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Linear modeling error and steady-state behaviour of nonlinear dynamical systems

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Abstract— **We consider different measures of nonlinearity that quantify the modeling error of optimal linear models for nonlinear systems. We show that linear dynamic approximations have no advantage over linear static approximations for static nonlinear functions in any of the considered measures. Simplified formulae for scalar nonlinearities are derived using the notion of a sector nonlinearity. We show that the steady state behaviour of a nonlinear system gives rise to a lower bound on the nonlinearity measures. Furthermore, some results for composite nonlinear systems are given.**

Index Terms— **Best linear approximation, Nonlinear system, Nonlinearity measure, Sector nonlinearity.**

I. Introduction

Linear techniques for systems analysis and control are well developed. For many control-related engineering problems, methods are available that are as well theoretically sound as practically implementable. Due to the diverse qualitative behaviour of nonlinear systems, tools for nonlinear systems analysis and control will probably never reach the same level of generality. To cope with nonlinear control problems, there are two alternative approaches. For highly nonlinear systems, special methods have to be developed that rely upon certain physical properties of the application or upon mathematical properties of a certain system class, like energy-shaping methods for mechanical control systems or feedback linearization. For mildly nonlinear systems, one can attempt to use a linear model and linear controller design methods, hoping that the nonlinear distortion is too small to destabilize the closed-loop system or to deteriorate the closed-loop performance. In order to provide a rigorous justification to this last approach, quantitative assessment of the nonlinearity is necessary.

Nonlinearity measures appeared for the first time in [1], where the induced gain of the difference between a nonlinear system and its best linear model is considered. Approaches to approximately compute such a measure are given in [2], [3], and a simplified procedure to calculate a rigorous lower bound is given in [3]. The idea of linear modeling for nonlinear systems is further developed for the discrete-time case in [4]. System gains for nonlinear systems are defined and an upper bound on the modeling error $\left\|\tilde{G} - G\right\|_{i,\infty}$ for discrete-time piecewise linear systems is given. In [5], the best linear models for discrete-time bi-gain systems is given with respect to the l_∞-norm and the existence of a best linear model for nonlinear finite impulse response filters is proven. The relative induced error $\left\|\tilde{G} - G\right\|_{i,\infty} / \|G\|_{i,\infty}$ as a measure of nonlinearity is mentioned. In [6], a procedure is proposed to approximately compute the L_2-gain of the error system defined as the difference between a nonlinear system and its Jacobi-linearization (both in continuous time). Other ways to measure the distance between two systems are given in [7].

There are also many approaches to nonlinearity assessment, that use other concepts than error system gains. Still close to the idea of induced norms is the relative error-like measure given in [3], [8]. In [9] the curvature of the steady state map is introduced as a measure of nonlinearity and in [10] an extension of this approach to dynamic systems is discussed. A different approach is presented in Ref. [11], where controllability and observability Gramians are used to quantify the degrees of input-to-state and state-to-output nonlinearity.

In this paper, we will consider both the error system gain from [1] and a version of the relative error-like measure from [3]. To this end, these quantities are defined in Section II together with some mathematical basics. In Section III, we give all main results. First, the error system gain for static nonlinearities is considered. An especially simple formula is given for scalar functions. Next, we state similar results for the relative error-like nonlinearity measure. The results are then extended to dynamical systems possessing a unique steady state output for a given steady state input. Finally, some results on composite nonlinear systems are derived. The paper ends with conclusions in Section IV.

II. Preliminaries

In this section we introduce the mathematical setup for the following studies. By $|\cdot|$ we denote the Euclidean vector norm in $\mathbb{R}^n$ and by $\mathbb{R}^+$ we denote the set of non-negative real numbers. The *Lebesgue-spaces* L_p^n are the sets of measurable functions $u : \mathbb{R}^+ \to \mathbb{R}^n$ for which $\|u\|_p = \left(\int_0^\infty |u|^p \,\mathrm{d}t\right)^{1/p} < \infty$ for $0 \le p < \infty$ and $\|u\|_\infty = \operatorname{ess\,sup}_{t\ge 0} |u(t)| < \infty$ for $p = \infty$ respectively. For convenience we will most of the time drop the subscript in $\|u\|_p$, well understood that for all calculations, the number p has a fixed value. The *truncation operator* u_T is defined as

$$u_T(t) = \begin{cases} u(t) & \text{for } t \le T \\ 0 & \text{for } t > T \end{cases}$$

and the *extended Lebesgue-spaces* L_{pe}^n are defined as the sets of all measurable functions u for which $u_T \in L_p^n$ for all $T > 0$. A mapping $N : L_{pe}^m \to L_{pe}^n$ is said to be *causal* if $(Nu)_T = (N(u_T))_T$, it is said to be (L_p-)*stable* if for every $u \in L_p^m$ the output satisfies $y = Nu \in L_p^n$ and it is said to be L_p-*stable with*

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finite gain (or short: to have finite gain) if there exists a finite constant γ such that $\|Nu\| \leq \gamma \|u\|$ for all u, or, equivalently, if the induced operator norm $\|N\|_i = \sup_{u \in L_p^m \setminus \{0\}} \frac{\|Nu\|}{\|u\|}$ is finite.

