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Closed loop analysis of control systems under scarce measurements

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Abstract— **In this paper, the analysis of the closed-loop dynamics of a control system with scarce random sampling is addressed. The control strategy is based on the use of a conventional single-rate controller combined with a model based output estimator that estimates the unmeasured inter-sampling outputs at the control (fast) rate. The proposed estimator takes into account the past measured outputs, as a difference with inferential control schemes. The separation principle between the estimator dynamics and the closed-loop one is demonstrated. The design of the control system is simplified, since the controller and the estimator can be designed separately. Some examples illustrate how an adequate estimator design improves the closed-loop performance with respect to the inferential control scheme (open loop estimator), solving the problem of instability that appears when the open loop system is unstable.**

I. INTRODUCTION

In many industrial applications the control signal is updated at a fixed rate T, but the output is measured with a different timing pattern and, sometimes, by various sensors, each one having a maybe different sampling rate, and reliability. In some practical cases the output is not available at every sampling time due to computer overload, communication errors, shared or slow sensors or event-driven sensors. Different authors have dealt with the modelling of such systems when the measurement pattern is periodic [1], [4] based on the definition of a model that relates outputs measured at one rate with inputs updated at another rate. This allows, for example, to tackle the problem of the design of a dual-rate control system [2], but the multirate approach cannot deal with random sampling.

A general approach to deal with missing data operation is to explicitly estimate the outputs at the instants when they are unavailable, in order to apply standard control or parameter estimation techniques. This is the idea of the inferential control, where a model of the plant is used to estimate the outputs. In many cases, like in [7], the missing data are estimated by running the plant model in open loop, using the measurements whenever they are available. It is clear that if the uncontrolled plant is unstable and the measurements are very scarce, the controlled plant stability cannot be guaranteed.

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This paper deals with the analysis of a control scheme where a conventional single-rate controller is fed back with the output of a model based estimator that estimates the unmeasured outputs at period T from the scarcely and irregularly measured data from the actual plant. This estimator (defined in [8] and [9]) takes into account the previous measurements, improving the disturbance rejection and the dynamic behaviour of the closed-loop. The open loop estimator used in [5], [6] and [7] is a particular case of the more general one used here.

The main contribution of the paper is to provide tools to determine the closed-loop stability of (unstable) controlled plants where an estimator is used to compute the output under scarce and irregular measurement operation. The main result is a separation principle, that allows the design of the conventional controller and the output estimator to be dealt with independently. Furthermore, some hints to design the estimator are presented.

The layout of this paper is as follows: In section 2 the control scheme is described, including the estimator algorithm. In section 3 the dynamics of the estimator is analysed, giving a design hint to compute the estimator gain. In section 4 the equation defining the dynamics of the closed-loop is derived. The separation principle is demonstrated in section 5. Some numerical examples are analysed on section 6 showing how the use of an adequate estimator improves the response of the controlled system. Finally on section 7 the main conclusions are summarized.

II. PROBLEM STATEMENT

A. Control scheme

Consider the digital control system shown in figure 1, where $G(s)$ is a continuous time (CT) SISO linear process whose input is updated at period T by a computer with a zero-order hold. A new input $u[t]$ arrives every control period, where t is the number of input update. The relationship between the discrete signal and the continuous one is

$$u[t] = u_c(t\,T). \tag{1}$$

The process output $y_c(\tau)$ is measured synchronously with the input update at a different and no necessarily constant period. The instants where the output $y[t]$ is available are given by the sequence $\{t_0, t_1, \ldots, t_k, \ldots\}$, where instant t_k indicates the time when the k-th output measurement is available ($y_k = y[t_k]$). The number of input updates (control periods) between measurements is $N_k = t_k - t_{k-1} \geq 1$.

In the control scheme, $C(z)$ represents the conventional linear digital controller that operates at period T and $\hat{y}[t]$ is

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the estimated output that is sent to the controller to compute the control action.

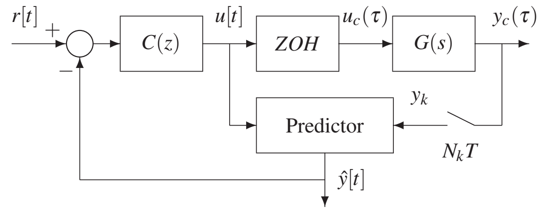

Fig. 1. Control scheme.

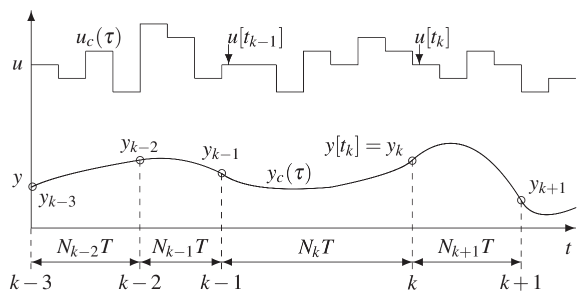

Fig. 2. Synchronous random sampling.

