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Control of unreliable cooperative multi-site production systems

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Abstract— **In this paper we consider a two site, cooperative, failure prone production system. The failure/repair process of each site is an independent Markov process. Each site may comply with possible shortage of the other site by sending to it a given quantity of products, with a penalty, modeling for example a transportation cost. General properties of the optimal policy are derived through the Hamilton Jacobi Bellman (HJB) equations, according to which the backlog/inventory space of the system is partitioned in regions, each characterized by a constant production rate. A general numerical procedure, which could be also exploited to face several problems approached through a HJB formulation, is developed and applied to the problem considered in this paper to derive the optimal policy and the corresponding optimal cost. An analysis of the effect of system parameters on the optimal policy as well as a comparison with analytical results known for the single site problem are also included in the paper.**

I. INTRODUCTION

The problem considered in this paper is in the context of multi-site production factories, where the sites have similar production characteristics and each of them attracts a territorial portion of the demand. Fluctuations in the production (as for example failures) or in the demand may force one of the sites to receive items from other sites, with some penalty, which may be imputed, for example, to transportation costs. Here, we consider the optimal production control of two cooperative sites producing the same kind of goods, where each site is subject to failure according to an independent Markov process. There is cooperation in the sense that each site can comply with possible shortages of the other by supplying it with its own production, but a reduced profit is made by selling goods not produced in site.

A large literature deals with single unreliable production sites. Usually, the common approach is the use of Hamilton Jacobi Bellman (HJB) equations to prove the optimality of some candidate policies. Analytical results have been obtained in some particular cases, starting with [8], [3] and [1] where it has been proved that, under a Markov description of the failure repair process of the system, the policy minimizing a long term, average, expected cost penalizing both the inventory surplus and the backlog is the *hedging point policy*, according to which the system is operated at full capacity until the inventory level hits a non negative safety stock, called *hedging level*, which is then maintained until the next failure event occurs.

This approach, if the number of system states is larger than two and/or the number of part types is larger than one, becomes intractable, and several numerical approaches have been considered. Among others, in [9] the solution of the HJB equations has been approximated through a quadratic function, in [14] the optimal policy is found by restricting the class of policies considered, in [12] the optimal policy is investigated under some assumptions on the shape of optimal switching surfaces, in [16] properties of the optimal control policy are identified that can be used to help formulate heuristic policies. A similar approach is used here for the problem considered here: the HJB equations are used to gain insight on the shape of the optimal policy.

The problem considered in this paper in fact presents many similarities with the multi-part multi-state single site problem, since the backlog/inventory state is not unidimensional and the presence of different sites can be described by a multi-state Markov chain, equivalent to the chain of a multi-state single site system. So, also in this case, the problem of finding the general solution of the HJB equations is intractable.

Usually, different approaches have been used to face the problem of multi-site cooperating systems. Among many others, we cite [4], where autonomous agents and holarchy concepts are used to integrate the activities of a supply chain and [11], where autonomous agents and Petri Nets are used to control a distributed system.

In this paper, on the contrary, we pursue a HJB approach. As stated above, we use this method to identify properties of the optimal policy, which allows, for a bounded backlog/inventory system, to apply numerical methods to derive the optimal control law and the corresponding optimal cost. To this end, a two-phases algorithm is developed, which in the first phase computes the optimal steady state cost J^*, a quantity which is needed in order to evaluate (second phase) the optimal feedback control law.

The HJB equations have been applied to a multi-site problem in [15] and in [7]. However, in these cases, a master production site exploits the production facilities of a number of slave sites, which are always available. The state space for those problems is unidimensional, and this allows to successfully apply the HJB approach. The problem considered in this paper deals with independent sites where, at each moment of time, depending on the joint backlog/inventory level, one may become subcontractor for the other. So, even if the formulation is similar to those presented in [15] and [7], the solution complexity and approach resembles those

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presented in [9] and [16].