When modeling nonlinear systems N by linear models G, one promising approach is to look at the induced norm of the error system $\|N - G\|_i$. A linear model is called a best linear approximation if it minimizes this error system gain [5]. For a signal set $\mathcal{U} \subseteq L_p^m$ let $\mathcal{U}_e = \{u | u_T \in \mathcal{U} \text{ for all } T > 0\}$. For any causal, L_p-stable mapping $N : \mathcal{U}_e \to L_{pe}^n$ we define the gain of the associated error system by[1]

$$\gamma_N^{\mathcal{U}} \triangleq \inf_{G \in \mathcal{G}} \sup_{u \in \mathcal{U}} \frac{\|Nu - Gu\|}{\|u\|} \tag{1}$$

and call this expression the *error gain nonlinearity measure of N on $\mathcal{U}$*. This quantity has been called *nonlinearity measure* in [1], [3], but we want to make clear that its origin is the error system gain. Here, $\mathcal{G}$ is the set of causal linear time-invariant (LTI) delay-free transfer operators $(Gu)(t) = G_{hf}u(t) + \int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau$ with the high frequency gain $G_{hf} \in \mathbb{R}^{n\times m}$ and a measurable kernel $g : \mathbb{R}^+ \to \mathbb{R}^{n\times m}$. Note that all results of the present paper apply also to the case when we restrict $\mathcal{G}$ to a suitable subset of the above definition, like the set of operators which correspond to proper real-rational transfer matrices. For notational convenience we henceforth understand that the supremum is taken for $u \neq 0$ without explicitly mentioning, and we define $\gamma_N^{\mathcal{U}} = \infty$ if $N0 \neq 0$. The expression $\gamma_N^{\mathcal{U}}$ can also be finite when calculated for unstable N (consider the case of an unstable linear system). In this paper, however, we will restrict ourselves to (stable) systems with finite gain. Observe that (along the lines of [12]) for causal, finite gain L_p-stable $P : \mathcal{U}_e \to L_{pe}^n$

$$\sup_{u \in \mathcal{U}_e, T>0} \frac{\|(Pu)_T\|}{\|u_T\|} = \sup_{u \in \mathcal{U}} \frac{\|Pu\|}{\|u\|} \tag{2}$$

and thus the following expressions are equivalent whenever they are finite

$$\begin{aligned} \inf_{G \in \mathcal{G}} \sup_{u \in \mathcal{U}_e, T>0} \frac{\|(Nu - Gu)_T\|}{\|u_T\|} &= \inf_{G \in \mathcal{G}} \sup_{u \in \mathcal{U}_e} \limsup_{T \to \infty} \frac{\|(Nu - Gu)_T\|}{\|u_T\|} \\ &= \inf_{G \in \mathcal{G}} \sup_{u \in \mathcal{U}} \frac{\|Nu - Gu\|}{\|u\|}. \end{aligned}$$

Thus we can consider persistent signals even for $p < \infty$ by using the extended spaces. Note that the extended signal spaces are no normed linear spaces. To operate with normed spaces for persistent signals, bounded-power like signal spaces can be used [13], [4], but we will not need them in the ensuing analysis. The error gain nonlinearity measure is an important and useful quantity, because it defines the modeling uncertainty associated with the best linear model for a nonlinear system.

But there are two reasons why it is not the only quantity we are interested in. Firstly, the error gain nonlinearity measure is not bounded by definition. For different systems, different gains indicate a "high degree of nonlinearity". For example, an additional scalar gain in the I/O-behaviour of a system results in a different measure, though the type and qualitative behaviour of the nonlinear system do not change. Secondly, the error gain nonlinearity measure can only be computed for stable systems in general while we might want to quantify the degree of nonlinear distortion also for unstable systems. Thus, for the analysis of nonlinear systems we want to introduce a second quantity.

Let therefore $N : \mathcal{U}_e \to L_{pe}^n$ be a causal, not necessarily stable mapping satisfying $\|(Nu)_T\| < \infty$ for all $T > 0$ and $u \in \mathcal{U}_e$ ("stability in finite time" or "no escape in finite time w.r.t. the L_p-norm"). We define the *relative nonlinearity measure of N on $\mathcal{U}_e$* by

$$\varphi_N^{\mathcal{U}_e} \triangleq \inf_{G \in \mathcal{G}} \sup_{u \in \mathcal{U}_e} \limsup_{T \to \infty} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|} \tag{3}$$

where the definitions of $\mathcal{G}$ and $\mathcal{U}_e$ are as above. We will understand the expression (3) in the sense of

$$\begin{aligned} \varphi_N^{\mathcal{U}_e} \quad = \quad & \inf\Big\{\varphi \in \mathbb{R} | \exists G \in \mathcal{G} : \\ & \limsup_{T \to \infty} \|(Nu - Gu)_T\| - \varphi \|(Nu)_T\| \leq 0 \; \forall u \in \mathcal{U}_e\Big\}. \end{aligned}$$

We will discuss some consequences of this formulation in Section III-B. If we restrict the set of considered inputs to $\mathcal{U}$ we obtain *for stable N* the equivalent definition

$$\varphi_N^{\mathcal{U}} = \theta_N^{\mathcal{U}} \triangleq \inf_{G \in \mathcal{G}} \sup_{u \in \mathcal{U}} \frac{\|Nu - Gu\|}{\|Nu\|} \tag{4}$$

which is well-known from [3], [8], [14]. Note that no analogous relationship like (2) exists and in general

$$\tilde{\varphi}_N^{\mathcal{U}_e} \triangleq \inf_{G \in \mathcal{G}} \sup_{u \in \mathcal{U}_e, T>0} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|} \neq \varphi_N^{\mathcal{U}} \tag{5}$$

but we trivially have the inequality $\tilde{\varphi}_N^{\mathcal{U}} \geq \varphi_N^{\mathcal{U}}$. By definition, all nonlinearity measures are bounded below by zero. On the other hand, as the zero operator $Z : u \mapsto 0$ is in $\mathcal{G}$, we have $\varphi \leq 1$. The value of the relative nonlinearity measures can be interpreted as the maximal *relative* deviation of the nonlinear system output from the output of the best linear approximation. Thus we can compare the degree of nonlinearity of different systems.