B. Plant

The discrete ZOH equivalent of $G(s)$ at period T, is defined by the difference equation

$$y[t] = -\boldsymbol{\theta}_a^{\intercal} Y[t-1] + \boldsymbol{\theta}_b^{\intercal} U[t-1], \tag{2}$$

where $\boldsymbol{\theta}_a = [a_1 \cdots a_n]^{\intercal}$ and $\boldsymbol{\theta}_b = [b_1 \cdots b_n]^{\intercal}$ are the parameter vectors, $Y[t-1] = [y[t-1] \cdots y[t-n]]^{\intercal}$ is the output vector and $U[t-1] = [u[t-1] \cdots u[t-n]]^{\intercal}$ is the input vector (whose elements are as defined in (1)). Equation (2) can be rewritten as[1]

$$\begin{aligned} Y[t] &= \begin{bmatrix} -\boldsymbol{\theta}_a^{\intercal} \\ \boldsymbol{I}_{(n-1)\times(n)} \end{bmatrix} Y[t-1] + \begin{bmatrix} \boldsymbol{\theta}_b^{\intercal} \\ \mathbf{0}_{(n-1)\times(n)} \end{bmatrix} U[t-1] \\ &= \boldsymbol{A} Y[t-1] + \boldsymbol{B} U[t-1]. \end{aligned} \tag{3}$$

C. Estimator

The digital controller $C(z)$ needs the sequence of outputs at period T, but the process output is measured irregularly at a slower rate. Therefore, the missing inter-sampling measurements have to be estimated. For that purpose, a model based output predictor has been added in the closed-loop. The estimator (introduced in [3] and analysed in depth in [8]) uses the parameters of the process model, and all the previous inputs, measurements and past output estimates to obtain the estimation of the unknown outputs. When there is no measurement available, the output is estimated running the model in open loop, leading to

$$\hat{Y}[t|t-1] = \boldsymbol{A}\hat{Y}[t-1] + \boldsymbol{B}U[t-1], \tag{4a}$$

where $\hat{Y}[t|t-1] = [\hat{y}[t|t-1] \cdots \hat{y}[t-n+1|t-1]]^{\intercal}$ is the initial estimation of the output regression vector, and $\hat{Y}[t] = \hat{Y}[t|t]$ represents the updated one. Depending on the availability of a new measurement, the estimated output regression vector is updated by

$$\hat{Y}[t] = \hat{Y}[t|t] = \hat{Y}[t|t-1] + \boldsymbol{l}\left(y[t] - \hat{y}[t|t-1]\right)\delta[t] \tag{4b}$$

where $\delta[t]$ is the availability factor ($\delta[t] = 1$ if the measurement is available and $\delta[t] = 0$ if not). The vector gain $\boldsymbol{l} = [l_1 \cdots l_n]^{\intercal}$ must be designed to assure the estimation error dynamics stability. The output estimation is the first value of $\hat{Y}[t]$ given by $\hat{y}[t] = \boldsymbol{h}\hat{Y}[t]$, with $\boldsymbol{h} = [1\,0 \cdots 0]$.

D. Digital controller

The conventional single-rate digital controller $C(z)$ is assumed to be a LTI SISO discrete system of order n, operating at period T. The reference signal will be assumed to be null (it is not relevant to obtain the closed-loop dynamics). The controller difference equation can then be expressed as

$$u[t] = -\boldsymbol{\theta}_p^{\intercal} U[t-1] + \boldsymbol{\theta}_q^{\intercal} \widetilde{Y}[t-1] + q_0\,\hat{y}[t], \tag{5}$$

where $\boldsymbol{\theta}_p = [p_1 \cdots p_n]^{\intercal}$, $\boldsymbol{\theta}_q = [q_1 \cdots q_n]^{\intercal}$ and q_0 are the controller parameters[2], and $\widetilde{Y}[t-1] = [\hat{y}[t-1] \cdots \hat{y}[t-n]]^{\intercal}$ is the history of the fed back output estimations at each control period (note the difference with $\hat{Y}[t]$). Equation (5) can be written in matrix form as

$$\begin{aligned} U[t] &= \begin{bmatrix} -\boldsymbol{\theta}_p^{\intercal} \\ \boldsymbol{I}_{(n-1)\times(n)} \end{bmatrix} U[t-1] + \begin{bmatrix} \boldsymbol{\theta}_q^{\intercal} \\ \mathbf{0}_{(n-1)\times(n)} \end{bmatrix} \widetilde{Y}[t-1] \\ &\quad + q_0 \boldsymbol{h}^{\intercal}\boldsymbol{h}\hat{Y}[t] \\ &= \boldsymbol{P}U[t-1] + \boldsymbol{Q}\widetilde{Y}[t-1] + q_0\boldsymbol{H}\hat{Y}[t], \end{aligned} \tag{6}$$

where $\boldsymbol{H} = \boldsymbol{h}^{\intercal}\boldsymbol{h}$ is a $n \times n$ null matrix with the element (1,1) equal 1.

III. Estimator dynamics

In this section, a sufficient condition for the estimator stability is obtained. First, the dynamics of the estimation error es obtained.