The contribution of this paper follows this baseline:

- First, general properties of the optimal policy are provided through the inspection of the HJB equations, yielding, for each operative state of the two sites, a partition of the state space defining regions characterized by a constant production control.
- We develop a numerical procedure to solve the HJB equations which allows to derive the optimal cost and the corresponding optimal control. This algorithm is general and could be exploited for optimization problems based on the HJB approach.
- The numerical procedure above, applied to the considered problem, allows to determine the optimal policy which is characterized by a positive hedging level when both sites are operative. An interesting and intuitive dependence of this hedging level and of the total cost on the transportation costs will be illustrated.

The paper is organized as follows: in Section II we introduce the notation and formulate the problem. In Section III the HJB equations are presented and the optimal policy is characterized in terms of switching regions. In Section IV we present the numerical methods adopted to evaluate the optimal production control and the corresponding optimal cost. Some examples are reported in Section V. Section VI concludes the paper.

II. PROBLEM FORMULATION

According to a standard notation, let $x_i(t)$, $i = 1, 2$, denote the buffer content of site i at time t, assumed unbounded, with $x_i(t) > 0$ representing an inventory surplus and $x_i(t) < 0$ a backlog of $-x_i(t)$. The up and down times of both sites are exponentially distributed random variables with average values $1/q_d$ and $1/q_u$, respectively. As each site can produce for its own customers or for the other site, the production rate for site i is given by $u_{ii}(t)+u_{ij}(t)$, $j \neq i$, where the first subscript denotes the producing site and the second denotes the destination site (e.g., site 1 produces at rate $u_{11}(t)$ for itself and at rate $u_{12}(t)$ for site 2) and the total production capacity for both sites is μ, i.e., $u_{ii}(t)+u_{ij}(t) \leq \mu,\ i = 1, 2,$ $j \neq i$. Adopting a fluid approximation model, for both sites it is assumed a constant demand rate d. The buffer level of site i, $i = 1, 2$, obeys the differential equation:

$$\dot{x_i} = u_{ii}(t) + u_{ji}(t) - d \tag{1}$$

Let $s = [i, j]^T,\ i, j \in \{0, 1\}$ be the *discrete* state[1] of the system indicating the up/down state of each site, with 0 (1) standing for machine/site being *down* (*up*). Denoting by $x := [x_1, x_2]^T$ the *continuous* state, $u(\cdot)$ the 2×2 matrix with entries $u_{ij}(t)$, $i, j = 1, 2$, and using d also for the column vector $[d, d]^T$ (the ambiguity being solved by the context) the dynamics of the cooperative system can be written as:

$$\dot{x} = u^T s - d \tag{2}$$

[1]This terminology, which refers to the state of a system as a combination of *discrete* – or *logical* – and *continuous* states is common in the literature of *hybrid systems*, see e.g. [2].

The scheduling problem considered in this paper is the determination of the optimal controls u which minimizes the following long term, average, expected cost:

$$J = \lim_{T \to \infty} \frac{1}{T} E\left[\int_0^T g(x(t), u(t))dt\right], \tag{3}$$

where $g(x(t), u(t)) = \sum_{i=1,2} c_p x_i^+ + c_m x_i^- + a(u_{12}(t) + u_{21}(t))$, $x_i^+ = \max\{0, x_i\}$, $x_i^- = \max\{0, -x_i\}$ and c_p, c_m, a are non-negative constants. For ease of notation, the shorthand $c(x) := \sum_{i=1,2} c_p x_i^+ + c_m x_i^-$ will be used in the following for the inventory/backlog costs of the cooperative system. The term $a(u_{12}(t) + u_{21}(t))$ in (3) models the reduced profit made by selling goods not produced in site, which may be due, for example, to transportation costs.

The problem will be studied under the following Assumption.

Assumption 1: The system has enough capacity to meet demand[2]: $\mu q_u - d(q_d + q_u) > 0$.

Notice, also, that heretofore a symmetric system was implicitly assumed, i.e., characterized by the same production capacities, demand rate, backlog and inventory costs. So, the problem will be formally stated as follow:

Problem 1: For the two-site cooperative, failure prone system described above, determine under Assumption 1 the production control $u^*(\cdot)$ minimizing (3).

