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# Boundary control of non-homogeneous systems of two conservation laws

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*Abstract*— **We investigate the problem of the stability of a system of two conservation laws perturbed by non-homogeneous terms. We assume that these non-homogeneous terms have a small $\mathcal{C}^1$-norm. By a Riemann coordinates approach we state a sufficient criterion for the stability in terms of the boundary conditions. This stability result is then applied to the problem of the regulation of the water level and the flow rate in an open channel. The flow in the channel is represented by the SaintVenant equations perturbed by small non-homogeneous terms that account for the friction effects as well as external water supplies or withdrawals.**

## I. Introduction

Many distributed physical systems are described by hyperbolic partial differential equations (PDE). The main property of this class of PDE is the existence of the so-called Riemann coordinates which are a successful tool for the proof of classical solutions, the analysis and the control among other properties, see e.g. [1], [7].

In this paper, we investigate the problem of the stability of such hyperbolic equations in presence of small non-homogeneous terms. In terms of a criterion on the boundary conditions, we state a sufficient condition for this stability property. The proof is based on an analysis of the influence (of these boundary conditions and of the non-homogenous terms) on the Riemann coordinates.

In [7], a sufficient condition for a stability property of non-homogeneous systems of conservation laws is also given. It is also written in terms of the boundary conditions but, in [7, Chap. 5], it is assumed that the non-homogeneous terms are continuously differentiable and that the non-homogeneous terms together with their derivative *vanish* at the equilibrium. In this paper, we assume only that the non-homogenenous terms vanish at the equilibrium and that their derivative function is *small* at the equilibrium. This important difference asks for special care in the proof of the existence of classical solutions and of the stability. However our result is stated here only for the case of two quasilinear conservative laws, and not for the case of $n$ quasilinear conservative laws, with $n \geq 2$, as considered in [7].

In this paper, we apply our general result on non-homogeneous systems of conservations laws to the design of a stabilizing boundary control of a channel. More precisely we address the problem of the regulation of the water level and the water flow rate in an open channel by using the gate opening as control action.

The model used is not strictly hyperbolic since we consider also the case of the presence of friction and some external small supplies or removal of water along the canal. The model is written in terms of Saint-Venant equations introduced in [15] and commonly used in hydraulics to describe the flow of water in open-channels (see e.g. the textbooks [3] or [5]). Here the Saint-Venant equations are perturbed by small non-homogenous terms that account for the friction effects as well as external water supplies or withdrawals.

This stability problem for the regulation of the flow in a channel has been considered for a long time in the literature as reported in the survey paper [13] which involves a comprehensive bibliography. For advanced control methods, see [4], [12] where discrete linear approximations of the perturbed Saint-Venant equations are used. See also [9], [10], [11] where a $H_\infty$ control design is developped. In [8] the perturbed Saint-Venant equations are linearized and an infinite dimensional controller is designed to suppress the oscillating modes over the canal. This paper can been seen as a generalization of [6], where Saint-Venant equations without non-homogenous term are considered.

The paper is organized as follows. First in Section II, we state our main result, namely a sufficient condition for the stability of two conservation laws. It is written in terms of the boundary conditions and non-homogeneous terms with small $\mathcal{C}^1$-norm are considered.

In Section III, the main result is applied to the boundary regulation of the water level and the water flow rate in a channel in presence of small friction and small external supplies or removal of water distributed along its length.

Due to the space limitation, the proofs are omitted. However we state the main steps of our proof in Section IV. For a complete proof see [14].

Section V contains some concluding remarks.

## II. Stability of hyperbolic systems with non-homogeneous terms

In all the following, we will denote $|\cdot|$ the norm in $\mathbb{R}^2$ defined, for all $(\xi_1, \xi_2) \in \mathbb{R}^2$, by

$$|(\xi_1, \xi_2)| = \max(|\xi_1|,\ |\xi_2|)\ ,$$

and $B(\varepsilon_0)$ the ball centered in 0 with radius $\varepsilon_0 > 0$. Let $L > 0$. Let us consider $\xi$: $[0, L] \times [0, +\infty) \to \mathbb{R}^2$ such that:

$$\partial_t \boldsymbol{\xi} + \Lambda(\boldsymbol{\xi}) \partial_x \boldsymbol{\xi} = \mathbf{h}(\boldsymbol{\xi}) \tag{1}$$

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where $\Lambda$: $B(\varepsilon_0) \to \mathbb{R}^{2\times 2}$ is a continously differentiable function such that

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2) \ ,$$

with

$$\lambda_1(0) < 0 < \lambda_2(0), \tag{2}$$

and $\mathbf{h} = (\mathbf{h}_1, \mathbf{h}_2)$: $B(\varepsilon_0) \to \mathbb{R}^2$ is a continously differentiable function such that

$$\mathbf{h}(0) = 0 \ . \tag{3}$$

The boundary conditions of (1) are

$$\begin{pmatrix} \boldsymbol{\xi}_1(L,t) \\ \boldsymbol{\xi}_2(0,t) \end{pmatrix} = \mathbf{g} \begin{pmatrix} \boldsymbol{\xi}_1(0,t) \\ \boldsymbol{\xi}_2(L,t) \end{pmatrix}, \tag{4}$$

where $\mathbf{g}$: $B(\varepsilon_0) \to \mathbb{R}^2$ is a continuously differentiable function satisfying $\mathbf{g}(0) = 0$.