III. Main Results

A. Error gain nonlinearity measures of static functions

First of all, we want to consider the error gain nonlinearity measure of static nonlinear mappings $f : \mathcal{V} \subseteq \mathbb{R}^m \to \mathbb{R}^n$. The following theorem about the gain of static nonlinear functions will turn out to be useful later. This fact is probably well known to experts but the authors do not know a reference and so we state it for the sake of completeness.

Theorem 1: (Gain of static nonlinear operators) Let the operator $N_f : u \in L_{pe}^m \mapsto y \in L_{pe}^n$ be given by

$$y(t) = (N_f u)(t) \triangleq f(u(t)) \; \forall t \tag{6}$$

where $f : \mathcal{V} \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is a function satisfying $|f(v)| < \infty$ (not necessarily uniformly) for all $v \in \mathcal{V}$. We require

[1]The symbol $\triangleq$ denotes definition of the left-hand term by the right-hand term.

that $\mathcal{V}$ contains at least an ε-ball around the origin. Let $\mathcal{U}_e(\mathcal{V}) \subseteq L^m_{pe}$ be the set of functions (time signals) defined on $\mathbb{R}^+$ having values in $\mathcal{V}$. Then the following equivalence holds

$$\sup_{u\in\mathcal{U}_e(\mathcal{V})} \limsup_{T\to\infty} \frac{\left\|(N_f u)_T\right\|}{\|u_T\|} = \sup_{v\in\mathcal{V}} \frac{|f(v)|}{|v|}. \tag{7}$$

Note that this equality holds regardless of the value of $p \in [1,\infty]$, and so all gains for N_f are equal.

Proof: Let $\gamma \triangleq \sup_{v\in\mathcal{V}} |f(v)| \,/\, |v| \geq 0$ and note that the inequality $|f(v)| \leq \gamma\,|v|$ holds for any $v \in \mathcal{V}$. For $p = \infty$

$$\frac{\left\|(N_f u)_T\right\|_\infty}{\|u_T\|_\infty} = \frac{\operatorname{ess\,sup}_{0\leq t\leq T} |f(u(t))|}{\operatorname{ess\,sup}_{0\leq t\leq T} |u(t)|} \leq \gamma$$

and similarly for $p < \infty$

$$\frac{\left\|(N_f u)_T\right\|}{\|u_T\|} = \frac{\left(\int_0^T |f(u(t))|^p \,\mathrm{d}t\right)^{\frac{1}{p}}}{\left(\int_0^T |u(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}} \leq \gamma$$

for all $u \in \mathcal{U}_e(\mathcal{V})$ and thus

$$\sup_{u\in\mathcal{U}_e(\mathcal{V})} \limsup_{T\to\infty} \frac{\left\|(N_f u)_T\right\|}{\|u_T\|} \leq \gamma = \sup_{v\in\mathcal{V}} \frac{|f(v)|}{|v|}. \tag{8}$$

Define the set $\mathcal{U}_1 \subset \mathcal{U}_e(\mathcal{V})$ that contains all the functions

$$u(t) = \begin{cases} v & 0 \leq t \leq 1 \\ 0 & 1 < t \end{cases}, v \in \mathcal{V}.$$

The relations

$$\sup_{u\in\mathcal{U}_1} \limsup_{T\to\infty} \frac{\left\|(N_f u)_T\right\|}{\|u_T\|} = \sup_{v\in\mathcal{V}} \frac{\left(\int_0^1 |f(v)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}}{\left(\int_0^1 |v|^p \,\mathrm{d}t\right)^{\frac{1}{p}}} = \gamma$$

for $p < \infty$ and

$$\sup_{u\in\mathcal{U}_1} \limsup_{T\to\infty} \frac{\left\|(N_f u)_T\right\|_\infty}{\|u_T\|_\infty} = \gamma$$

show that the equality in (8) must hold and the proof is complete. ■

We now turn to the first main result, that states that a dynamic linear approximation is never better than a static linear approximation in terms of the error gain nonlinearity measure, and the minimax problem over function spaces is reduced to a minimax problem over real numbers.

Theorem 2: (Equivalence of static formulation for error gain nonlinearity measure) Let the operator $N_f : u \in L^m_{pe} \mapsto y \in L^n_{pe}$ be given as in Theorem 1. Then the following equivalence holds

$$\inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}_e(\mathcal{V})} \limsup_{T\to\infty} \frac{\left\|(N_f u - Gu)_T\right\|}{\|u_T\|} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v) - Kv|}{|v|}. \tag{9}$$

Again, this equality holds regardless of the value of $p \in [1,\infty]$ and the value of the nonlinearity measure of N_f does not depend on p.

Proof: We have to prove that there is no difference in considering all dynamic LTI operators in $\mathcal{G}$ or just the subset of static operators $\mathcal{G}_0$. The principle used in the proof is as follows: By taking signals that remain constant for an arbitrarily large time interval, the dynamic response of the linear system will die out and the steady state response will dominate the error gain. We thus can restrict ourselves to only consider the steady state response in the first place, being equivalent to a static linear system described by a gain matrix. The rest follows by Theorem 1.