III. HJB EQUATIONS AND CHARACTERIZATION OF THE OPTIMAL POLICY

Recall that for a single machine manufacturing system subject to a Markovian failure/repair process, i.e. each site of our problem if non interacting with the other, with dynamics $\dot{x}(t) = u(t) - d$, the optimal control is a hedging policy [3]:

$$u^*(x) = \begin{cases} \mu & x < z^* \\ d & x = z^* \\ 0 & x > z^* \end{cases} \tag{4}$$

The procedure that leads to this result is well established and based on suitable regularity conditions on the control (see for example [13]), which guarantee that the optimal feedback control of the system is stationary and characterized by the HJB equations reported in [3]. If J^* denotes the optimal cost for Problem 1 (i.e. the minimum value of (3)), for system dynamics (2), following the procedure presented in [5], yields the following HJB equations:

$$\begin{aligned} J^* = c(x) &+ (V_{01}(x) - 2V_{11}(x) + V_{10}(x))q_d \\ &- \Big(\frac{\partial V_{11}(x)}{\partial x_1} + \frac{\partial V_{11}(x)}{\partial x_2}\Big)d \\ &+ \min_{u \in \mathcal{U}_{11}} \Big(\frac{\partial V_{11}(x)}{\partial x_1}(u_{11} + u_{21}) \\ &+ \frac{\partial V_{11}(x)}{\partial x_2}(u_{22} + u_{12}) + a(u_{21} + u_{12})\Big) \end{aligned} \tag{5}$$

[2]Notice that the stability requirements for the cooperative system are the same as for a single site.

for both sites/machines operative,

$$J^* = c(x) - V_{01}(x)(q_u + q_d) + V_{11}(x)q_u + V_{00}(x)q_d - \Big(\frac{\partial V_{01}(x)}{\partial x_1} + \frac{\partial V_{01}(x)}{\partial x_2}\Big)d + \min_{u\in\mathcal{U}_{01}} \left(\Big(\frac{\partial V_{01}(x)}{\partial x_1} + a\Big)u_{21} + \frac{\partial V_{01}(x)}{\partial x_2}u_{22}\right) \quad (6)$$

when site 1 is down and site 2 is up (similar equations hold for the symmetric situation);

$$J^* = c(x) + (V_{01}(x) + V_{10}(x) - 2V_{00}(x))q_u - \Big(\frac{\partial V_{00}(x)}{\partial x_1} + \frac{\partial V_{00}(x)}{\partial x_2}\Big)d \quad (7)$$

when both machines/sites are down. As in [3] for the single site problem, if it is possible to find a J^*, a control policy $u^*(\cdot)$ and four C^1 functions $V_{ij}(\cdot)$, $i, j = 0, 1$, such that the HJB equations (5)-(7) are satisfied, then $u^*(\cdot)$ is optimal and J^* is the optimal cost.

Unfortunately, as mentioned above, the solution of these equations (i.e. the determination of the functions $V_{ij}(\cdot)$) is not an easy task, even for quite small problems, and in particular if one wants to extend the results to more than two production sites. The first problem is that in general J^* is not known and its value depends on the optimal control which in turn could be derived if the $V_{ij}(\cdot)$ functions were known. For this reason, the usual way to use the HJB equations is to prove the optimality of a tentative candidate policy or, alternatively, the HJB equations are used under some assumptions or approximations on the optimal control. We pursue a different approach and provide in Section IV a numeric procedure to approach the HJB equations which will allow, for a bounded problem, to compute J^* and the optimal control. In this section we provide some preliminary general properties of the optimal control based on the inspection of the HJB equations (5)-(7). This analysis will be necessary to develop the numeric procedure described in Section IV.