We define the compatibility condition for (1) and (4):

*Definition 2.1:* A function $\boldsymbol{\xi}^{\#} \in C^1(0, L; \mathbb{R}^2)$ satisfies the compatibility condition $\mathcal{C}$ if

$$\begin{pmatrix} \boldsymbol{\xi}_1^{\#}(L) \\ \boldsymbol{\xi}_2^{\#}(0) \end{pmatrix} = \mathbf{g} \begin{pmatrix} \boldsymbol{\xi}_1^{\#}(0) \\ \boldsymbol{\xi}_2^{\#}(L) \end{pmatrix},$$

and

$$\begin{pmatrix} \lambda_1(\boldsymbol{\xi}^{\#}(L))\partial_x \boldsymbol{\xi}_1^{\#}(L) - \mathbf{h}_1(\boldsymbol{\xi}^{\#}(L)) \\ \lambda_2(\boldsymbol{\xi}^{\#}(0))\partial_x \boldsymbol{\xi}_2^{\#}(0) - \mathbf{h}_2(\boldsymbol{\xi}^{\#}(0)) \end{pmatrix} =$$

$$\nabla \mathbf{g} \begin{pmatrix} \boldsymbol{\xi}_1^{\#}(0) \\ \boldsymbol{\xi}_2^{\#}(L) \end{pmatrix} \begin{pmatrix} \lambda_1(\boldsymbol{\xi}^{\#}(0))\partial_x \boldsymbol{\xi}_1^{\#}(0) - \mathbf{h}_1(\boldsymbol{\xi}^{\#}(0)) \\ \lambda_2(\boldsymbol{\xi}^{\#}(L))\partial_x \boldsymbol{\xi}_2^{\#}(L) - \mathbf{h}_2(\boldsymbol{\xi}^{\#}(L)) \end{pmatrix}.$$

We denote by $B_{\mathcal{C}}(\varepsilon_0)$ the set of functions $\xi^{\#}$: $[0.L] \to B(\varepsilon_0)$ of class $\mathcal{C}^1$ satisfying the compatibility assumption $\mathcal{C}$. To state the following result, we need to define the classical norms on $C^0(0, L)$ and $C^1(0, L)$. Given $\Phi$ continuous on $[0, L]$ and $\Psi$ continuously differentiable on $[0, L]$, we denote

$$\begin{aligned} |\Phi|_{C^0(0,L)} &= \max_{x\in[0,L]} |\Phi(x)| \ , \\ |\Psi|_{C^1(0,L)} &= |\Psi|_{C^0(0,L)} + |\Psi'|_{C^0(0,L)} \end{aligned}$$

In addition, for a given matrix $A = (a_{ij})$, $\rho(A)$ denotes its spectral radius and $\mathrm{abs}(A)$ is the matrix defined by $\mathrm{abs}(A) = (|a_{ij}|)$.

The main result of this paper is the following

*Theorem 2.1:* Let $\varepsilon_0 > 0$. If

$$\rho(\mathrm{abs}(\nabla \mathbf{g}(0)) < 1, \tag{5}$$

then there exist $\varepsilon_1 \in (0, \varepsilon_0)$, and $H_1 > 0$ such that, for all continuously differentiable functions $\mathbf{h} : B(\varepsilon_1) \to \mathbb{R}^2$ such that (3) holds together with

$$|\nabla \mathbf{h}(0)| \leq H_1 \ , \tag{6}$$

for all $\boldsymbol{\xi}^{\#} \in B_{\mathcal{C}}(\varepsilon_1)$, there exists one and only one function $\boldsymbol{\xi} \in C^1([0, L] \times [0, +\infty) \, ; \mathbb{R}^2)$ satisfying (1), (4) and

$$\boldsymbol{\xi}(x, 0) = \boldsymbol{\xi}^{\#}(x) \ , \forall x \in [0, L]. \tag{7}$$

Moreover, there exist $\mu > 0$ and $C_1 > 0$ such that this function satisfies

$$|\boldsymbol{\xi}(., t)|_{C^1(0,L)} \leq C_1 e^{-\mu t} |\boldsymbol{\xi}^{\#}|_{C^1(0,L)} \ , \forall t \geq 0. \tag{8}$$

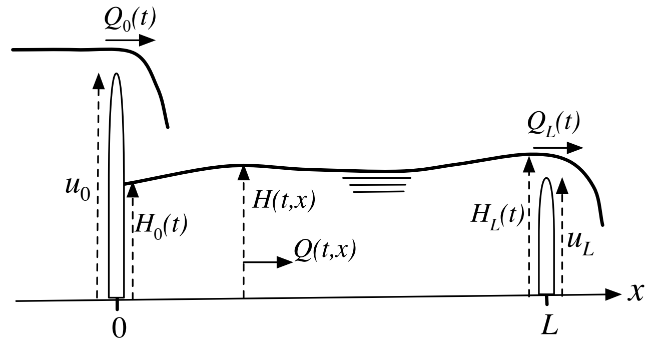


Fig. 1. A reach of an open channel delimited by two adjustable overflow spillways

The proof of this result is based on an estimation of the influence of the non-homogeneous terms on the evolutions of the Riemann coordinates. In particular, we have to prove that the damping condition (5) is strong enough to manage the non-homogeneous terms, whose derivative is assumed to be small at the origin due to (6). In fact we prove this result for a particular structure of the boundary conditions (see (15) below) and we deduce our main result for the boundary conditions (4).

