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Disturbance attenuation by dynamic output feedback for input-delay systems

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***Abstract*— This paper addresses the disturbance attenuation problem by output feedback for multivariable linear systems with delayed inputs. To solve this problem, a feedback compensator is used, which is decomposed into an observer part, a state predictive part, and a static feedback part. Then, the analysis of the closed loop system is made on an equivalent linear system without delay. Based on the geometric approach, we solve two different disturbance attenuation problems, providing necessary and sufficient conditions for their solvability.**

I. Introduction

This work addresses the disturbance attenuation problem by output feedback in linear multivariable systems with input delay. Time–delays appear frequently in industrial processes, economical, physiological and biological systems [14], and their presence is a consequence of delays in the process itself, or is caused by controllers (transport, communication, processing, ...).

Disturbance attenuation is a topic of recurrent interest. Among different methods well developed in the literature for solving this problem, geometric approach is an effective tool. Various versions of this problem have been solved [16], [19], [20], [21]. The solutions consist in necessary and sufficient conditions in terms of certain subspaces associated to the considered system. The computation of the subspaces effectively permits to check the solvability and to construct a solution controller.

For time-delays systems, necessary and sufficient conditions are also established for static or dynamic output feedback. The corresponding closed-loop systems have in general an infinite number of poles. Consider a linear multivariable system with delayed input

$$\dot{x}(t) = Ax(t) + Bu(t-h) + Ew(t),$$

where $h \in \mathbb{R}_+$ is the delay. The problem is to make $z = Gx$, with G of appropriate dimension, insensitive in closed–loop to the disturbance w which is not available by measurement, and where all the state is not measured.

Following Smith, Olbrot and Manitius [11], if all the state is measured, a prediction $x_p(t)$ of the state vector $x(t+h)$, is given by

$$x_p(t) = \mathrm{e}^{Ah}x(t) + \int_{t-h}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau,$$

which is available at time t. This prediction is established without taking into account the disturbance, and assuming that all the state is measured. We refer to [10] where the same idea is used to solve the disturbance decoupling problem by static state feedback, or the disturbance decoupled estimation problem by output injection.

If all the state is not measured, an observer is used to estimate the state at the current time, and then a prediction is based on this estimation.

The motivation to using such control laws is their simplicity, with a static gain feedback, and the induced properties of the closed–loop system, as invariant factors [12].

In this paper, we are interested in solving problems of disturbance attenuation by such a static output feedback coupled with a prediction equation, for linear input delay systems.

A decomposition of the closed–loop transfer function from w to z allows us to reduce this problem to a disturbance decoupling problem without delay. Geometric conditions are also given to solve them.

Different versions of the disturbance attenuation problem by output feedback are considered, namely the so-called exact and almost disturbance decoupling.

The paper is organized as follows. In Section 2, we formulate the problem under consideration. Section 3 is devoted to the analysis of the closed–loop transfer matrix. A time–decomposition of impulse response in closed–loop is established in Section 4. Necessary and sufficient geometric conditions are given in Section 5, to solve problems under interest.

Notations. We denote $A_F = A + BF$, $A^L = A + LC$. We define $\Psi_M(s) = (sI - M)^{-1}$, for any square matrix M with real entries. The Laplace transform of $(\cdot)$ is denoted by $(\hat{\cdot})(s)$. $\mathcal{A}$ denotes the Wiener algebra [5]. L_p denotes the set of the complex-valued measurable functions $g(t)$ on the non-negative real axis such that $\|g\|_{L_p}^p = \int_0^\infty |g(t)|^p\,\mathrm{d}t < \infty$, for $1 \le p < \infty$, and such that $\|g\|_{L_\infty} = \operatorname{ess\,sup}_{t\in\mathbb{R}_+} |g(t)| < \infty$.

II. Disturbance attenuation problem

Consider a linear, time-invariant, input-delay system (Σ)

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t-h) + Ew(t) \\ y(t) = Cx(t) \\ z(t) = Gx(t) \end{array} \right. , \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $w \in \mathbb{R}^d$ is an unknown disturbance, $h \in \mathbb{R}_+$ is the delay, $y \in \mathbb{R}^p$ is the measure, and $z \in \mathbb{R}^c$ is the output to be controlled. Matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $E \in \mathbb{R}^{n\times d}$, $C \in \mathbb{R}^{p\times n}$ and $G \in \mathbb{R}^{c\times n}$ have real entries. We will suppose that the disturbance w is not available by measurement, and that the pairs (A, B) and (C, A) of (1) are stabilizable and detectable respectively [15].