It is convenient to analyze first Eq. (6); there the optimal control actions are easily characterized in terms of the signs or the relative ordering of $v_1 := \frac{\partial V_{01}(x)}{\partial x_1}$ and of $v_2 := \frac{\partial V_{01}(x)}{\partial x_2}$. Indeed, $v_i > 0$ implies $u_{2i} = 0$, whereas $v_1 + a < \min\{v_2, 0\}$ implies $u_{21} = \mu$ and $u_{22} = 0$ (conversely $v_2 < \min\{v_1 + a, 0\}$ implies $u_{21} = 0$ and $u_{22} = \mu$). Similar considerations hold when site 1 is up, site 2 is down. The conditions outlined above, for $s = [0, 1]^T$, partition the state space of the operative machine into three regions each characterized by a typical "behavior". In the first region the site idles, in the second it produces for itself at full capacity, in the third it produces at full capacity for the other site.

When both sites are operative, the system behavior is characterized by (5), and in particular by the terms inside the $\min$ operation, which we rewrite for convenience as

$$v_1 u_{11} + (v_1 + a)u_{21} + v_2 u_{22} + (v_2 + a)u_{12}, \quad (8)$$

where v_1 and v_2 are used in this case for the partial derivatives of V_{11}, i.e., $v_1 = \frac{\partial V_{11}(x)}{\partial x_1}$ and $v_2 = \frac{\partial V_{11}(x)}{\partial x_2}$.

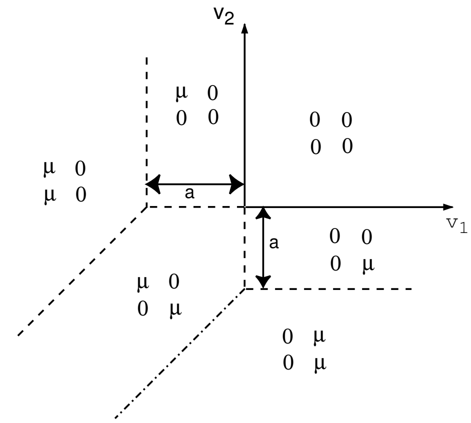


Fig. 1. Optimal control of the cooperative system at state $s = [1, 1]^T$ as a function of the partial derivative terms of V_{11}.

When $v_i > 0$, $u_{1i} = u_{2i} = 0$, in other words, site i needs not any productive supply. Such situation may be associated to a buffer level of site i which is above a certain "security" level. Such a security level turns out to be dependent on the degree of cooperation between the two sites, and when $a \to \infty$ this is equivalent, for each site, to the hedging point for a single site unreliable system, as the following Section will illustrate. For this reason, the security level evaluated when both sites are operative is defined, for a generic value of a, as the *cooperative hedging level*, and denoted as z_c^*. Due to the symmetry of the system, the point $Z_{11} := [z_c^*, z_c^*]$, is defined the cooperative hedging point for the system in consideration. When $-a < v_1 < 0$ and $v_2 > 0$ the optimal control action is to set $u_{11} = \mu$ and $u_{21} = u_{12} = u_{22} = 0$. In this case, while the buffer of site 2 is above its security level, the buffer of site 1 needs replenishment, but not so urgently to pay the additive costs associated to production coming from the other site, hence $u_{21} = 0$. Finally, if $v_1 < -a$, the emergency makes the additive transportation costs appealing/affordable but in this case the state of the other site must be checked: cooperation takes place only when $v_1 + a < v_2$, and in this case site 2 devotes its whole production capacity to site 1. By the symmetry in system parameters a similar analysis is valid relatively to site 2; the behavior of the system is depicted in figure 1, which shows the optimal u matrix for each region in the (v_1, v_2) space. Notice that such regions have slight relationships with those in the space of the joint inventory/backlog level; such result will be presented in the next Section (see Figs. 2,3).

The analysis above identifies the possible optimal control actions for each discrete state, restricting them to six possible choices for $s = [1, 1]^T$ (i.e., those illustrated in figure 1), and three for each discrete states $s = [0, 1]^T$ and $s = [1, 0]^T$. Denote by $\mathcal{U}_{ij}^{opt}$, $i, j \in \{0, 1\}$, such possible optimal control actions at state $s = [i, j]^T$. The sets $\mathcal{U}_{ij}^{opt}$, $i, j \in \{0, 1\}$, will be used in the numeric procedure defined in the following section.