The sketch of the proof of this Theorem is postponed in Section IV.

## III. Application to level and flow control in an horizontal reach of an open channel

We consider the special case of a reach of an open channel delimited by two overflow spillways as represented in Figure 1.

We assume that 1) the channel is horizontal, 2) the channel is prismatic with a constant rectangular section and a **unit** width, 3) the channel is subject to time-invariant spatially distributed water supplies or removals that do not modify the momentum conservation.

The flow dynamics are described by a nonhomogeneous system of two laws of conservation (Saint-Venant or shallow water equations), namely a law of mass conservation :

$$\partial_t H(x, t) + \partial_x (Q(x, t)) = q(x), \tag{9}$$

and a law of momentum conservation :

$$\partial_t Q(x, t) + \partial_x \left(\frac{Q^2(x, t)}{H(x, t)} + \frac{g H^2(x, t)}{2}\right) = -C_f \frac{Q^2(x, t)}{H^2(x, t)}. \tag{10}$$

where $H(x, t)$ represents the water level and $Q(x, t)$ the water flow rate in the reach, $g$ denotes the gravitation constant, $q(x)$ the water supply/removal function, $C_f$ the friction coefficient. The system is written in matrix form as follows :

$$\partial_t \begin{pmatrix} H \\ Q \end{pmatrix} + A(H, Q) \partial_x \begin{pmatrix} H \\ Q \end{pmatrix} = \begin{pmatrix} q(x) \\ -C_f Q^2/H^2 \end{pmatrix}$$

with the matrix $A(H,Q)$ defined as :

$$A(H,Q) = \begin{pmatrix} 0 & 1 \\ gH - (Q^2/H^2) & 2Q/H \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix $A(H,Q)$ :

$$\begin{aligned} \lambda_1(H,V) &= (Q/H) - \sqrt{gH} \\ \lambda_2(H,V) &= (Q/H) + \sqrt{gH} \end{aligned}$$

are generally called *characteristic velocities* . The flow is said to be *fluvial* (or subcritical) when the characteristic velocities have opposite signs :

$$\lambda_1(H,Q) < 0 < \lambda_2(H,Q).$$

### A. Steady-state solution

Under constant boundary conditions $Q(0,t) = \bar{Q}_0$ and $H(L,t) = \bar{H}_L$ $\forall t$, there exists a steady-state solution :

$$H(x,t) = \bar{H}(x) \quad \text{and} \quad Q(x,t) = \bar{Q}(x) \quad x \in [0,L] \ \ \forall t$$

which satisfies the differential equations :

$$\begin{aligned} \partial_x \bar{Q}(x) &= q(x) \\ \partial_x \bar{H}(x) &= \frac{2q(x)}{\bar{H}(x)\bar{Q}(x)} + \frac{C_f}{\bar{H}^2(x)} \end{aligned}$$

### B. Control design

The control objective is to stabilise the level $H(x,t)$ and the flow rate $Q(x,t)$ at the steady state profiles $\bar{H}(x)$ and $\bar{Q}(x)$ corresponding to set points $\bar{H}_L$ and $\bar{Q}_0$. We assume that the boundary flow rates $Q(0,t)$ and $Q(L,t)$ are the control actions at the user's disposal because they can be assigned by the positions $u_0$ and $u_L$ of the spillways. It is also assumed that the water levels at the boundaries $H_0(t) = H(0,t)$ and $H_L(t) = H(L,t)$ are the only available on-line measurements. In order to satisfy this control objective, and in light of the control laws designed in [6] for the homogeneous case, the following control laws are proposed :

$$\begin{aligned} Q_0 &= \frac{\bar{Q}_0}{\bar{H}_0} H_0 - \alpha_0 H_0 \left( 2\sqrt{gH_0} - 2\sqrt{g\bar{H}_0} \right) && (11) \\ Q_L &= \frac{\bar{Q}_L}{\bar{H}_L} H_L + \alpha_L H_L \left( 2\sqrt{gH_L} - 2\sqrt{g\bar{H}_L} \right) && (12) \end{aligned}$$

with:

$$0 < \alpha_0 < 1 \quad \text{and } 0 < \alpha_L < 1.$$

The parameters $\alpha_0$ and $\alpha_L$ are tuning parameters at the user's disposal. It can be seen that both controls have the form of a state feedback at the two boundaries. In addition, it can be emphasized that the implementation of the controls is particularly simple since only measurements of the levels $H_0(t)$ et $H_L(t)$ at the two spillways are required. This means that the feedback implementation does not require neither level measurements inside the pool nor any flow rate measurements.