The problem under interest is the synthesis of a control law which guarantees an (optimal) attenuation of the disturbance effect on the output z. For this aim, the compensator which is used, is decomposed into two parts. The first one is an observer-predictor. From the measured output y, the state at time t is estimated, and a prediction based on this estimation is done. The second one is a static feedback control law, based on this state prediction. This kind of compensator involves an output feedback distributed control law.

More precisely, considering the system (Σ) described in (1), the estimate $x_o(t)$ of the state $x(t)$ at time $t \geq 0$ is given by

$$\dot{x}_o(t) = Ax_o(t) + Bu(t-h) - L(y(t) - Cx_o(t)), \qquad (2)$$

where $L \in \mathbb{R}^{n\times p}$ is such that $A^L = A + LC$ is stable.

The prediction $x_p(t)$ of $x(t+h)$, based on the estimate $x_o(t)$, and without taking into account the disturbance effect, is described, as in [13], by

$$x_p(t) = \mathrm{e}^{Ah} x_o(t) + \int_{t-h}^{t} \mathrm{e}^{A(t-\tau)} Bu(\tau)\, \mathrm{d}\tau. \qquad (3)$$

Then, the static feedback control law is build from the prediction $x_p(t)$, i.e.

$$u(t) = x_f(t) + v(t), \qquad (4)$$

where $x_f(t) = Fx_p(t)$, with a feedback matrix $F \in \mathbb{R}^{m\times n}$ such that $A_F = A + BF$ is stable, and v is a new input for the closed–loop system.

The observer-predictor compensator (Σ_c) is described by (2)-(3), and the feedback compensator (Σ_f) by (4). The synthesis of a compensator (Σ_c)-(Σ_f) to make the output to be controlled z insensitive in closed loop to the disturbance w is represented in Fig. 1. Denote $T_{wz}(s)$ the closed-loop transfer matrix from the disturbance w and the controlled output with the compensator (Σ_c)-(Σ_f) described by (2)-(3)-(4). The problem under interest is to find constant matrices F and L such that

$$\|T_{wz}\|_1 \triangleq \sup_{w\in L_1,\, w\neq 0} \frac{\|z\|_{L_1}}{\|w\|_{L_1}}$$

is minimal.

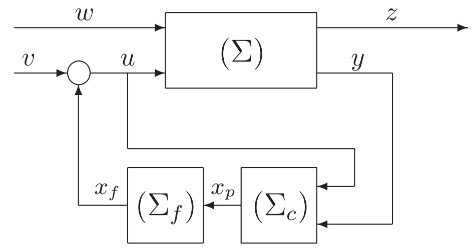

Fig. 1. Closed loop system in the disturbance attenuation problem.

III. Closed loop system

This section is devoted to characterize the input–output evolution in closed–loop from the disturbance and the controlled output. We take also $v = 0$ in (4).

Lemma 3.1: Consider the input delay system described by (1). With the compensator (Σ_c)-(Σ_f) described by (2)-(3)-(4), the closed loop transfer matrix from w and z is

$$T_{wz}(s) = T_1(s) + \mathrm{e}^{-sh} T_2(s), \qquad (5)$$

with

$$\begin{array}{rcl} T_1(s) & = & G\Psi_A(s)(I - \mathrm{e}^{-sh}\mathrm{e}^{Ah})E, \\ T_2(s) & = & G\Psi_{A_F}(s)\left[I - BF\Psi_\Xi(s)\right]\mathrm{e}^{Ah}E, \end{array}$$

where $\Xi = A + \widetilde{L}\widetilde{C}$, and

$$\begin{array}{rcl} \widetilde{L} & = & \mathrm{e}^{Ah}L \\ \widetilde{C} & = & C\mathrm{e}^{-Ah} \end{array} .$$

Proof. Denote

$$\varphi(t) = \int_{t-h}^{t} \mathrm{e}^{A(t-\theta)} Ew(\theta)\, \mathrm{d}\theta, \qquad (6)$$

and define

$$\begin{array}{rcll} e_o(t) & = & x(t) - x_o(t), & (7) \\ e_p(t) & = & x(t) - x_p(t-h), & (8) \end{array}$$

which are respectively the estimation error and the prediction error.