Clearly, the infimum on the left side of (9) will not be approached by unstable G as we otherwise can always find a bounded $u \in \mathcal{U}_e(\mathcal{V})$ that results in an unbounded $\left\|N_f u - Gu\right\|$. We may thus restrict ourselves to stable linear systems with $\int_0^t |g(\tau)|\,\mathrm{d}\tau < \infty$. We split up the response of the linear system by $Gu = G_0 u + G_d u$ into a static part $(G_0 u)(t) = G_{hf} u(t) + \int_0^\infty g(\tau)u(t)\mathrm{d}\tau = Ku(t)$ for some $K \in \mathbb{R}^{n\times m}$ and into a dynamic part $(G_d u)(t) = \int_0^\infty g(\tau)\,(u(t-\tau) - u(t))\,\mathrm{d}\tau$. Now, whenever $u(t) = u_0 = \text{const.}$ for $t \geq 0$ we have that $u(t-\tau) - u(t) = 0$ for $\tau < t$ and $u(t-\tau) = 0$ for $\tau > t$ and thus by stability of the linear system $(G_d u)(t) = -\left(\int_t^\infty g(\tau)\mathrm{d}\tau\right) u_0 \to 0$ for $t \to \infty$. Consider the set of functions $\mathcal{U}^* = \{u^v | u^v(t) \equiv v,\ v \in \mathcal{V}\} \subset \mathcal{U}_e(\mathcal{V})$ and note that $\|(u^v)_T\| = |v|\, T^{\frac{1}{p}}$. As the response of the dynamic part G_d tends to zero, for any $\varepsilon > 0$ there exists a $T_1 \geq 0$ such that $|(G_d u)(t)| < \varepsilon$ for all $t > T_1$ and

$$\begin{aligned} \left(\lim_{T\to\infty} \frac{\|(G_d u^v)_T\|}{\left\|u^v_T\right\|}\right)^p &\leq \lim_{T\to\infty} \frac{\left\|(G_d u^v)_{T_1}\right\|^p + \varepsilon^p (T - T_1)}{|v|^p\, T} \\ &= \left(\frac{\varepsilon}{|v|}\right)^p . \end{aligned}$$

As ε was chosen arbitrarily it must be true that $\lim_{T\to\infty} \frac{\|(G_d u^v)_T\|}{\|(u^v)_T\|} = 0$. We have the following sequence of (in)equalities

$$\begin{aligned} &\inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}_e(\mathcal{V})} \limsup_{T\to\infty} \frac{\left\|(N_f u - Gu)_T\right\|}{\|u_T\|} \\ \geq\ &\inf_{G\in\mathcal{G}} \sup_{u^v\in\mathcal{U}^*} \lim_{T\to\infty} \frac{\left\|(N_f u^v - Gu^v)_T\right\|}{\|(u^v)_T\|} \\ =\ &\inf_{G_0\in\mathcal{G}_0} \sup_{u^v\in\mathcal{U}^*} \lim_{T\to\infty} \frac{\left\|(N_f u^v - G_0 u^v)_T\right\|}{\|(u^v)_T\|} \\ =\ &\inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v) - Kv|}{|v|} \end{aligned}$$

where the first inequality follows from $\mathcal{U}^* \subset \mathcal{U}_e(\mathcal{V})$, the second equality is justified by $\left\|(N_f u - G_0 u)_T\right\| - \|(G_d u)_T\| \leq \left\|(N_f u - Gu)_T\right\| \leq \left\|(N_f u - G_0 u)_T\right\| + \|(G_d u)_T\|$ and $\lim_{T\to\infty} \|(G_d u^v)_T\| \,/\, \|(u^v)_T\| = 0$ and the last equality can be obtained by straightforward calculation. On the other hand, as $\mathcal{G}_0 \subset \mathcal{G}$ and by Theorem 1

$$\begin{aligned} &\inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}_e(\mathcal{V})} \limsup_{T\to\infty} \frac{\left\|(N_f u - Gu)_T\right\|}{\|u_T\|} \\ \leq\ &\inf_{G_0\in\mathcal{G}_0} \sup_{u\in\mathcal{U}_e(\mathcal{V})} \lim_{T\to\infty} \frac{\left\|(N_f u - G_0 u)_T\right\|}{\|u_T\|} \\ =\ &\inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v) - Kv|}{|v|} \end{aligned}$$

and the result follows for $p < \infty$. As $\lim_{T\to\infty}(Gu^v)(t) = Kv$ and $(N_f u^v)(t) = f(v)$ for all t,

$$\begin{aligned}\lim_{T\to\infty} \frac{\left\|(N_f u^v - Gu^v)_T\right\|_\infty}{\|(u^v)_T\|_\infty} &\geq \frac{|f(v) - Kv|}{|v|} \\ &= \lim_{T\to\infty} \frac{\left\|(N_f u^v - G_0 u^v)_T\right\|_\infty}{\|(u^v)_T\|_\infty}\end{aligned}$$

which proves in a similar fashion that a dynamic approximation can do no better for the case $p = \infty$, and the proof is complete. ■

In the case of a scalar function $f : \mathbb{R} \to \mathbb{R}$ the nonlinearity measure can be determined analytically and graphically. This result is established by the following theorem.

Theorem 3: (Error gain nonlinearity measure for scalar functions) For any given function $f : \mathcal{V} \subseteq \mathbb{R} \to \mathbb{R}$ define

$$\begin{aligned} k^+ &\triangleq \sup_{v\in\mathcal{V}\setminus\{0\}} \frac{f(v)}{v} \\ k^- &\triangleq \inf_{v\in\mathcal{V}\setminus\{0\}} \frac{f(v)}{v}\end{aligned}$$

and define k^* by

$$k^* \triangleq \begin{cases} \frac{1}{2}(k^+ + k^-) & \text{if } |k^+ k^-| < \infty \\ 0 & \text{else} \end{cases}.$$

Then,

$$\inf_{k\in\mathbb{R}} \sup_{v\in\mathcal{V}\setminus\{0\}} \frac{|f(v) - kv|}{|v|} = \begin{cases} \frac{1}{2}(k^+ - k^-) & \text{if } |k^+|, |k^-| < \infty \\ \infty & \text{else} \end{cases}$$

and if the infimum is finite it is uniquely achieved for $k = k^*$.