IV. NUMERICAL METHODS

As discussed in the Introduction, the characterization of the optimal control policies of the system is carried through numerical integration of the HJB equations (5)-(7). This will be performed assuming a bounded backlog/inventory space.

Assumption 2: The backlog/inventory space of each site is bounded, i.e. there exist two positive constants L^- and L^+ such that $-L^- \leq x_i \leq L^+$, $i = 1, 2$. We will denote the bounded space by $\mathcal{X}_L := [-L^-, L^+] \times [-L^-, L^+]$.

With a bound on the backlog/inventory space we also introduce a cost associated with rejected demands, so that an additional contribute $R \cdot \mathbf{1}(x_i = -L^-)$, where $\mathbf{1}$ is the indicator function, is added to the instantaneous cost $g(x, u)$.

Remark 1: If the bounds are large enough (and the system is stable as assumed in Assumption 1), the optimal policy near the origin and the optimal cost are as in the unbounded case. This observation is based on experimental results of this case and also on the analytical studies [10] of a single site system. For this reason, the considered bounded problem and the numerical approach to solve it provide a satisfactory approximation also for the original problem, i.e. Problem 1.

Now, for the original unbounded system, we know that, if the Markov chain characterizing the joint operating state of the two sites is irreducible and a stationary policy keeps the state bounded (as it is possible in this case thanks to Assumption 1), then the process $x(t)$ is ergodic and the limit value for (3) is well defined. This also holds for the bounded system. The limit value of (3), independent of initial conditions, can also be seen (see [5]) as the steady-state expected value of the instantaneous cost $g(x, u)$ when a stationary stable policy is applied, i.e. $J = E[g(x, u)]$. Then, given a stable stationary policy starting from any given state s and buffer level x, it is possible to define as in [5] a *differential cost* $V_s(x)$ which gives the deviation of the cost from its expected value J, in such a way that the corresponding long run expected (but not average) cost in $[0, T]$ can be written as $J \cdot T + V_s(x)$. So, $V_s(x)$ can also be seen as the transient cost incurred by the considered policy from the initial state (s, x) to carry the system at steady state and can be written as follows:

$$V_s(x) = \lim_{T\to\infty} E \int_0^T [g(x(t), u(t)) - J]\, dt|_{x(0)=x, s(0)=s}. \tag{9}$$

Clearly this differential cost depends on initial conditions. To solve the problem for the bounded system (hence to obtain indications on the original Problem 1), we write the discrete dynamic programming equations corresponding to the two cost criteria defined in (3) and (9) and solve them numerically. To this purpose, the state space $\mathcal{X}_L$ is mapped into a set of N^2 points $\mathcal{X}_L^D := \{x_{lm} : l, m = 0, 1, \ldots, N - 1\}$ equally spaced by $\Delta x = (L^+ + L^-)/(N - 1)$, i.e., $x_{lm} = (-L^- + l\Delta x, -L^- + m\Delta x)$. Setting v_{gcd} as the greatest common divisor of $\{d, \mu - d, 2\mu - d\}$ i.e., all possible "velocities" of the system, identifies $\Delta t = \Delta x / v_{gcd}$ as the time step for the numerical integration of (2), given by:

$$x(u)' = B(x + (u^T s - d)\Delta t) \tag{10}$$

where the operator B bounds the state inside the limited state space $\mathcal{X}_L$, e.g. $B(x) = [b(x_1), b(x_2)]^T$, $b(x_i) = \min\{\max\{x_i, -L^-\}, L^+\}$. Notice that $x \in \mathcal{X}_L^D$ implies $x'(u) \in \mathcal{X}_L^D$ for any possible control action u, by the choice made for Δt. In the following we will always consider states belonging to the discretized state space, which implies that functions of x are represented by $N \times N$ matrices. Define $J_s^{(k)}(x)$ as the minimum *average* expected cost on a time horizon of length $k\Delta t$, incurred by the system if starting from the (hybrid) state (s, x). Then we can write:

$$J_s^{(k+1)}(x) = \frac{1}{k+1} \min_{u \in \mathcal{U}_s^{opt}} \{g(x, u) + \sum_{s'} p_{ss'}\, k\, J_{s'}^{(k)}(x'(u))\} \tag{11}$$