Lemma 1 (Estimation error dynamics): The prediction error dynamics of the algorithm (4) applied to system (3) when there is no modelling error is defined by the linear time-variant system (updated every measuring instant)

$$E_k = (\boldsymbol{I} - \boldsymbol{l}\boldsymbol{h})\boldsymbol{A}^{N_k} E_{k-1} = \boldsymbol{M}_k E_{k-1} \tag{7}$$

where N_k may change arbitrarily with time. The estimation error vector is defined when the measurement is available ($t = t_k$) as $E_k = Y[t_k] - \hat{Y}[t_k]$.

Proof 1: When $t = t_k$, equation (4b) can be expressed as

$$\begin{aligned} \hat{Y}[t_k] = &\boldsymbol{A}\hat{Y}[t_k-1] + \boldsymbol{B}U[t-1] \\ &+ \boldsymbol{l}\boldsymbol{h}\left(Y[t_k] - \boldsymbol{A}\hat{Y}[t_k-1] - \boldsymbol{B}U[t_k-1]\right). \end{aligned}$$

[1] $\boldsymbol{I}_{(n-1)\times(n)}$ are the $n-1$ first rows of the nxn identity matrix, and $\mathbf{0}_{(n-1)\times(n)}$ is a null matrix of order $(n-1)\times(n)$.

[2] If the process and the controller polynomials have different orders, n will be the higher one, and the vectors of parameters are assumed to be completed by zeros.

Subtracting this expression from $Y[t_k]$, one obtains

$$E_k = (\boldsymbol{I} - \boldsymbol{l}\boldsymbol{h})\left(Y[t_k] - \boldsymbol{A}\hat{Y}[t_k-1] - \boldsymbol{B}U[t_k-1]\right).$$

If equations (3) and (4a) are applied recursively, $Y[t_k]$ and $\hat{Y}[t_k-1]$ can be expressed as a function of the regression vectors in the previous sampling time (i.e, $Y[t_{k-1}]$ and $\hat{Y}[t_{k-1}]$) leading to

$$E_k = (\boldsymbol{I} - \boldsymbol{l}\boldsymbol{h})\boldsymbol{A}^{N_k}\left(Y[t_{k-1}] - \hat{Y}[t_{k-1}]\right).$$

When a periodic sampling is addressed ($N_k = N$, where N is a constant value), the predictor error dynamics of the algorithm (4) is stable if and only if the eigenvalues of the matrix

$$\boldsymbol{M} = (\boldsymbol{I} - \boldsymbol{l}\boldsymbol{h})\boldsymbol{A}^N$$

are inside the unit circle. When a time-varying irregular sampling is addressed (N_k varies arbitrarily with time), the previous condition established for all k is not necessary neither sufficient, and the nature of the time-variant matrix $\boldsymbol{M}_k$ in (7) must be taken into account. The number of input updates between available measurements is assumed to vary in a given finite set

$$N_k \in \mathscr{N} = \{\nu_1, \ldots, \nu_m\}. \tag{8}$$

Theorem 1: Consider the system (3). If there exist one matrix $\boldsymbol{P} = \boldsymbol{P}^\intercal \in \mathbb{R}^{n\times n}$ such that the set of LMIs

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{P}(\boldsymbol{I}-\boldsymbol{l}\boldsymbol{h})\boldsymbol{A}^{\nu_i} \\ ((\boldsymbol{I}-\boldsymbol{l}\boldsymbol{h})\boldsymbol{A}^{\nu_i})^\intercal\boldsymbol{P} & \boldsymbol{P} \end{bmatrix} \succ 0, \tag{9}$$

for $i = 1, \ldots, m$ is feasible, then the prediction error of the algorithm defined by (4) when there is one measurement available every N_k, belonging to the set (8), converges asymptotically to zero.

Proof 2: Applying Schur complements in (9)

$$((\boldsymbol{I}-\boldsymbol{l}\boldsymbol{h})\boldsymbol{A}^{\nu_i})^\intercal\boldsymbol{P}(\boldsymbol{I}-\boldsymbol{l}\boldsymbol{h})\boldsymbol{A}^{\nu_i} - \boldsymbol{P} \prec 0, \tag{10}$$

holds for $i = 1, \ldots, m$. Defining the Lyapunov function $L_k = E_k^\intercal\boldsymbol{P}E_k$, the existence of a matrix $\boldsymbol{P} \succ 0$ such that $L_k < L_{k-1}, \forall k$ is a sufficient condition to assure the convergence of the estimation error. Using the estimation error dynamic equation (7), this is equivalent to condition

$$\boldsymbol{M}_k^\intercal\boldsymbol{P}\boldsymbol{M}_k - \boldsymbol{P} \prec 0, \tag{11}$$

for any instant k. As (10) holds, and as (9) also implies $\boldsymbol{P} \succ 0$, condition (11) also holds and quadratic stability is then proved.

Remark 1: The design of the output estimator gain can be addressed by pole placement using a constant value N (for example the average of the possible values of N_k), and then checking the stability over all possible N_k with the help of theorem 1. If the set of LMI is not feasible, then the assigned poles, or the constant N can be changed till one gain l is found such that the estimator is stable. Other LMI based design strategies are being developed but they are out of the goal of this work.