Note that we do not require continuity or differentiability of f, but for consistency we define $k^+ = \infty$ or $k^- = -\infty$ respectively if $f(0) \neq 0$. The error gain nonlinearity measure is defined to be infinite in this case.

Proof: In the scalar case, the errror gain nonlinearity measure of a static function can be reformulated as

$$\inf_{k\in\mathbb{R}} \sup_{v\in\mathcal{V}\setminus\{0\}} \frac{|f(v) - kv|}{|v|} = \inf_{k\in\mathbb{R}} \sup_{v\in\mathcal{V}\setminus\{0\}} \left|\frac{f(v)}{v} - k\right|.$$

The cases in which $|k^+| = \infty$ or $|k^-| = \infty$ are obvious. In all other cases we have for $k = k^*$

$$\sup_{v\in\mathcal{V}\setminus\{0\}} \left|\frac{f(v)}{v} - k^*\right| = \left|k^+ - k^*\right| = \left|k^- - k^*\right| = \frac{1}{2}(k^+ - k^-).$$

Note that we always have $k^- \leq k^* \leq k^+$. Now, for any k we either have $k^- \leq k^* < k$ and thus

$$\sup_{v\in\mathcal{V}\setminus\{0\}} \left|\frac{f(v)}{v} - k\right| \geq k - k^- > k^* - k^- = \frac{1}{2}(k^+ - k^-)$$

or $k < k^* \leq k^+$ and thus

$$\sup_{v\in\mathcal{V}\setminus\{0\}} \left|\frac{f(v)}{v} - k\right| \geq k^+ - k > k^+ - k^* = \frac{1}{2}(k^+ - k^-)$$

We have shown that no other k than k^* can achieve the same or a smaller value and the proof is complete. ■

What we rediscovered here using a novel framework are of course the famous sector conditions used in the theory of abolute stability, e.g. to derive the circle criterion [15]. As

$$k^- v \leq f(v) \leq k^+ v \; \forall v \in \mathcal{V}$$

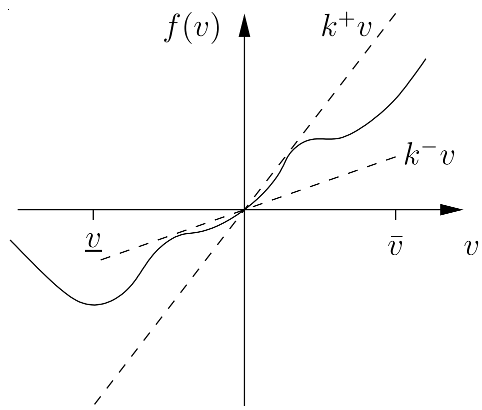

Fig. 1. A nonlinear function and its sector bounds for $\mathcal{V} = [\underline{v}, \bar{v}]$.

the sector in which f lies is given by the slopes k^+ and k^- and by definition, those bounds are the tightest bounds possible. Note that f is to lie in the sector $[k^-, k^+]$ only for $v \in \mathcal{V}$ and may lie outside for $v \neq \mathcal{V}$ (Fig. 1 shows an illustration of this concept with the set $\mathcal{V}$ being an interval on the v-axis).

B. Relative nonlinearity measures of static functions

By considering the same arguments as in Theorem 2, one can obtain a similar result for the relative nonlinearity measures.

Theorem 4: (Equivalence of static formulation for relative nonlinearity measures) Assume the same conditions as in Theorem 2, but relax the assumption that N_f be L_p-stable to the requirement that $\|(Nu)_T\| < \infty$ for all $T > 0$. *Then,*

$$\inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}_e(\mathcal{V})} \limsup_{T\to\infty} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v) - Kv|}{|f(v)|}. \tag{10}$$

According to the definition of the relative nonlinearity measure, we will understand the expression on the right hand side of (10) in the sense of

$$\inf\{\varphi \in \mathbb{R} | \exists K \in \mathbb{R}^{n\times m} : |f(v) - Kv| \leq \varphi |f(v)| \, \forall v \in \mathcal{V}\}.$$

As a consequence, either the set $\mathcal{V}_0$ of all points v for which $f(v) = 0$ (and all limit points $\bar{v}$ of sequences $\{v_i\}$ for which $f(v_i) \to 0$) is equal to $\ker K \cap \mathcal{V}$ or we have $\varphi_{N_f}^{\mathcal{U}_e(\mathcal{V})} = 1$. For rectangular systems with more inputs than outputs ($m > n$), the relation $f(v) = 0 \leftrightarrow Kv = 0$ is very unlikely to hold for any K and thus the realtive nonlinearity measure, being almost always equal to one, is of restricted use for such systems. In the cases where $m \leq n$, the set $\mathcal{V}_0$ will usually have only few elements, and the nonlinearity measure is likely to deliver useful information.

Proof: The proof of the result for $\varphi_N^{\mathcal{U}(\mathcal{V})}$ goes through as in Theorem 2 with obvious modifications and is omitted. ■

In the case of scalar functions, we can again obtain an explicit solution.