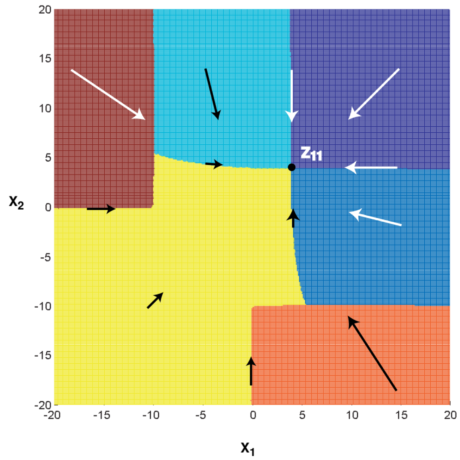
where $J_s^{(0)}(x) = 0$ for all s and x, and $p_{ss'}$ are the probability of going from s to s'. It is possible to verify that as $k \to \infty$, a limit value of $J_s^{(k)}(x)$ which is constant with respect to (s, x) satisfies (11). Actually, the iterative equation (11) converges with k to J^* (in fact (11) is nothing but a numerical integration of (3) when the optimal policy is applied).

Once J^* has been evaluated, the discrete dynamic programming iterative equation for the index (9) can be written by introducing a function $V_s^{(k)}(x)$ which represents the minimum long term expected (*non average*) cost $V_s(x)$ on a time horizon $[0, k\Delta t]$. We have:

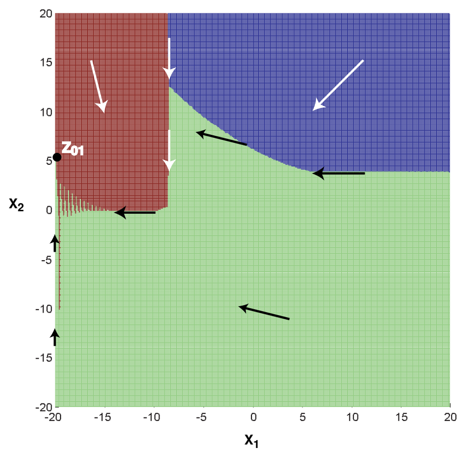
$$V_s^{(k+1)}(x) = -J^* + \min_{u \in \mathcal{U}_s^{opt}} \{g(x, u) + \sum_{s'} p_{ss'}\, V_{s'}^{(k)}(x'(u))\} \tag{12}$$

with the same meaning for variables as in (11) and $V_s^{(0)}(x) = 0$ for all s and x. It is important to remark that the iterative equation (12) does *not* converge to a constant value for all x and s: a different transient cost is paid from different initial conditions. In this case, the functions $V_s^{(k)}(x)$ converge to non constant functions from which it is straightforward to get the optimal production rates at each (hybrid) state (s, x). We also remark that, as $\Delta t \to 0$, the iterative equations in (12) provide the HJB equations (5)-(7). This was expected according to the analysis reported in [5].

It is possible to characterize the convergence of (12) in terms of the difference $\Delta V_s^{(k)}(x) := V_s^{(k+1)}(x) - V_s^{(k)}(x)$, and in particular for those s and x which give the difference with largest absolute value. Denote this quantity by (the scalar) $e(k)$, i.e., if $(\bar{s}(k), \bar{x}(k)) := \arg\max_{s,x} |\Delta V_s^{(k)}(x)|$, then $e(k) := \Delta V_{\bar{s}(k)}^{(k)}(\bar{x}(k))$. If a wrong estimate of J^* is used in (12), convergence of $V_s^{(k)}(x)$ is not achieved, and in particular, $e(k)$ tends to a negative (resp. positive) constant if $J > J^*$ (resp. $J < J^*$). This fact could be exploited to define a fast procedure to evaluate the optimal J^* if an upper bound J_M and a lower bound J_m of J^* are available. This procedure is a bisection algorithm which evaluates $e(k)$ after a large number of iterations for a tentative $\hat{J}^* = \frac{J_m + J_M}{2}$. If this is positive, we set $J_m = \hat{J}^*$ otherwise we set $J_M = \hat{J}^*$. Due to the slow convergence rate of $J_s^{(k)}(x)$ to J^*, this procedure could provide a faster approach to evaluate J^*. The numerical optimization is formally stated

