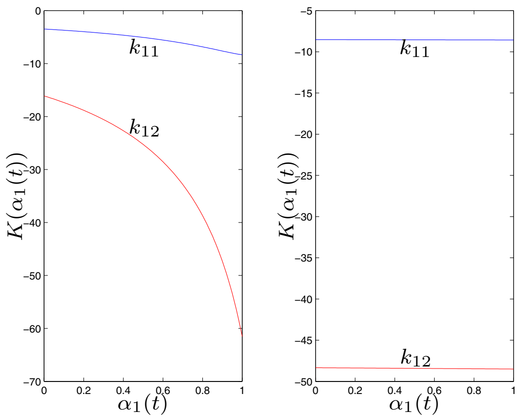


Fig. 2. Behavior of the entries of the parameter-dependent gains $K(\alpha_1(t))$ given by Theorem 1 for $\rho_1 = 0$ (left) and $\rho_1 = 3$ (right) assuring closed-loop system stability with an $\mathcal{H}_\infty$ guaranteed cost for system (1)-(2) with vertices given by (32)-(35).

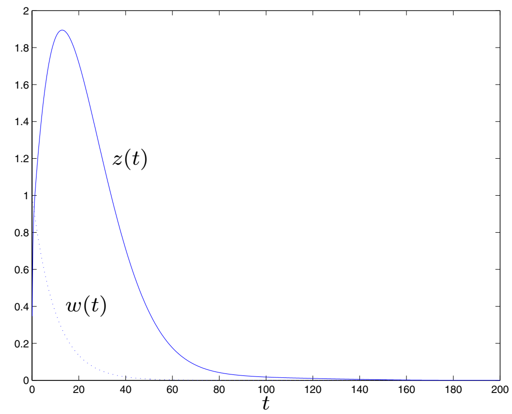


Fig. 4. Output $z(t)$ of system (1)-(2) with vertices given by (37)-(40) for the control gains calculated through the conditions of Theorem 1 with $\rho_1 = 0.01$, for uncertain time-varying parameters given by (42), with $\lambda = 0.01$ and with the disturbance signal (41).

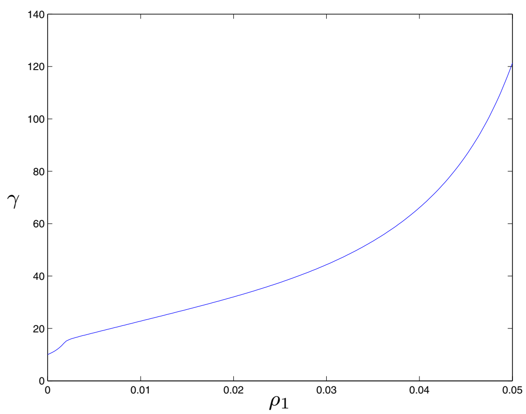


Fig. 3. $\mathcal{H}_\infty$ guaranteed costs provided by parameter-dependent gains obtained through the conditions of Theorem 1 for system (1)-(2) with vertices given by (37)-(40).

through parameter-dependent control gains with $\mathcal{H}_\infty$ guaranteed costs shown in Figure 3.

Assume that the system is affected by the disturbance signal

$$w(t) = \exp(-0.01t) \tag{41}$$

with the uncertain parameter given by

$$\alpha_1(t) = 0.5 + 0.5\sin(2\lambda t) \tag{42}$$

which leads to $|\dot{\alpha}_1(t)| \le \lambda$. Using the conditions (12)-(14) for $\rho_1 = \lambda = 0.01$, one has that this system, not quadratically stabilizable through fixed gains, can be stabilized through parameter-dependent gains, with an $\mathcal{H}_\infty$ guaranteed cost given by $\gamma = 22.7882$ and with the closed-loop time response shown in Figure 4.

As a final example, consider a design example applied to a single flexible link studied in [8] and [12]. Here, the model is slightly modified to address the problem of actuator failures, being described by the following equations

$$\dot{x}(t) = \bar{A}(\theta(t))x(t) + \bar{B}_1 w(t) + \bar{B}_2(f(t))u(t) \tag{43}$$

$$z(t) = \bar{C}x(t) + \bar{D}_1 w(t) + \bar{D}_2(f(t))u(t) \tag{44}$$

$$\bar{A}(\theta(t)) = \bar{A}_o + \theta(t)\bar{A}_1 \tag{45}$$

$$\bar{B}_2(f(t)) = f(t)\bar{B}_2 \quad , \quad \bar{D}_2(f(t)) = f(t)\bar{D}_2 \tag{46}$$

The system matrices are given by

$$\bar{A}_o = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \; \bar{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 9.8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{47}$$

$$\bar{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.01 \end{bmatrix}, \; \bar{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{48}$$

$$\bar{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; \bar{D}_2 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} \tag{49}$$

Matrix $\bar{D}_1$ is supposed equal to zero, and the focus is on the design of $\mathcal{H}_\infty$ controllers considering that the system is subject to the time-varying parameters $\theta(t)$ and $f(t)$ (which models failures of the actuator) lying in the intervals

$$\theta(t) \in [0, 0.5] \quad , \quad f(t) \in [0.1, 1] \tag{50}$$

For instance, when $f(t) = 0.1$, the actuator operates with $10\%$ of its full strength.