### C. Stability analysis

We shall now show that the stability of this control system can be analysed with the theorem reported in Section II. In order to transform the model (9)-(10) into characteristic form (1), the following characteristic (Riemann) coordinates are considered :

$$\begin{aligned} \xi_1 &= (Q/H) - 2\sqrt{gH} - (\bar{Q}/\bar{H}) + 2\sqrt{g\bar{H}} \\ \xi_2 &= (Q/H) + 2\sqrt{gH} - (\bar{Q}/\bar{H}) - 2\sqrt{g\bar{H}} \end{aligned}$$

Observe that these coordinates can be inverted to give :

$$H(\boldsymbol{\xi}) = \frac{\left(\xi_2 - \xi_1 + 4\sqrt{g\bar{H}}\right)^2}{16g} \qquad (13)$$

and

$$Q(\boldsymbol{\xi}) = \frac{\left(\xi_1 + \xi_2 + 2(\bar{Q}/\bar{H})\right)}{2} \frac{\left(\xi_2 - \xi_1 + 4\sqrt{g\bar{H}}\right)^2}{16g} \qquad (14)$$

With these coordinates, it is then readily shown that the model (9)-(10) can be written in the characteristic form (1) with the characteristic velocities $\lambda_1(\boldsymbol{\xi})$ and $\lambda_2(\boldsymbol{\xi})$ expressed in the Riemann coordinates by using the inverse transformation (13)-(14) and the following definitions for $h_1(\boldsymbol{\xi})$ and $h_2(\boldsymbol{\xi})$ :

$$\begin{aligned} \mathbf{h}_1(\boldsymbol{\xi}) &= \left( \frac{\bar{Q}}{\bar{H}} + \sqrt{g\bar{H}} - \frac{3\xi_1 + \xi_2}{4} \right) \left( \frac{\bar{Q}}{\bar{H}} - 2\sqrt{g\bar{H}} \right) \\ &\quad + \frac{g\left[q(x) - 4C_f(\xi_1 + \xi_2 + 2(\bar{Q}/\bar{H}))\right]}{\xi_2 - \xi_1 + \sqrt{g\bar{H}}} \\ \mathbf{h}_2(\boldsymbol{\xi}) &= \left( \frac{\bar{Q}}{\bar{H}} - \sqrt{g\bar{H}} - \frac{3\xi_1 + \xi_2}{4} \right) \left( \frac{\bar{Q}}{\bar{H}} + 2\sqrt{g\bar{H}} \right) \\ &\quad - \frac{g\left[q(x) + 4C_f(\xi_1 + \xi_2 + 2(\bar{Q}/\bar{H}))\right]}{\xi_2 - \xi_1 + \sqrt{g\bar{H}}} \end{aligned}$$

Observe also that $\mathbf{h}_1(0) = \mathbf{h}_2(0) = 0$.

Moreover, in the characteristic coordinates, the control laws (11)-(12) can be shown to be equivalent to the following boundary conditions :

$$\xi_1(L,t) = -k_L \xi_2(L,t) \quad \text{with} \quad k_L = \frac{1-\alpha_L}{1+\alpha_L}$$

and

$$\xi_2(0,t) = -k_L \xi_1(0,t) \quad \text{with} \quad k_0 = \frac{1-\alpha_0}{1+\alpha_0}$$

It is easily seen that these boundary conditions are in the form (3) and that, in this special case,

$$\rho(\text{abs}(\nabla g(0)) = k_0 k_L.$$

Hence the closed loop control system is exactly set in a form which allows to apply Theorem 2.1. This means that, provided the conditions of that theorem are satisfied (in particular provided the tuning paramaters $\alpha_0$ and $\alpha_L$ are chosen such that $|k_0 k_L| < 1$), under the control laws (11)-(12), the level $H(x,t)$ and the flow rate $Q(x,t)$ are guaranteed to smoothly exponentially converge to the desired steady-state profiles $\bar{H}(x)$ and $\bar{Q}(x)$ respectively.

## IV. SKETCH OF THE PROOF OF THEOREM 2.1

This section is devoted to the proof of Theorem 2.1. First we assume that the boundary conditions have a particular form (see (15) below). It allows us to set down a more natural machinery to prove intermediate technical lemmas. In Section IV-A, we recall an existence result of a solution in finite time. In Sections IV-B and IV-C, estimates of $|\boldsymbol{\xi}(\cdot,t)|_{\mathcal{C}^0(0,L)}$ and $|\boldsymbol{\xi}(\cdot,t)|_{\mathcal{C}^1(0,L)}$ are derived, and we conclude the proof of Theorem 2.1 in Section IV-E (in particular the assumption on the form of the boundary conditions is removed).

### A. Existence result

In all the following, except in Section IV-E below for the proof of Theorem 2.1, we assume that the boundary conditons (4) are of the particular form

$$\begin{pmatrix} \boldsymbol{\xi}_1(L,t) \\ \boldsymbol{\xi}_2(0,t) \end{pmatrix} = \begin{pmatrix} \mathbf{g}_1(\boldsymbol{\xi}_2(L,t)) \\ \mathbf{g}_2(\boldsymbol{\xi}_1(0,t)) \end{pmatrix}, \qquad (15)$$

where the functions $\mathbf{g}_1$ and $\mathbf{g}_2$ are continuously differentiable on a neighborhood of 0.