IV. Closed loop dynamics

In this section, the time varying matrix defining the global closed-loop dynamics is obtained. For this purpose a relationship between the vectors of inputs, outputs and estimation errors is established, first at the control period T and then at the global time varying period N_kT.

A. Closed loop dynamics under standard sampling

Consider a standard sampled data closed-loop system running at period T with the output measured every time the control input is updated and therefore, no predictor is used. In this case the controller only receives measured outputs ($\hat{y}[t] = y[t]$), so E_k is a null vector. The controller equation (6) can be combined with equation (3) leading to

$$U[t] = (\boldsymbol{P} + q_0\boldsymbol{H}\boldsymbol{B})U[t-1] + (\boldsymbol{Q} + q_0\boldsymbol{H}\boldsymbol{A})Y[t-1]. \tag{12}$$

Combining the process (3) and the controller (12) equations, the closed-loop dynamics of the system can be expressed as

$$\begin{bmatrix} Y[t+1] \\ U[t+1] \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{Q}+q_0\boldsymbol{H}\boldsymbol{A} & \boldsymbol{P}+q_0\boldsymbol{H}\boldsymbol{B} \end{bmatrix} \begin{bmatrix} Y[t] \\ U[t] \end{bmatrix} \equiv \boldsymbol{M}_{\text{CL}} \begin{bmatrix} Y[t] \\ U[t] \end{bmatrix}, \tag{13}$$

where the eigenvalues of matrix $\boldsymbol{M}_{\text{CL}}$ define the closed-loop behaviour. If they are inside the unit circle, stability is assured.

B. Feedback signal reconstruction

In order to express the closed-loop dynamics of the system with missing outputs estimator, the four equations (3), (4) and (6) must be joined together. For this purpose the feedback signal vector $\widetilde{Y}[t]$ must be expressed as a function of the ouput vector $Y[t]$ and the estimation error $E[t]$. As $\hat{Y}[t]$ contains the updated estimations $\hat{y}[t-i|t]$, they must be related with the fedback ones $\hat{y}[t-i] = \hat{y}[t-i|t-i]$. The second element in $\hat{Y}[t]$ is

$$\hat{y}[t-1|t] = \hat{y}[t-1] + l_2\left(y[t] - \hat{y}[t|t-1]\right)\delta[t],$$

being

$$\hat{y}[t|t-1] = \frac{1}{1 - l_1\delta[t]}(\hat{y}[t] - l_1 y[t]\delta[t]),$$

that leads to

$$\hat{y}[t-1|t] = \hat{y}[t-1] + \frac{l_2\delta[t]}{1-l_1}(y[t] - \hat{y}[t]).$$

When expressing the third element of $\hat{Y}[t]$ as

$$\hat{y}[t-2|t] = \hat{y}[t-2|t-1] + l_3\left(y[t] - \hat{y}[t|t-1]\right)\delta[t]$$

the elements $\hat{y}[t|t-1]$ and $\hat{y}[t-2|t-1]$ must be obtained as a function of $\hat{y}[t]$, $\hat{y}[t-1]$ and $\hat{y}[t-2]$ leading to

$$\hat{y}[t-2|t] = \hat{y}[t-2] + \frac{l_3(y[t] - \hat{y}[t])\delta[t] + l_2(y[t-1] - \hat{y}[t-1])\delta[t-1]}{1-l_1}.$$

For a generic entry in $\hat{Y}[t]$ the above expression takes the form

$$\hat{y}[t-i|t] = \hat{y}[t-i] + \sum_{j=0}^{i-1}\frac{l_{i-j+1}}{1-l_1}\left(y[t-j] - \hat{y}[t-j]\right)\delta[t-j]$$

$i = 1, \ldots, n-1$. This set of equalities can be joined in the expression

$$\hat{Y}[t] = \boldsymbol{R}[t]\widetilde{Y}[t] + (\boldsymbol{I} - \boldsymbol{R}[t])Y[t]$$

with

$$\boldsymbol{R}[t] = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \frac{-l_2\,\delta[t]}{1-l_1} & 1 & \ddots & & \vdots \\ \frac{-l_3\,\delta[t]}{1-l_1} & \frac{-l_2\,\delta[t-1]}{1-l_1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{-l_n\,\delta[t]}{1-l_1} & \frac{-l_{n-1}\,\delta[t-1]}{1-l_1} & \cdots & \frac{-l_2\,\delta[t-n+2]}{1-l_1} & 1 \end{bmatrix}. \quad (14)$$

The relationship between feedback regressor and estimation error can then be established as

$$\widetilde{Y}[t] = Y[t] - \boldsymbol{W}[t]E[t] \quad (15)$$

where $\boldsymbol{W}[t]$ is the inverse of $\boldsymbol{R}[t]$ defined as

$$\boldsymbol{W}[t] = \boldsymbol{R}[t]^{-1} = 2\boldsymbol{I} - \boldsymbol{R}[t], \quad (16)$$