This system can be represented by a four-vertex polytope, obtained from all possible combinations of maximum and minimum values of $(\theta(t), f(t))$, yielding

$$|\dot{\theta}(t)| \le 0.5(\rho_1 + \rho_2) \triangleq \delta_1 \; , \; |\dot{f}(t)| \le 0.9(\rho_1 + \rho_3) \triangleq \delta_2 \tag{51}$$

The linear relationships between the bounds on the rates of parametric variations of the original affine system $\delta_1$, $\delta_2$, and the bounds $\rho_1$, $\rho_2$, $\rho_3$ in its polytopic representation is apparent.

The quadratic stability condition allows to determine a robust state feedback gain that stabilizes system (43)-(50) with an $\mathcal{H}_\infty$ guaranteed cost given by $\gamma_Q = 0.7720$, for arbitrary rates of parametric variations $\dot{\theta}(t)$, $\dot{f}(t)$.

TABLE I

REDUCTION ON THE $\mathcal{H}_\infty$ GUARANTEED COSTS GIVEN BY (52) FOR SYSTEM (43)-(50) WITH $|\dot{\theta}(t)| \leq \delta_1$, $|\dot{f}(t)| \leq \delta_2$. THE SYMBOL $\infty$ DENOTES ARBITRARILY HIGH VALUES.

| $\delta_1$ | $\delta_2$ | $\Delta\gamma(\%)$ |
|---|---|---|
| 0 | 0 | 18.4 |
| 0 | $\infty$ | 18.4 |
| $\infty$ | 0 | 16.9 |
| 1 | 1 | 14.8 |
| 10 | 10 | 9.2 |

It is possible to improve the performance of the system using the gain scheduled strategy proposed in Theorem 1. A performance comparison is summarized in Table I, where each row provides the bounds on $|\dot{\theta}(t)|$ and $|\dot{f}(t)|$, the corresponding value of reduction in the $\mathcal{H}_\infty$ guaranteed costs of the closed-loop system provided by Theorem 1 with respect to the results of robust $\mathcal{H}_\infty$ quadratic stabilizing gains, defined as

$$\Delta\gamma = \frac{\gamma_Q - \gamma_{T1}}{\gamma_Q} \tag{52}$$

The first three rows of Table I show that when at least one of the parameters $\theta(t)$, $f(t)$ is time-invariant, the reductions on the $\mathcal{H}_\infty$ guaranteed costs provided by Theorem 1 are higher. Particularly, the second row indicates that the performance assured by Theorem 1 does not depend on how fast the failures on the system actuator $f(t)$ may occur when $\theta(t)$ is considered as time-invariant. The last two rows of Table I point out that when both parameters are considered as time-varying, the values of $\Delta\gamma$ decrease as the bounds on the rates of parametric variations $\delta_1$ and $\delta_2$ increase. When $\delta_1$ and $\delta_2$ are arbitrarily high, Theorem 1 provides an almost LPV gain (as stated in the second remark of Theorem 1) with an $\mathcal{H}_\infty$ guaranteed cost given by $0.7052$.

Finally, it is important to stress that conditions from the literature that consider fixed time-invariant control matrices cannot cope with the problem of control under actuator failures studied in this example. On the other hand, the conditions from Theorem 1 can handle this problem efficiently.

## V. CONCLUSION

This paper has presented sufficient convex conditions to design $\mathcal{H}_\infty$ state feedback gain scheduled controllers for linear time-varying systems with parameters that have bounded rates of variation and lie inside a polytope. If the rates of variation are known *a priori*, a convex test based on LMIs allows to determine the stabilizing $\mathcal{H}_\infty$ parameter-dependent gain, that always provides a better performance than fixed gains given by quadratic stability. Differently from other gain scheduled approaches in the literature, all the system matrices are supposed to be time-varying here, which allows to address problems of actuator failures, for instance. Moreover, there is no need of grids on the space of parameters neither restrictive assumptions on the parameter structure. Numerical examples including an application of control subject to actuator failures illustrate the efficiency of the proposed conditions in the design of $\mathcal{H}_\infty$ parameter-dependent state feedback controllers.