Combining the existence result of [7, Chap. 5, Theo. 1.1] together with the result of the continuity with respect to parameters as given in [1, Chap. 3], we get the following existence result:

*Theorem 4.1:* Let $T > 0$. There exist $\varepsilon(T) > 0$, $c(T) > 0$ and $H(T)$ such that, for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon(T))$ and for all continuously differentiable functions $\mathbf{h}$: $B(\varepsilon(T)) \to \mathbb{R}^2$ such that (3) holds and

$$|\nabla \mathbf{h}(0)| \leq H(T) \qquad (16)$$

there exists one and only one function $\boldsymbol{\xi} \in C^1([0,L] \times [0,T], \mathbb{R}^2)$ satisfying the PDE (1) with boundary conditions (15) and initial condition (7). Moreover, this function $\boldsymbol{\xi}$ satisfies, $\forall t \in [0,T]$,

$$|\boldsymbol{\xi}(.,t)|_{C^0(0,L)} \leq c(T)|\boldsymbol{\xi}^\#|_{C^0(0,L)}, \qquad (17)$$

$$|\boldsymbol{\xi}(.,t)|_{C^1(0,L)} \leq c(T)|\boldsymbol{\xi}^\#|_{C^1(0,L)}. \qquad (18)$$

In the following, we apply Theorem 4.1 several times. This allows us to define two decreasing sequences of positive numbers $\varepsilon_2$, $\varepsilon_3$, ... and $H_2$, $H_3$, ... We consider initial conditions $\boldsymbol{\xi}^\#$ successively in $B_{\mathcal{C}}(\varepsilon_2)$, $B_{\mathcal{C}}(\varepsilon_3)$, ...

Let, for $i \in \{1,2\}$,

$$s_i = \tfrac{L}{|\lambda_i(0)|}, \qquad (19)$$

$$\tau_1 > \max\{s_1,\ s_2\}. \qquad (20)$$

Let $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$ and $a > 1$ such that

$$|(\nabla \mathbf{g})_{ij}(0)| < a_{ij} < a, \quad \forall (i,j) \in \{1,2\}^2, \qquad (21)$$

$$\rho(A) < 1. \qquad (22)$$

Let

$$\tau_2 := (K+2)\tau_1. \qquad (23)$$

From (22), there exists an integer $K \geq 2$ such that such that $c(2\tau_1)\sum_{k\geq K}|A^k| < 1$. Let $\mu$, $\nu$ and $\omega > 1$ be such that

$$\nu = c(2\tau_1)\textstyle\sum_{k\geq K}|A^k| < 1, \qquad (24)$$

$$\mu = c(\tau_2)\tau_2(2a)^{\frac{\tau_2-\tau_1}{\min\{s_1,s_2\}}+1} + c(\tau_2)\tau_2, \qquad (25)$$

$$\omega \geq |(\Lambda'(\xi^\#))_{ij}|, \forall (i,j) \in \{1,2\}^2, \forall \xi^\# \in B_{\mathcal{C}}(\varepsilon(2\tau_1)) \qquad (26)$$

$$\omega \geq |(\bar{\Lambda})_{ij}|, \forall (i,j) \in \{1,2\}^2. \qquad (27)$$

where $\bar{\Lambda} = \Lambda(0)$ and $c(2\tau_1)$, $c(\tau_2)$ and $\varepsilon(2\tau_1)$ are given by Theorem 4.1.

### B. Estimation of $|\boldsymbol{\xi}(.,t)|_{C^0(0,L)}$

Let $\varepsilon_2 = \varepsilon(\tau_2)$ and $H_2 = H(\tau_2)$ given by Theorem 4.1. For all $0 < H < H_2$, for all continuously differentiable functions $\mathbf{h}$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (3) and

$$|\nabla \mathbf{h}(0)| \leq H, \qquad (28)$$

for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_2)$, there exists a solution $\boldsymbol{\xi} \in C^1([0,L] \times [0,\tau_2]; \mathbb{R}^2)$ of (1), (15) and (7).

Due to (2), we may assume without loss of generality (i.e. with $\varepsilon_2$ sufficiently small) that

$$\lambda_1(\boldsymbol{\xi}(x,t)) < 0 < \lambda_2(\boldsymbol{\xi}(x,t)). \qquad (29)$$

The aim of this section is to establish the following

*Lemma 4.2:* We have the existence of $H_6 > 0$ and $\varepsilon_6 \in (0,\varepsilon_2)$, such that for all continuously differentiable functions $\mathbf{h}$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (3) and (28), for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_6)$, the following inequalities holds:

$$|\boldsymbol{\xi}(.,\tau_2)|_{C^0(0,L)} \leq (\nu+\mu)(H+\varepsilon_2)|\boldsymbol{\xi}^\#|_{C^0(0,L)}. \qquad (30)$$

Before sketching the proof of this lemma, let us state a series of intermediate results.

Let $x \in [0,L]$.

We define, for $k \in \mathbb{N}\setminus\{0\}$ and for $(i_1,...,i_k) \in \{1,2\}^k$, $t_{i_1\ldots i_k} \in [0,T]$ and $p_{i_1\ldots i_k} \in [0,L]\times\{t_{i_1\ldots i_k}\}$ by recursion on $k$ as follows

- For $k=1$, let us consider the solution $y_1$ of the Cauchy problem

  $$\overset{\bullet}{y}_1(t) = \lambda_1(\boldsymbol{\xi}(y_1(t),t)), \qquad y_1(\tau_2) = x.$$

  Due to (29), it allows us to define the time instant $t_1 \leq \tau_2$ by $y_1(t_1) = L$, and we set $p_1 = (L,t_1)$.
- For $k=1$ again, let us consider the solution $y_2$ of the Cauchy problem

  $$\overset{\bullet}{y}_2(t) = \lambda_2(\boldsymbol{\xi}(y_2(t),t)), \qquad y_2(\tau_2) = x.$$

  Due to (29), it allows us to define the time instant $t_2 \leq \tau_2$ by $y_2(t_2) = 0$. and we set $p_2 = (0,t_2)$.