Theorem 5: (Relative nonlinearity measure of scalar functions) Let f, k^+ and k^- be as above and define k^* by

$$\begin{cases} \frac{1}{k^*} \triangleq \frac{1}{2}\left(\frac{1}{k^+} + \frac{1}{k^-}\right) & \text{if } 0 < k^+k^-, |k^+|, |k^-| < \infty \\ k^* \triangleq 0 & \text{else} \end{cases}.$$

Then

$$\varphi_f^{\mathcal{V}} \triangleq \inf_{k\in\mathbb{R}} \sup_{v\in\mathcal{V}} \frac{|f(v) - kv|}{|f(v)|} = \begin{cases} \left|\frac{k^+-k^-}{k^++k^-}\right| & \text{if } 0 < k^+k^-, |k^+|, |k^-| < \infty \\ 0 & \text{if } k^+ = k^- = 0 \\ 1 & \text{else} \end{cases}.$$

and the infimum is uniquely achieved for $k = k^*$ if $k^+$and k^- are finite .

Proof: In the scalar case, the relative nonlinearity measure of a static function can be reformulated as

$$\inf_{k\in\mathbb{R}} \sup_{v\in\mathcal{V}} \frac{|f(v) - kv|}{|f(v)|} = \inf_{k\in\mathbb{R}} \sup_{v\in\mathcal{V}} \left|1 - \frac{k}{f(v)/v}\right|.$$

We first consider the case $k^+k^- \leq 0$. The case $k^+ = k^- = 0$ is obvious. If either $k^+ = 0$ or $k^- = 0$ then $k = 0$ is the only admissible value for k, leading to $\varphi_f^V = 1$ (see remark after Theorem 4), so it suffices to consider the case $k^- < 0 < k^+$. If $k = 0$ we have again $\varphi_f^V = 1$. If $k > 0$ then $\varphi_f^V \geq |1 - k/k^-| > 1$ and if $k < 0$ then $\varphi_f^V \geq |1 - k/k^+| > 1$. Hence $\varphi_f^V = 1$ and this value is uniquely achieved for $k = 0$. Next, consider the case when either $|k^+| = \infty$ or $|k^-| = \infty$. Clearly, for any $k \neq 0$: $\sup_{v\in\mathcal{V}} \left|1 - \frac{k}{f(v)/v}\right| \geq 1$ but $\sup_{v\in\mathcal{V}} |f(v) - kv| \,/\, |f(v)| = 1$ for $k = 0$ (but this time there may be other k achieving the same value).

We have now to verify the case $0 < k^+k^-$. For $k = k^*$ we have

$$\sup_{v\in\mathcal{V}} \left|1 - \frac{k^*}{f(v)/v}\right| = \left|1 - \frac{k^*}{k^+}\right| = \left|1 - \frac{k^*}{k^-}\right| = \left|\frac{k^+ - k^-}{k^+ + k^-}\right|.$$

Note that we always have $k^- \leq k^* \leq k^+$. We can now distinguish the two cases

$$k^- \leq k^* < k: \quad \sup_{v\in\mathcal{V}} \left|1 - \frac{k}{f(v)/v}\right| \geq \left|1 - \frac{k}{k^-}\right| = \left|\frac{1}{k^-}\right|(k - k^-) > \left|\frac{1}{k^-}\right|(k^* - k^-) = \left|\frac{k^+ - k^-}{k^+ + k^-}\right|$$

$$k < k^* \leq k^+: \quad \sup_{v\in\mathcal{V}} \left|1 - \frac{k}{f(v)/v}\right| \geq \left|1 - \frac{k}{k^+}\right| = \left|\frac{1}{k^+}\right|(k^+ - k) > \left|\frac{1}{k^+}\right|(k^+ - k^*) = \left|\frac{k^+ - k^-}{k^+ + k^-}\right|.$$

We have shown that no other k than k^* can achieve the same or a smaller value and the proof is complete. ■

C. Steady-state behaviour of nonlinear dynamical systems

Next, we turn our attention to stable nonlinear systems featuring a unique steady state response to a steady state input. The intuitive idea is that the nonlinearity measures are bounded from below by the corresponding quantities of the steady state locus. This idea is formalized as follows.

Theorem 6: Consider a nonlinear system defined by the causal, L_p-stable mapping $N : u \in L_{pe}^m \mapsto y \in L_{pe}^n$. Consider the function $f : \mathcal{V} \subseteq \mathbb{R}^m \to \mathbb{R}^n$ satisfying $|f(v)| < \infty$ for all $v \in \mathcal{V}$ and assume that N has a unique steady state locus given by f in the sense that $y(t) = (Nu)(t) \to f(v)$ whenever $u(t) \to v$ for $t \to \infty$. Then

$$\gamma_N^{\mathcal{U}_e(\mathcal{V})} = \inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}_e(\mathcal{V})} \limsup_{T\to\infty} \frac{\|(Nu - Gu)_T\|}{\|u_T\|} \tag{11}$$

$$\geq \gamma_{N_f}^{\mathcal{U}_e(\mathcal{V})} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v) - Kv|}{|v|}. \tag{12}$$