C. Closed loop dynamics under random sampling

Theorem 2: Consider the control scheme shown in figure 1 where the process is assumed to be a CT LTI SISO system defined by equation (3). The controller is assumed to be a discrete LTI system operating at period T and defined by equation (6), and the estimator is defined by equations (4). Assume also a synchronous random sampling scenario as shown in figure 2. Then, the closed-loop dynamics of the resulting linear time-variant system updated every measuring instant is defined by

$$\boldsymbol{\xi}[t_k] = \boldsymbol{\Gamma}\left(\prod_{j=1}^{N_k} \boldsymbol{M}[t_k - j]\right)\boldsymbol{\xi}[t_{k-1}] \equiv \boldsymbol{M}_{\text{pred}}(k)\,\boldsymbol{\xi}[t_{k-1}], \quad (17)$$

with the global state vector $\boldsymbol{\xi}[t] = [E[t]^\intercal\, Y[t]^\intercal\, U[t]^\intercal]^\intercal$ and where matrix $\boldsymbol{M}[t]$ is defined by

$$\boldsymbol{M}[t] = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{B} \\ -(\boldsymbol{Q}\boldsymbol{W}[t] + q_0\boldsymbol{H}\boldsymbol{A}) & \boldsymbol{Q} + q_0\boldsymbol{H}\boldsymbol{A} & \boldsymbol{P} + q_0\boldsymbol{H}\boldsymbol{B} \end{bmatrix}, \quad (18)$$

and matrix $\boldsymbol{\Gamma}$ is defined by

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{I} - \boldsymbol{l}\boldsymbol{h} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ q_0\, l_1\, \boldsymbol{H} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \quad (19)$$

Proof 3: For an arbitrary instant $t \neq t_k$ without measurement, predictor (4) is applied with $\delta[t] = 0$ and the controller equation (6) is expressed as

$$U[t] = \boldsymbol{P}U[t-1] + \boldsymbol{Q}\widetilde{Y}[t-1] + q_0\boldsymbol{H}(\boldsymbol{A}\hat{Y}[t-1] + \boldsymbol{B}U[t-1]). \quad (20)$$

Applying expression (15) to vector $\widetilde{Y}[t-1]$

$$\begin{aligned} U[t] = &(\boldsymbol{P} + q_0\boldsymbol{H}\boldsymbol{B})U[t-1] + (\boldsymbol{Q} + q_0\boldsymbol{H}\boldsymbol{A})Y[t-1] \\ &- (\boldsymbol{Q}\boldsymbol{W}[t-1] + q_0\boldsymbol{H}\boldsymbol{A})E[t-1]. \end{aligned} \quad (21)$$

and grouping equations (3), (4) and (21) the following expression is obtained when there is no measurement available, i.e., when $t = t_{k-1}+1, \ldots, t_k - 1$

$$\boldsymbol{\xi}[t] = \boldsymbol{M}[t-1]\,\boldsymbol{\xi}[t-1] \quad (22)$$

When a new measurement is available at instant $t = t_k$, expression (6) leads to

$$\begin{aligned} U[t] = &(\boldsymbol{P} + q_0\boldsymbol{H}\boldsymbol{B})U[t-1] + (\boldsymbol{Q} + q_0\boldsymbol{H}\boldsymbol{A})Y[t-1] \\ &- (\boldsymbol{Q}\boldsymbol{W}[t-1] + q_0\boldsymbol{H}\boldsymbol{A})E[t-1] + q_0\boldsymbol{H}\boldsymbol{l}\boldsymbol{h}\boldsymbol{A}E[t-1], \end{aligned} \quad (23)$$

where $q_0\,\boldsymbol{H}\,\boldsymbol{l}\,\boldsymbol{h} = q_0\,l_1\,\boldsymbol{H}$ due to the special form of matrices $\boldsymbol{h}$ and $\boldsymbol{H}$. Grouping equations (3), (4) and (23) the next expression can be written

$$\boldsymbol{\xi}[t] = \boldsymbol{\Gamma}\,\boldsymbol{M}[t-1]\,\boldsymbol{\xi}[t-1] \quad (24)$$

Applying equation (22) recursively for $t = t_{k-1}+1, \ldots, t_k - 1$ and equation (24) for $t = t_k$, the desired expression (17) is obtained

$$\boldsymbol{\xi}[t_k] = \boldsymbol{\Gamma}\,\boldsymbol{M}[t_k - 1]\,\boldsymbol{M}[t_k - 2]\cdots\boldsymbol{M}[t_{k-1}]\,\boldsymbol{\xi}[t_{k-1}].$$

V. Separation Principle

In this section, the control scheme of figure 1 is shown to fulfil a separation principle, in the sense that the stability of the estimator (4), and the standard closed-loop system (13) are necessary and sufficient conditions to ensure the stability of the full system (17). First, an important result of a class of linear time variant systems is presented.