We can prove the following

*Claim 4.1:* There exist $\varepsilon_3 \in (0,\varepsilon_2)$ and $H_3 \in (0,H_2)$ such that, for all $0 < H < H_3$, for all continuously differentiable $\mathbf{h}$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (3) and (28), for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_3)$, we have, $\forall i_1 \in \{1,2\}$

$$|\xi_{i_1}(x,\tau_2)| \leq |\xi_{i_1}(p_{i_1})| + \tau_2 c(\tau_2)(H+\varepsilon_2)|\boldsymbol{\xi}^\#|_{\mathcal{C}^0(0,L)}. \quad (31)$$

Suppose that $t_{i_1 \dots i_k} \in [0,\tau_2]$ and $p_{i_1\dots i_k} \in [0,L] \times \{t_{i_1\dots i_k}\}$ are defined and, for $i_{k+1} \in \{1,2\}$, we define $t_{i_1\dots i_{k+1}} \in [0,\tau_2]$ and $p_{i_1\dots i_{k+1}} \in [0,L] \times \{t_{i_1\dots i_{k+1}}\}$. Again, we consider two cases:

- We consider the Cauchy problem
$$\overset{\bullet}{y}_1(t) = \lambda_1(\boldsymbol{\xi}(y_1(t),t)), \qquad y_1(t_{i_1\dots i_k}) = 0$$
and we define $t_{i_1\dots i_k 1} \in [0, t_{i_1\dots i_k})$ by $y_1(t_{i_1\dots i_k 1}) = L$. If such $t_{i_1\dots i_k 1}$ exists, it is unique and we define then $p_{i_1\dots i_k 1}$ such that $p_{i_1\dots i_k 1} = (L, t_{i_1\dots i_k 1})$. In contrast, if such $t_{i_1\dots i_k 1}$ does not exist, we do not define $t_{i_1\dots i_k 1}$, nor even $p_{i_1\dots i_k 1}$, nor $t_{i_1\dots i_k 1\dots i_l}$ and $p_{i_1\dots i_k 1\dots i_l}$ for $l > k+1$.
- We consider the Cauchy problem
$$\overset{\bullet}{y}_2(t) = \lambda_2(\boldsymbol{\xi}(y_2(t),t)), \qquad y_2(t_{i_1\dots i_k}) = L,$$
and we define $t_{i_1\dots i_k 2} \in [0, t_{i_1\dots i_k})$ by $y_2(t_{i_1\dots i_k 2}) = 0$. Again, if such $t_{i_1\dots i_k 2}$ exists, it is unique and we define then $p_{i_1\dots i_k 2}$ such that $p_{i_1\dots i_k 2} = (0, t_{i_1\dots i_k 2})$. However if such $t_{i_1\dots i_k 2}$ does not exist, we do not define $t_{i_1\dots i_k 2}$, nor even $p_{i_1\dots i_k 2}$, nor $t_{i_1\dots i_k 2\dots i_l}$ and $p_{i_1\dots i_k 2\dots i_l}$ for $l > k+1$.

Similarly to Claim 4.1, using (1), (3), Theorem 4.1 and by construction of $t_{i_1\dots i_{k+1}}$ and $p_{i_1\dots i_{k+1}}$, we have

*Claim 4.2:* There exist $\varepsilon_4$ in $(0,\varepsilon_3)$ and $H_4 \in (0,H_3)$ such that, for all $0 < H < H_4$, for all continuously differentiable functions $\mathbf{h}$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (3) and (28), and for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_4)$,

$$|\xi_1(0,t_{i_1\dots i_k})| \leq |\xi_1(p_{i_1\dots i_k 1})| + c(\tau_2)\tau_2(H+\varepsilon_2)|\boldsymbol{\xi}^\#|_{\mathcal{C}^0(0,L)}, \quad (32)$$

and

$$|\xi_2(L,t_{i_1\dots i_k})| \leq |\xi_2(p_{i_1\dots i_k 2})| + c(\tau_2)\tau_2(H+\varepsilon_2)|\boldsymbol{\xi}^\#|_{\mathcal{C}^0(0,L)}. \quad (33)$$

Note that, due to (19), there exists a finite number of $k \geq 1$ such that

$$s_{i_1} + \dots + s_{i_k} \leq \tau_2 - \tau_1 .$$

Similarly to Claim 4.1, by using Equations (15), (19), (20), (31), (32), (33), and Theorem 4.1, we can prove the following technical result.