Proof: We define the set $\mathcal{U}^*$ of functions $u^* \in \mathcal{U}_e(\mathcal{V})$ that satisfy $u^*(t) \to v$, $v \in \mathcal{V}\backslash\{0\}$ for $t \to \infty$ and $|u^*(t)| \leq v$ for all $t \geq 0$. Consider the following (in)equalities

$$\begin{aligned} &\inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}_e(\mathcal{V})} \limsup_{T\to\infty} \frac{\|(Nu - Gu)_T\|}{\|u_T\|} \\ \geq\ &\inf_{G\in\mathcal{G}} \sup_{u^*\in\mathcal{U}^*} \limsup_{T\to\infty} \frac{\|(Nu^* - Gu^*)_T\|}{\|u_T^*\|} \\ =\ &\inf_{G\in\mathcal{G}} \sup_{u^*\in\mathcal{U}^*} \limsup_{T\to\infty} \frac{\|(N_f u^* - Gu^*)_T\|}{\|u_T^*\|} \\ =\ &\inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v) - Kv|}{|v|} \end{aligned}$$

where N_f denotes the static nonlinear transfer operator associated with f. The first inequality is trivial and the last equality follows from the proof of Theorem 2. Thus, if we can prove the second equality, the result follows. We first consider the case $p < \infty$. For any $u^* \in \mathcal{U}^*$ we have $(Nu^*)(t) \to (N_f u^*)(t)$ as $t \to \infty$. Thus, there is a constant $c > 0$ such that for any $\varepsilon > 0$ there exist times $T_2 \geq T_1 \geq 0$ such that $\left|(Nu^*)(t) - (N_f u^*)(t)\right| < 1$ and $|u^*(t)| > c$ for all $t > T_1$ and $\left|(Nu^*)(t) - (N_f u^*)(t)\right| < \varepsilon$ for all $t > T_2$. Hence

$$\begin{aligned} &\left(\limsup_{T\to\infty} \frac{\|(N_f u^* - Nu^*)_T\|}{\|u_T^*\|}\right)^p \\ =\ &\limsup_{T\to\infty} \frac{\|(N_f u^* - Nu^*)_T\|^p}{\|u_T^*\|^p} \\ \leq\ &\limsup_{T\to\infty} \frac{\|(N_f u^* - Nu^*)_{T_1}\|^p + (T_2 - T_1) + \varepsilon^p(T - T_2)}{\|u_{T_1}^*\|^p + c^p(T - T_1)} \\ =\ &\left(\frac{\varepsilon}{c}\right)^p \end{aligned}$$

for a fixed $c > 0$ and an arbitrary $\varepsilon > 0$, so the limit must vanish. As $\|(N_f u^* - Nu^*)_T\| \geq \left|\|(N_f u^* - Gu^*)_T\| - \|(Nu^* - Gu^*)_T\|\right|$ we get

$$\limsup_{T\to\infty} \frac{\|(Nu^* - Gu^*)_T\|}{\|u_T^*\|} = \limsup_{T\to\infty} \frac{\|(N_f u^* - Gu^*)_T\|}{\|u_T^*\|}$$

for all $u^* \in \mathcal{U}^*$ and the result follows for $p < \infty$. If $p = \infty$ we have $\|u_T^*\|_\infty \leq |v|$ and $\lim_{T\to\infty} \|(Nu^* - Gu^*)_T\|_\infty \geq |f(v) - Kv|$ and we thus directly get

$$\limsup_{T\to\infty} \frac{\|(Nu^* - Gu^*)_T\|_\infty}{\|u_T^*\|_\infty} \geq \frac{|f(v) - Kv|}{|v|}$$

and the proof is complete. ■

Proceeding the same way, a similar result can again be obtained for the relative nonlinearity measure and we give it without proof.

Theorem 7: Consider a nonlinear system defined by the causal mapping $N : u \in L_{pe}^m \mapsto y \in L_{pe}^n$ *satisfying* $\|(Nu)_T\| < \infty$ *for all* $T > 0$. *Consider the function* $f : \mathcal{V} \subseteq \mathbb{R}^m \to \mathbb{R}^n$ *satisfying* $|f(v)| < \infty$ *for all* $v \in \mathcal{V}$ *and assume that* N *has a unique steady state locus given by* f *in the sense that* $y(t) = (Nu)(t) \to f(v)$ *whenever* $u(t) \to v$ *for* $t \to \infty$. *Then*

$$\varphi_N^{\mathcal{U}_e(\mathcal{V})} = \inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}(\mathcal{V})} \lim_{T\to\infty} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|} \qquad (13)$$

$$\geq \quad \varphi_{N_f}^{\mathcal{U}_e(\mathcal{V})} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v) - Kv|}{|f(v)|}. \qquad (14)$$

D. Results for composite nonlinear systems

Finally, we have a look at systems that are composed of two subsystems. We first consider the case of a parallel connection of two L_p-stable mappings N_1 and N_2. We have

$$\begin{aligned}
\gamma_{N_1+N_2}^{\mathcal{U}} &= \inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}} \frac{\|(N_1+N_2)u - Gu\|}{\|u\|} \\
&= \inf_{G_1,G_2\in\mathcal{G}} \sup_{u\in\mathcal{U}} \frac{\|(N_1+N_2)u - (G_1+G_2)u\|}{\|u\|} \\
&\leq \inf_{G_1,G_2\in\mathcal{G}} \sup_{u\in\mathcal{U}} \frac{\|N_1u - G_1u\| + \|N_2u - G_2u\|}{\|u\|} \\
&\leq \inf_{G_1,G_2\in\mathcal{G}} \sup_{u,w\in\mathcal{U}} \left(\frac{\|N_1u - G_1u\|}{\|u\|} + \frac{\|N_2w - G_2w\|}{\|w\|} \right) \\
&= \gamma_{N_1}^{\mathcal{U}} + \gamma_{N_2}^{\mathcal{U}}
\end{aligned}$$

and thus for two causal, L_p-stable mappings $N_1, N_2 : \mathcal{U}_e \to L_{pe}$ the triangle-like inequality $\gamma_{N_1+N_2}^{\mathcal{U}} \leq \gamma_{N_1}^{\mathcal{U}} + \gamma_{N_2}^{\mathcal{U}}$ holds. If one of the systems is linear, say $N_2 \in \mathcal{G}$, we trivially get $\gamma_{N_1+N_2}^{\mathcal{U}} = \gamma_{N_1}^{\mathcal{U}}$.