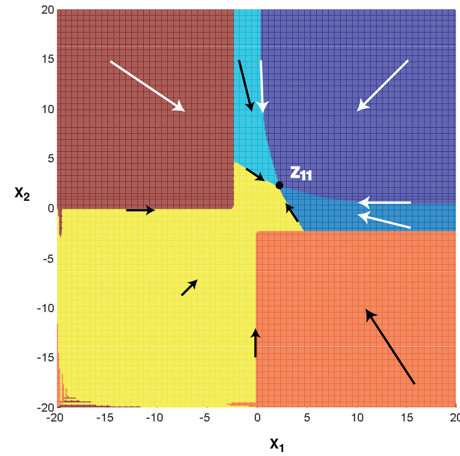


(a) $s = (1,1)^T$

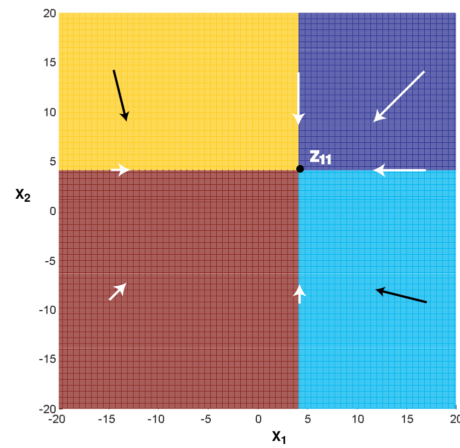


(b) $s = (0,1)^T$

Fig. 2. Optimal control of the cooperative system as a function of the joint backlog/inventory level. Arrows define the flow of the optimal feedback system ($a = 50$)



(a) $a = 10$



(b) $a = \infty$

Fig. 3. Optimal control in state $(1,1)$ of the cooperative system as a function of the joint backlog/inventory level. Arrows define the flow of the optimal feedback system

in the following Algorithm.

Algorithm 1: Select N to perform the time-state discretization of $\mathcal{X}_L$ described above. Select a small quantity ε and a large integer K. Let J_m and J_M be a lower bound and an upper bound on J^*, respectively. For all $x \in \mathcal{X}_L^D$ and $s \in \{0,1\} \times \{0,1\}$ initialize $V_s^{(0)}(x) = 0$

While $J_M - J_m > \varepsilon$,

1) Let $\hat{J}^* = \frac{J_m + J_M}{2}$.
2) Evaluate K times (12) with $\hat{J}^*$ in place of J^*.
3) If $e(K) > 0$, set $J_m = \hat{J}^*$, else set $J_M = \hat{J}^*$. □

Clearly a proper selection of the constants in Algorithm 1 should be performed. The constant ε gives the accuracy in the computation of J^*. Such accuracy, however, also depends on N, which cannot be increased too much, being the complexity of the algorithm proportional to N^2. The choice of K should also be performed as a tradeoff between the computation time and the minimum number of iterations which allows to understand if the considered $\hat{J}^*$ is larger or smaller than J^*.