Lemma 2: Consider a time variant system defined by the partitioned matrix

$$\boldsymbol{M}_k = \begin{bmatrix} \boldsymbol{A}_k & \boldsymbol{0}_{(n\times 2n)} \\ \boldsymbol{B}_k & \boldsymbol{C}_k \end{bmatrix} \quad (25)$$

where the matrices $\boldsymbol{A}_k$ and $\boldsymbol{C}_k$ represent two subsystems that are part of the global system represented by $\boldsymbol{M}_k$. If there exist two matrices $\boldsymbol{Q}_1 \succ 0$ and $\boldsymbol{P}_3 \succ 0$ such that

$$\boldsymbol{A}_k\boldsymbol{Q}_1\boldsymbol{A}_k^\intercal - \boldsymbol{Q}_1 \prec 0, \quad (26)$$

$$\boldsymbol{C}_k^\intercal\boldsymbol{P}_3\boldsymbol{C}_k - \boldsymbol{P}_3 \prec 0, \quad (27)$$

then there exists one matrix $\boldsymbol{P} \succ 0$ such that

$$\boldsymbol{M}_k^\intercal\boldsymbol{P}\boldsymbol{M}_k - \boldsymbol{P} \prec 0, \quad (28)$$

and, therefore, the system $\boldsymbol{M}_k$ is stable.

Conversely, if there exists one matrix $\boldsymbol{P} \succ 0$ such that (28) holds, then there exist two matrices $\boldsymbol{Q}_1 \succ 0$ and $\boldsymbol{P}_3 \succ 0$ such that (26) and (27) are true, and, therefore, the subsystems $\boldsymbol{A}_k$ and $\boldsymbol{C}_k$ are stable.

Proof 4: First assume that (28) has a solution $\boldsymbol{P} \succ 0$, with $\boldsymbol{P}$ partitioned as blocks $\boldsymbol{P} = [\boldsymbol{P}_1, \boldsymbol{P}_2^\intercal; \boldsymbol{P}_2, \boldsymbol{P}_3]$. Due to the special structure of matrix (25) it is easily checked that the inequality (28) implies (27). If (28) has a solution $\boldsymbol{P} \succ 0$, then there also exists a solution $\boldsymbol{Q} \succ 0$ to the equation

$$\boldsymbol{M}_k\boldsymbol{Q}\boldsymbol{M}_k^\intercal - \boldsymbol{Q} \prec 0, \quad (29)$$

with $\boldsymbol{Q}$ partitioned as blocks $\boldsymbol{Q} = [\boldsymbol{Q}_1, \boldsymbol{Q}_2^\intercal; \boldsymbol{Q}_2, \boldsymbol{Q}_3]$. Again, it is easily checked that the inequality (29) implies (26).

Conversely, suppose now that there are two matrices $\boldsymbol{Q}_1 \succ 0$ and $\boldsymbol{P}_3 \succ 0$ such that (26) and (27) hold. It is evident that there also exists a solution $\boldsymbol{Z} \succ 0$ to the equation

$$\boldsymbol{A}_k^{\intercal} \boldsymbol{Z} \boldsymbol{A}_k - \boldsymbol{Z} \prec 0. \tag{30}$$

Moreover, the existence of a positive γ such that the matrix

$$\boldsymbol{P} = \begin{bmatrix} \gamma \boldsymbol{Z}_{(n\times n)} & \boldsymbol{0}_{(n\times 2n)} \\ \boldsymbol{0}_{(2n\times n)} & \boldsymbol{P}_{3(2n\times 2n)} \end{bmatrix}$$

satisfies (28), will be demonstrated, proving the stability of the closed-loop system (25). Equation (28) can be written as

$$\boldsymbol{M}_k^{\intercal} \boldsymbol{P} \boldsymbol{M}_k - \boldsymbol{P} = \begin{bmatrix} \gamma \boldsymbol{A}_k^{\intercal} \boldsymbol{Z} \boldsymbol{A}_k + \boldsymbol{B}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{B}_k - \gamma \boldsymbol{Z} & \boldsymbol{B}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{C}_k \\ \boldsymbol{C}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{B}_k & \boldsymbol{C}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{C}_k - \boldsymbol{P}_3 \end{bmatrix} \prec 0.$$

Using Schur complements it is easy to show that the above condition holds (and so, the closed-loop system (25) is stable) if γ satisfies

$$\gamma\left(\boldsymbol{A}_k^{\intercal} \boldsymbol{Z} \boldsymbol{A}_k - \boldsymbol{Z}\right) + \boldsymbol{B}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{B}_k - \boldsymbol{B}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{C}_k \left(\boldsymbol{C}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{C}_k - \boldsymbol{P}_3\right)^{-1} \boldsymbol{C}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{B}_k \prec 0. \tag{31}$$

Since (30) is satisfied and $\gamma > 0$, the first addend is negative definite. The term $\boldsymbol{B}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{B}_k$ is positive definite because $\boldsymbol{P}_3 \succ 0$. The last addend is positive definite because the inverse of a positive definite matrix is also positive definite, and because it is pre and postmultiplied by a matrix ($\boldsymbol{B}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{C}_k$) and its transpose ($\boldsymbol{C}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{B}_k$). Condition (31) is then satisfied for any $\gamma > 0$ such that $\gamma \cdot \mu > \nu$, where

$$\mu = \lambda_{\min}\left(\boldsymbol{Z} - \boldsymbol{A}_k^{\intercal} \boldsymbol{Z} \boldsymbol{A}_k\right)$$

and

$$\nu = \lambda_{\max}\left(\boldsymbol{B}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{B}_k - \boldsymbol{B}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{C}_k \left(\boldsymbol{C}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{C}_k - \boldsymbol{P}_3\right)^{-1} \boldsymbol{C}_k^{\intercal} \boldsymbol{P}_3 \boldsymbol{B}_k\right).$$

Since (30) and (27) are satisfied, such a γ always exists.