*Claim 4.3:* There exist $\varepsilon_5 \in (0,\varepsilon_4)$ and $H_5 \in (0,H_4)$, such that, for all $0 < H < H_5$, for all continuously differentiable functions $\mathbf{h}$: $B(\varepsilon_2) \to \mathbb{R}$ satisfying (3) and (28), for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_5)$, for all integer $k \geq 1$, and for all $(i_1,\dots,i_k,i_{k+1}) \in \{1,2\}^{k+1}$ such that $s_{i_1} + \dots + s_{i_k} \leq \tau_2 - \tau_1$, we have the existence of $t_{i_1\dots i_k i_{k+1}}$ and $p_{i_1\dots i_k i_{k+1}}$. Moreover

$$|\xi_{i_k}(p_{i_1\dots i_k})| \leq \textstyle\sum_{j\neq i_k} a_{i_k j}|\xi_j(p_{i_1\dots i_k j})| + c(\tau_2)\tau_2(H+\varepsilon_2)|\boldsymbol{\xi}^\#|_{\mathcal{C}^0(0,L)}.$$

By using Theorem 4.1, Equations (17), (19) and (20), we may prove the follwing

*Claim 4.4:* There exist $\varepsilon_6 \in (0,\varepsilon_5)$ and $H_6 \in (0,H_5)$, such that, for all $0 < H < H_6$, for all continuously differentiable functions $\mathbf{h}$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (3) and (28), for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_6)$, for all integer $k \geq 1$, and for all $(i_1,\dots,i_k,i_{k+1}) \in \{1,2\}^{k+1}$ such that

$$\tau_2 - 2\tau_1 \leq s_{i_1} + \dots + s_{i_k} \leq \tau_2 - \tau_1 ,$$

we have the existence of $t_{i_1\dots i_k i_{k+1}}$ and the estimation $t_{i_1\dots i_k} \in [0, 2\tau_1]$. Moreover

$$|\boldsymbol{\xi}(p_{i_1\dots i_k})| \leq c(2\tau_1)|\boldsymbol{\xi}^\#|_{C^0(0,L)}. \quad (34)$$

Let us state the following

*Claim 4.5:* For all $l$ in $\mathbb{N}$, there exists $\delta_l > 0$ such that we have $(\mathcal{P}_l)$:

For all $0 < H < H_6$, for all continuously differentiable functions $\mathbf{h}$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (3) and (28), for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_6)$, $\forall (i_1,\dots,i_l) \in \{1,2\}^l$ such that $s_{i_1} + \dots + s_{i_l} \leq \tau_2 - 2\tau_1$, we have

$$\begin{aligned}|\xi_{i_l}(p_{i_1\dots i_l})| \leq \sum_{k\geq l} \sum_{\tau_2 - 2\tau_1 \leq s_{i_1}+\dots+s_{i_l}+s_{i_{l+1}}+\dots+s_{i_k} \leq \tau_2-\tau_1} \\ \sum_{j=1}^{2} a_{i_l i_{l+1}} a_{i_{l+1} i_{l+2}} \cdots a_{i_k j} |\xi_j(p_{i_1\dots i_k j})| \\ + \delta_l (H+\varepsilon_2)|\boldsymbol{\xi}^\#|_{\mathcal{C}^0(0,L)} .\end{aligned}$$

Moreover we may assume that

$$\delta_1 = c(\tau_2)\tau_2(2a)^{\frac{\tau_2-\tau_1}{\min\{s_1,s_2\}}+1}. \quad (35)$$

*C. Estimation of $|\partial_x \boldsymbol{\xi}(.,t)|_{C^0(0,L)}$*

Let $\boldsymbol{\eta}$: $[0,L]\times[0,\tau_2] \to \mathbb{R}^2$ be defined by $\boldsymbol{\eta} = \bar{\Lambda}\frac{\partial \boldsymbol{\xi}}{\partial x}$ where $\boldsymbol{\xi} \in C^1([0,L]\times[0,\tau_2];\mathbb{R}^2)$ is defined by $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_6)$, (1), (15) and (7) as above.

Similarly let us define $\boldsymbol{\eta}_1$: $[0,L]\times[0,\tau_2] \to \mathbb{R}$ and $\boldsymbol{\eta}_2$: $[0,L]\times[0,\tau_2] \to \mathbb{R}$ defined respectively by $\boldsymbol{\eta} = (\boldsymbol{\eta}_1,\ \boldsymbol{\eta}_2)^T$.

Differentiating (1) with respect to $x$, it follows that

$$\partial_t \boldsymbol{\eta} + \bar{\Lambda}\Lambda(\boldsymbol{\xi})\bar{\Lambda}^{-1}\partial_x \boldsymbol{\eta} = -\bar{\Lambda}(\Lambda'(\boldsymbol{\xi})\partial_x\boldsymbol{\xi})\partial_x\boldsymbol{\xi} + \bar{\Lambda}\nabla\mathbf{h}(\boldsymbol{\xi})\partial_x\boldsymbol{\xi} , \quad (36)$$

along the characteristics.