## VI. ACKNOWLEDGMENTS

The authors would like to thank the Brazilian agencies CNPq, CAPES and FAPESP for the financial support. Vinícius F. Montagner thanks the Department of Telematics of the School of Electrical and Computer Engineering, University of Campinas, where he is currently an associate researcher.

## REFERENCES

[1] K. Zhou, K. Glover, and J. C. Doyle, *Robust and Optimal Control*. Upper Saddle River, NJ, USA: Prentice Hall, 1996.

[2] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. Francis, “State space solutions to the standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems,” *IEEE Trans. Automat. Contr.*, vol. 34, no. 8, pp. 831–847, August 1989.

[3] P. L. D. Peres, J. C. Geromel, and S. R. Souza, “$\mathcal{H}_\infty$ guaranteed cost control for uncertain continuous-time linear systems,” *Syst. Contr. Lett.*, vol. 20, no. 6, pp. 413–418, June 1993.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM Studies in Applied Mathematics, 1994.

[5] W. J. Rugh and J. S. Shamma, “Research on gain scheduling,” *Automatica*, vol. 36, no. 10, pp. 1401–1425, October 2000.

[6] G. Scorletti and L. El Ghaoui, “Improved LMI conditions for gain scheduling and related control problems,” *Int. J. Robust Nonlinear Contr.*, vol. 8, no. 10, pp. 845–877, August 1998.

[7] P. Apkarian and R. J. Adams, “Advanced gain-scheduling techniques for uncertain systems,” *IEEE Trans. Trans. Contr. Syst. Technol.*, vol. 6, no. 1, pp. 21–32, 1998.

[8] J. S. Shamma and M. Athans, “Gain scheduling: potential hazards and possible remedies,” *IEEE Control Systems Magazine*, vol. 12, no. 3, pp. 101–107, June 1992.

[9] D. Liberzon and A. S. Morse, “Basic problems in stability and design of switched systems,” *IEEE Control Syst. Mag.*, vol. 19, no. 5, pp. 59–70, October 1999.

[10] G. Becker and A. Packard, “Robust performance of linear parametrically varying systems using parametrically-dependent linear feedback,” *Syst. Contr. Lett.*, vol. 23, pp. 205–215, 1994.

[11] C. Courties, J. Bernussou, and G. Garcia, “LPV control by dynamic output feedback,” in *Proc. 1999 Amer. Control Conf.*, vol. 4, San Diego, CA, June 1999, pp. 2267–2271.

[12] J. Yu and A. Sideris, “$\mathcal{H}_\infty$ control with parametric Lyapunov functions,” *Syst. Contr. Lett.*, vol. 30, pp. 57–69, 1997.

[13] C. W. Scherer, “LPV control and full block multipliers,” *Automatica*, vol. 37, no. 3, pp. 361–375, March 2001.

[14] V. J. S. Leite and P. L. D. Peres, “An improved LMI condition for robust $\mathcal{D}$-stability of uncertain polytopic systems,” *IEEE Trans. Automat. Contr.*, vol. 48, no. 3, pp. 500–504, March 2003.

[15] P. Apkarian and H. D. Tuan, “Parametrized LMIs in control theory,” *SIAM J. Control Optim.*, vol. 38, no. 4, pp. 1241–1264, May 2000.

[16] P. Gahinet, P. Apkarian, and M. Chilali, “Affine parameter-dependent Lyapunov functions and real parametric uncertainty,” *IEEE Trans. Automat. Contr.*, vol. 41, no. 3, pp. 436–442, March 1996.

[17] V. F. Montagner and P. L. D. Peres, “A new LMI condition for the robust stability of linear time-varying systems,” in *Proc. 42nd IEEE Conf. Decision Contr.*, Maui, HI, USA, December 2003, pp. 6133–6138.

[18] F. Wu, “A generalized LPV system analysis and control synthesis framework,” *Int. J. Contr.*, vol. 74, no. 7, pp. 745–759, May 2001.

[19] A. Packard, “Gain scheduling via linear fractional transformations,” *Syst. Contr. Lett.*, vol. 22, no. 2, pp. 79–92, February 1994.

[20] A. V. Savkin and R. J. Evans, *Hybrid Dynamical Systems. Controller and Sensor Switching Problems*. Boston, MA: Birkhäuser, 2002.

[21] V. F. Montagner and P. L. D. Peres, “State feedback gain scheduling for linear systems with time-varying parameters,” in *Proc. 2004 Amer. Control Conf.*, Boston, MA, USA, June/July 2004, pp. 2004–2009.

[22] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, *LMI Control Toolbox User's Guide*. The Math Works Inc., Natick, MA, 1995.