Theorem 3: Consider the same hypothesis described in theorem 2. Assume that the next conditions hold

(i) there exists one matrix $\boldsymbol{P} \succ 0$ such that (11) is satisfied (the estimator is stable), and

(ii) there exists one matrix $\boldsymbol{Q} \succ 0$ such that

$$\left(\boldsymbol{M}_{\text{CL}}{}^{N_k}\right)^{\intercal} \boldsymbol{Q} \boldsymbol{M}_{\text{CL}}{}^{N_k} - \boldsymbol{Q} \prec 0, \tag{32}$$

i.e., the conventional sampling closed loop system (defined by $\boldsymbol{M}_{\text{CL}}$) is stable.

Then, there exists one matrix $X \succ 0$ such that

$$\boldsymbol{M}_{\text{pred}}(k)^{\intercal} \boldsymbol{X} \boldsymbol{M}_{\text{pred}}(k) - \boldsymbol{X} \prec 0, \tag{33}$$

i.e., the closed loop system defined in (17) is stable.

Conversely, if there exists one matrix $\boldsymbol{X} \succ 0$ such that condition (33) is satisfied, then the estimator subsystem (7), and the conventional closed-loop subsystem (13) satisfy conditions (i) and (ii) (and therefore both are stable).

Proof 5: First, let us rewrite matrix (18) as

$$\boldsymbol{M}[t] = \left[\begin{array}{c|c} \boldsymbol{A} & [\boldsymbol{0} \quad \boldsymbol{0}] \\ \hline \boldsymbol{\Phi}[t] & \boldsymbol{M}_{\text{CL}} \end{array}\right],$$

with

$$\boldsymbol{\Phi}[t] = \begin{bmatrix} \boldsymbol{0} \\ -(\boldsymbol{Q}\boldsymbol{W}[t] + q_0 \boldsymbol{H}\boldsymbol{A}) \end{bmatrix}$$

and $\boldsymbol{M}_{\text{CL}}$ the conventional closed-loop dynamic matrix (13). The closed-loop dynamic equation (17) is then defined by the matrix

$$\boldsymbol{M}_{\text{pred}}(k) = \boldsymbol{\Gamma} \boldsymbol{M}[t_k - 1] \cdots \boldsymbol{M}[t_{k-1}] = \left[\begin{array}{c|c} (\boldsymbol{I} - \boldsymbol{l}\boldsymbol{h}) \boldsymbol{A}^{N_k} & [\boldsymbol{0} \quad \boldsymbol{0}] \\ \hline \begin{bmatrix} \boldsymbol{0} \\ q_0 l_1 \boldsymbol{H} \end{bmatrix} \boldsymbol{A}^{N_k} + \boldsymbol{\Delta}(k) & \boldsymbol{M}_{\text{CL}}{}^{N_k} \end{array}\right], \tag{34}$$

with

$$\boldsymbol{\Delta}(k) = \sum_{j=0}^{N_k} \boldsymbol{M}_{\text{CL}}{}^{N_k - j} \boldsymbol{\Phi}[t+j] A^j.$$

The matrix (34) describes a linear time variant system with predefined structure defined by all the possible combinations of the sequence $\{N_k\}$. Applying the lemma 2 to this matrix, the separation principle for the system (17) is easily demonstrated.

The previous result simplifies the procedure to check the stability of the whole system, because it is sufficient to prove the stability of the estimator and the conventional control loop. The design of the whole system is also simplified, because the estimator and the controller can be designed separately.

The control scheme studied in this paper (with the proposed estimator that takes into account the previous measurements) does not need the plant to be open loop stable in order to reach a closed-loop stable system. This is illustrated on the examples.

VI. EXAMPLES

To illustrate the previous results, the closed-loop behaviour is analysed in two different examples.

Example 1 (Control of a stable system): Consider a stable system that can be approximated by the second order transfer function $G(s) = 10/(s^2 + 2s + 8)$. The input is assumed to be updated at constant period $T = 0.06$ seconds. The DT ZOH equivalent transfer function is $G(z) = (0.0173z^{-1} + 0.0166z^{-2})/(1 - 1.86z^{-1} + 0.8869z^{-2})$. A digital controller $C(z)$ is designed by pole placement for closed-loop poles $z = \{0.85, 0.7, 0.4, 0.3, 0.2\}$, leading to $C(z) = (18.32 - 31.79z^{-1} + 13.92z^{-2})/((1 - z^{-1})(1 + 0.0937z^{-1} + 0.0161z^{-2}))$.