V. EXPERIMENTAL RESULTS

Consider a system with the following parameters: $R = 2500$, $\mu = 5$, $d = 4$, $q_u = 1$, $q_d = 0.01$, $c_m = 50$, $c_p = 1$ and a defined below. The system is feasible according to Assumption 1. Applying algorithm 1 (with $N = 400$, $\varepsilon = 0.02$ and $K = 800$) to the considered system with $a = 50$, $L^- = L^+ = 20$ gives the results reported in Fig. 2 for the production control in state $(1,1)$ and $(0,1)$ respectively. The dimension of the arrows is proportional to the optimal flow rate (i.e. $u^*(\cdot) - d$). In the figures it is also reported the limit point for each state which is the hedging point $Z_{11}^* = (3.95, 3.95)$ for $s = (1,1)^T$ and the point $Z_{01}^* = (-19.9, 5.05)$ for $s = (0,1)^T$. Observe that in the state $(0,1)$, which is not feasible, the limit point is on the border of the state space (the trajectory would go to $-\infty$ if no bound was effective). Also observe that, according to intuition and to the discussion in Section III, one site helps the other if its inventory level is large enough and the other site has a very large backlog (see the up-left and the down-right areas in Fig. 2(a)). As the transportation cost a is reduced, in particular

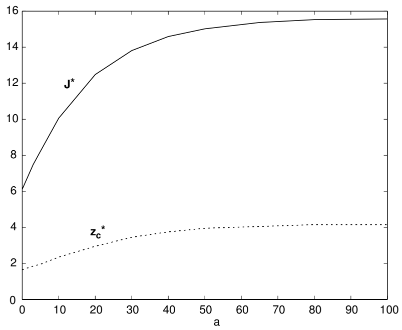


Fig. 4. Optimal safety stock z_c^* and cost J^* derived through the numerical procedure of Algorithm 1 for the system in Section V as a function of a

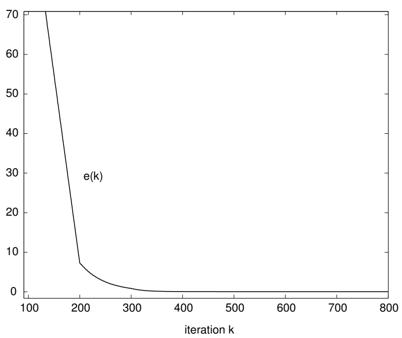


Fig. 5. The difference $e(k)$ as a function of the last iterations of (12) for the $a = 50$ example

$a = 10$, the collaborative area of the state space increases and the optimal safety stock $Z_{11}^* = (2.35, 2.35)$ is reduced (see Fig. 3(a)). On the contrary, if $a \to \infty$, the collaborative area vanishes, according to intuition and a larger safety stock $Z_{11}^* = (4.15, 4.15)$ and cost $J^* = 15.57$ is found (see Fig. 3(b)). According to [10], with the same parameters, the optimal safety stock is given by $z^* = 3.8$ (close to the value obtained considering the approximation introduced, indeed it was observed $z_c^* = 3.95$ for $N = 800$) and $J^* = 7.73$ (very close to one half of the J^* obtained for the two site system).

In other words, if the transportation cost goes to infinity, the sites become isolated and the considered problem becomes equivalent to the replication of single-site problems with the cost for the two site system twice the cost of the single site system. As a validation of the numerical methods proposed in this paper, observe that in such case they provide, for each site, a good approximation to the optimal analytical solution computed in [10] for the single site system.

Fig. 4 illustrates z_c^* and J^* as a function of a ranging in $[0, 100]$. Both increase with a meaning that the collaboration of the two sites, when not too expensive, gives robustness to the whole system against failures. Finally, we also report, in Fig. 5, the value of the differences $e(k)$ as a function of the last iterations of (12) for the $a = 50$ example.

VI. CONCLUSIONS

We have considered a two-site, cooperative, failure prone, Markov production system. General properties of the optimal policy have been highlighted while the optimal solution has been evaluated through a general numerical procedure developed in this paper. Such numerical methods may find possible applications to other problems tackled by the HJB approach. A set of examples to illustrate this procedure and to analyze the effects of some system parameters are included in the paper and compared with known analytical results for the single site problem. As future research the extension of the results obtained to systems comprising more than 2 sites could certainly be considered as well as a strategy to select in a proper manner the constants used in Algorithm 1.

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