Moreover, differentiating (15) and using (1), it gives

$$\begin{pmatrix} (-\Lambda(\xi)\bar{\Lambda}^{-1}\boldsymbol{\eta} + \mathbf{h}(\boldsymbol{\xi}))_1(L,t) \\ (-\Lambda(\xi)\bar{\Lambda}^{-1}\boldsymbol{\eta} + \mathbf{h}(\boldsymbol{\xi}))_2(0,t) \end{pmatrix} = \nabla\mathbf{g}\begin{pmatrix} \boldsymbol{\xi}_1(0,t) \\ \boldsymbol{\xi}_2(L,t) \end{pmatrix}\begin{pmatrix} (-\Lambda(\xi)\bar{\Lambda}^{-1}\boldsymbol{\eta} + \mathbf{h}(\boldsymbol{\xi}))_1(0,t) \\ (-\Lambda(\xi)\bar{\Lambda}^{-1}\boldsymbol{\eta} + \mathbf{h}(\boldsymbol{\xi}))_2(L,t) \end{pmatrix} .$$

A development similar to $\boldsymbol{\xi}$ can be used as for $\xi_i$ along the trajectories of (36). It can be shown from Theorem 4.1, and (18), (26) and (27), that there exist $\varepsilon_7 \in (0,\varepsilon_6)$ and $H_7 \in (0,H_6)$ such that, for all $0 < H < H_7$, for all continuously differentiable functions $\mathbf{h}$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (3) and (28), for all $\boldsymbol{\xi}^\# \in B_{\mathcal{C}}(\varepsilon_7)$ then, we have with (36), for all $i \in \{1,2\}$,

$$\begin{aligned}|\eta^i(y(t_2),t_2) - \eta^i(y(t_1),t_1)| \leq \omega|\boldsymbol{\xi}^\#|^2_{C^1(0,L)}|t_2-t_1| \\ + \omega(H+\varepsilon_2)|\boldsymbol{\xi}^\#|_{C^1(0,L)}|t_2-t_1|.\end{aligned}$$

Hence, using the computations of Section IV-B, it follows the following

*Lemma 4.3:* There exist $\varepsilon_7$ and $H_7 > 0$ such that, for all $0 < H < H_7$, for all continuously differentiable functions $\mathbf{h}$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (3) and (28), for all $\boldsymbol{\xi}^{\#} \in B_{\mathcal{C}}(\varepsilon_7)$, we have

$$|\boldsymbol{\eta}(., \tau_2)|_{C^0(0,L)} \leq (\nu + 2\omega)(H + \varepsilon_2)|\boldsymbol{\xi}^{\#}|_{C^1(0,L)}. \quad (37)$$

*D. Sketch of the proof of Theorem 2.1 for boundary conditions (15)*

In this section, we sketch the proof of Theorem 2.1 for the special boundary conditions (15) instead of (4).

Let $\nu' \in (0, 1)$. Up to reducing $\varepsilon_2$, there exists $H_8 \in (0, H_7)$ such that

$$(\nu + \max(\mu, 2\omega))(H_8 + \varepsilon_2) < \nu' \ .$$

We combine (30) and (37) to get the existence of $\varepsilon_8 \in (0, \varepsilon_7)$, such that, for all $0 < H < H_8$, for all continuously differentiable functions $\mathbf{h}$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (3) and (28), for all $\boldsymbol{\xi}^{\#} \in B_{\mathcal{C}}(\varepsilon_8)$, we have

$$|\boldsymbol{\xi}(., \tau_2)|_{\mathcal{C}^1(0,L)} \leq \nu' |\boldsymbol{\xi}^{\#}|_{\mathcal{C}^1(0,L)} \ .$$

This estimate allows a repeated application of Theorem 4.1 to give, for all $0 < H < H_8$, for all continuously differentiable functions $\mathbf{h}$: $B(\varepsilon_2) \to \mathbb{R}^2$ satisfying (3) and (28), for all $\boldsymbol{\xi}^{\#} \in B_{\mathcal{C}}(\varepsilon_8)$, the existence of a unique solution of (1), (7) and (15) over any interval $[0, n\tau_2]$ with $n \in \mathbf{N} \setminus \{0\}$ and

$$|\boldsymbol{\xi}(., n\tau_2)|_{\mathcal{C}^1(0,L)} \leq \nu'^n |\boldsymbol{\xi}^{\#}|_{\mathcal{C}^1(0,L)} \ .$$

Thus, by letting $C_1 = \max(c(\tau_2), 1)e^{-\ln \nu'}$ and $\mu = -\frac{\ln(\nu')}{\tau_2}$, we get (8).

*E. Conclusion of the proof of Theorem 2.1*

In the previous section, we have proved Theorem 2.1 if the boundary conditions have the special form (15). To prove Theorem 2.1 for the boundary conditions (4), the size of the state can be doubled as done in [6, Proof of Theorem 6]. For more details see [14].

## V. Conclusion

The aim of this paper is to address the two following problems.

Firstly, we state a sufficient condition for the stability of non-homogeneous systems of two conservation laws, when the non-homogeneous part is small in $\mathcal{C}^1$-norm. This sufficient criterion is written in terms of the boundary conditions and it is proved thanks to an analysis of the Riemann coordinates.

Secondly, we apply this general condition to the case of the regulation of the water level and the water flow rate in an open channel. The evolution of the flow is described by using the Saint-Venant equations perturbed by small nonhomegeneous terms that account for the friction effects as well as external supplies or withdrawals. Our general sufficient condition allows us to design stabilizing boundary controls of the canal.

The complete version of this paper [14] will also investigate the generalization of these results to a network of $n$ open-channels with $n > 2$. This asks for a study of hyperbolic systems of larger dimension than those considered in this paper.

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