Assume that the outputs are measured randomly every ν input periods, with $\nu \in \mathscr{N} = \{1, \ldots, 5\}$. An estimator is designed by pole placement for stable eigenvalues (0.7,0.7) at period $N = 3$, leading to $\boldsymbol{l} = [0.8314 \; 0.7921]^{\intercal}$. In order to guarantee the stability, LMI (9) must be verified. With the help of standard LMI solvers a positive definite matrix is found

$$\boldsymbol{P} = \begin{bmatrix} 0.7068 & -0.0286 \\ -0.0286 & 0.7797 \end{bmatrix} \succ 0.$$

Applying the theorem 3 one concludes that the closed loop system is also stable (the estimator and the conventional closed loop are stable and hence the equation (33) is ver-

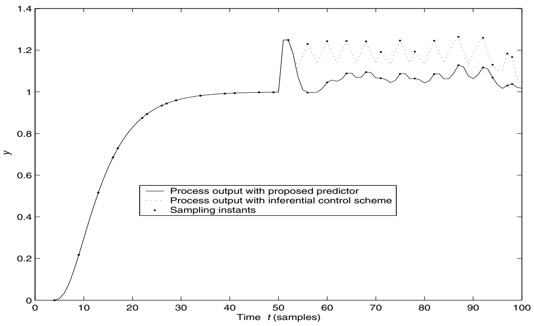

Fig. 3. Step response and output disturbance rejection (example 1).

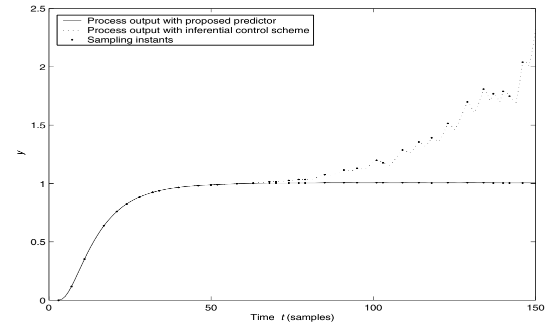

Fig. 4. Step response and input disturbance rejection (example 2).

ified). The step response and output disturbance rejection of the control system is shown in the figure 3 and is compared to the inferential control (estimator running in open loop, $\boldsymbol{l} = \mathbf{0}$).

Example 2 (Control of an unstable system): Consider now an unstable system defined by $G(s) = 10/(s^2 - s)$ with the input being updated at constant period $T = 0.03$ seconds. The DT ZOH equivalent transfer function is $G(z) = (0.0045z^{-1} + 0.0046z^{-2})/(1 - 2.0305z^{-1} + 1.0305z^{-2})$. Assume that the outputs are measured randomly every 2, 3, 4, 5 or 6 input periods, with the same probability. A digital controller $C(z)$ is designed by pole placement assigning closed-loop poles at $z = \{0.9, 0.7, 0.5\}$, leading to $C(z) = (27.05 - 25.407z^{-1})/(1 - 0.1925z^{-1})$. For the estimator, the gain of the estimator (4b) is designed for eigenvalues $\{0.7, 0.7\}$ assuming $N = 4$, leading to $\boldsymbol{l} = [0.9489\ 0.787]^{\mathsf{T}}$. The stability of the closed loop is checked by looking for feasibility on the LMI problem (9). Using standard LMI solvers it is found a matrix $\boldsymbol{P}$

$$\boldsymbol{P} = \begin{bmatrix} 20.6907 & -0.1227 \\ -0.1227 & 19.3365 \end{bmatrix} \succ 0,$$

that is positive definite. The stability of the closed loop is therefore concluded applying theorem 3. In this case, the set of LMI (9) is not feasible when running the estimator in open loop ($\boldsymbol{l} = \mathbf{0}$) because the plant is unstable, and thus the closed loop system with that strategy is also unstable.

The response of the closed-loop to a step change on the reference (at $t = 0$), and on the input disturbance (at $t = 50$) is shown in figure 4. The behaviour (disturbance rejection and reference tracking) of the closed-loop estimator strategy (–) is correct. The open loop estimator strategy (- -) leads to an unstable response due to the unstable dynamics of the process included in the closed-loop.

VII. Conclusions

In this paper, the analysis of the closed-loop dynamics of a control system with synchronous random sampling has been addressed. A control scheme consisting of a conventional single-rate controller combined with a model based output estimator that estimates the unmeasured inter-sampling outputs at the fast rate has been analysed. The proposed estimator takes into account the previous measurements (as a difference with inferential control schemes).

A sufficient condition for stability of the output estimator has been obtained in terms of the feasibility of a set of LMI. A predictor gain design strategy has been proposed based on this result.

The global closed-loop dynamic equation of the controller-estimator system has been derived, related to the measuring instants (variable hyper-period).

The main result is the demonstration of the separation principle between estimator and controller, in the sense that the stability of the estimator and the conventional control loop imply the stability of the global closed loop system. The estimator and the controller can thus be designed separately.

The proposed scheme can lead to stable closed loop even if the process is open loop unstable, as a difference with inferential control schemes (where instability of open loop process implies instability of the closed loop).

Finally, some examples illustrate the main results and show the improvement of the closed-loop performance with respect to the inferential control scheme (open loop estimator), when an adequate closed-loop estimator is designed (solving the instability problem that appears when the open loop system is unstable).

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