Next, we consider the series connection of a nonlinear system N_1 and a linear system $N_2 \in \mathcal{G}$. We get

$$\begin{aligned}
\gamma_{N_2N_1}^{\mathcal{U}} &= \inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}} \frac{\|N_2(N_1u) - Gu\|}{\|u\|} \\
&\leq \inf_{\tilde{G}\in\mathcal{G}} \sup_{u\in\mathcal{U}} \frac{\left\|N_2(N_1u) - N_2(\tilde{G}u)\right\|}{\|u\|} \\
&\leq \|N_2\|_{i,p}\, \gamma_{N_1}^{\mathcal{U}}
\end{aligned}$$

where $\|N_2\|_{i,p}$ is the L_p-induced norm of N_2. If the nonlinear subsystem is static, the structure of the composed system is of Hammerstein-type and we get $\gamma_{N_2N_1}^{\mathcal{U}(\mathcal{V})} \leq \|N_2\|_{i,p}\, \gamma_f^{\mathcal{V}}$. For these relations, no equivalent properties exist for relative nonlinearity measures.

IV. Conclusions

When using linear controller design techniques for nonlinear systems, it is important to quantify the degree of nonlinearity of the process under consideration. We studied two such measures. The first measure, called *error gain nonlinearity measure*, corresponds to an error system gain where the error signal is the difference between the output of the nonlinear system and its appropriately defined best linear approximation. The second measure, termed *relative nonlinearity measure*, is somewhat similar, but rather corresponds to a relative deviation and allows the comparison of different systems and the analysis of unstable systems. For both measures we have given simpler formulae in the case of static nonlinearities. A particularly simple case was obtained for scalar nonlinearities, where both quantities are determined by the slopes of the bounding straight lines. The most useful result is that for dynamical systems, the values of both quantities are bounded from below by the corresponding quantities of the function describing the steady state. Moreover, two results for interconnections of systems were obtained for the error gain nonlinearity measure.

The obtained results give a very easily derivable lower bound on the nonlinearity measures and will thus help to render those quantities more useful both in theoretical studies and analysis of real-world problems. For the future, further results on bounds for special system structures as well as guidelines on how to use the results of the nonlinearity assessment for controller design purposes are desirable.

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References

[1] C. A. Desoer and Y.-T. Wang, "Foundations of feedback theory for nonlinear dynamical systems," *IEEE Trans. on Circuits and System*, vol. CAS-27, no. 2, pp. 104–123, February 1980.

[2] D. Sourlas and V. Manousiouthakis, "On the computation of the nonlinearity measure," in *Proc. 37th IEEE Conf. Decision Contr.*, 1998, pp. 1434–39.

[3] F. Allgöwer, "Definition and computation of a nonlinearity measure," in *3rd IFAC Nonlinear Control Systems Design Symposium*, Lake Tahoe, CA, 1995, pp. 279–284.

[4] J. R. Partington and P. M. Mäkilä, "On system gains for linear and nonlinear systems," *Systems & Control Letters*, vol. 46, pp. 129–136, 2002.

[5] P. M. Mäkilä and J. R. Partington, "On linear models for nonlinear systems," *Automatica*, vol. 39, no. 1, pp. 1–13, 2003.

[6] D. Kihas and H. J. Marquez, "Computing the distance between a nonlinear model and its linear approximation: an l_2 approach," *Comp. and Chem. Eng.*, vol. 28, no. 12, pp. 2659–2666, 2004.

[7] T. T. Georgiou and M. C. Smith, "Robustness analysis of nonlinear feedback systems: An input-ouput approach." *IEEE Trans. Automat. Contr.*, vol. 42, no. 9, pp. 1200–1221, September 1997.

[8] A. Helbig, W. Marquardt, and F. Allgöwer, "Nonlinearity measures: definition, computation and applications," *J. Proc. Contr.*, vol. 10, pp. 113–123, 2000.

[9] M. Guay, P. J. McLellan, and D. W. Bacon, "Measurement of nonlinearity in chemical process control systems: The steady state map," *Can. J. Chem. Eng.*, vol. 73, pp. 868–882, 1995.

[10] ——, "Measure of closed-loop nonlinearity and interaction for nonlinear chemical processes," *AIChE J.*, vol. 43, no. 9, pp. 2261–2278, 1997.

[11] J. Hahn and T. F. Edgar, "A Gramian based approach to nonlinearity quantification and model classification," *Ind. Eng. Chem. Res.*, vol. 40, pp. 5724–31, 2001.

[12] M. Vidyasagar, *Nonlinear System Analysis (2nd ed.)*. Philadelphia, PA: SIAM, 2002.

[13] P. M. Mäkilä, J. R. Partington, and T. Norlander, "Bounded power signal spaces for robust control and modeling," *SIAM J. Control Optim.*, vol. 37, no. 1, pp. 92–117, 1998.

[14] F. Allgöwer, *Näherungsweise Ein-/Ausgangs-Linearisierung nichtlinearer Systeme*. Düsseldorf: Fortschr.-Ber. VDI Reihe 8 Nr. 582, VDI Verlag, 1996.

[15] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems – Part II," *IEEE Trans. Automat. Contr.*, vol. 11(3), pp. 465–476, 1966.