By direct calculations, we obtain

$$\begin{array}{rcll} \dot{e}_o(t) & = & A^L e_o(t) + Ew(t), & (9) \\ e_p(t) & = & \mathrm{e}^{Ah} e_o(t-h) + \varphi(t), & (10) \end{array}$$

where $\varphi(t)$ is given in (6). Furthermore, in closed loop, the state prediction $x_p(t)$ is governed by the differential equation

$$\dot{x}_p(t) = A_F x_p(t) - \mathrm{e}^{Ah} LCe_o(t). \qquad (11)$$

Since $x(t) = e_p(t) + x_p(t-h)$, the state $x(t)$ at time t can be written into the form

$$x(t) = \mathrm{e}^{Ah} e_o(t-h) + \varphi(t) + x_p(t-h). \qquad (12)$$

By Laplace transform, we have

$$\hat{x}(s) = \hat{\varphi}(s) + \mathrm{e}^{-sh}(\mathrm{e}^{Ah}\hat{e}_o(s) + \hat{x}_p(s)). \qquad (13)$$

With $\hat{\varphi}(s) = (sI - A)^{-1}(I - e^{Ah}e^{-sh})E\hat{w}(s)$, we obtain directly in (5) that

$$T_1(s)\hat{w}(s) = G\hat{\varphi}(s).$$

Furthermore, equations (9), (11), and (13) yield to

$$\begin{aligned} T_2(s) &= Ge^{Ah}\Psi_{A^L}(s)E \\ &\quad -G\Psi_{A_F}(s)e^{Ah}LC\Psi_{A^L}(s)E \\ &= G\Psi_{A_F}(s)\left[\Psi_{A_F}^{-1}(s)e^{Ah} - e^{Ah}LC\right]\Psi_{A^L}(s)E \\ &= G\Psi_{A_F}(s)\left[e^{Ah} - BFe^{Ah}\Psi_{A^L}(s)\right]E \\ &= G\Psi_{A_F}(s)\left[I - BF\Psi_{\Xi}(s)\right]e^{Ah}E, \end{aligned}$$

where we introduce the notation $\Xi = A + e^{Ah}LCe^{-Ah}$. □

This decomposition of the transfer function from the disturbance and the controlled output is also presented in [13], and it is shown that it allows to characterize all stabilizing controllers of the delayed system (1). The reader is also referred to [18], where the same idea is used, and in [10], where the authors solved the problem of static state distributed feedback law, and the dual case of output injection.

Remark 3.2: In the decomposition (5), the second term $T_2(s)$ can be factorized in two forms. In fact, we have

$$\begin{aligned} T_2(s) &= G\Psi_{A_F}(s)\left[I - BF\Psi_{\Xi}(s)\right]e^{Ah}E \\ &= G\Psi_{A_F}(s)\left[e^{Ah} - BFe^{Ah}\Psi_{A^L}(s)\right]E \\ &= G\Psi_{A_F}(s)\left[e^{Ah}\Psi_{A^L}^{-1}(s) - BFe^{Ah}\right]\Psi_{A^L}(s)E \\ &= G\left[e^{Ah} - \Psi_{A_F}(s)e^{Ah}LC\right]\Psi_{A^L}(s)E \\ &= Ge^{Ah}\left[I - \Psi_{\Lambda}(s)LC\right]\Psi_{A^L}(s)E, \end{aligned}$$

with $\Lambda = A + \widetilde{B}\widetilde{F}$, $\widetilde{B} = e^{-Ah}B$ and $\widetilde{F} = Fe^{Ah}$.

These two factorizations of $T_2(s)$ in (5) will be analyzed in Section V. It is also fundamental to see that in (5), the first part $T_1(s)$ is independent from any control action, like feedback or injection, whereas the second part $T_2(s)$ depends on the compensator structure. Then, the analysis of the transfer matrix $T_{wz}(s)$ can be expressed in terms of a system without delay.

Lemma 3.3: The transfer matrix $T_2(s)$ in (5) is the closed loop transfer matrix of a system without delay (Σ_2'), with a static output feedback compensator $(\Sigma_{c,2}')$.

Proof. By Lemma 3.1, we showed that in the closed loop system under interest,

$$T_2(s) = G\Psi_{A_F}(s)\left[I - BF\Psi_{\Xi}(s)\right]e^{Ah}E.$$

Clearly, this closed loop transfer function is independent from any delay. Furthermore, consider the system (Σ_2') described by

$$\left\{\begin{array}{l} \dot{\zeta}(t) = A\zeta(t) + Bu(t) + e^{Ah}Ew(t) \\ y_2(t) = Ce^{-Ah}\zeta(t) \\ z_2(t) = G\zeta(t) \end{array}\right. , \tag{14}$$

and the feedback compensator $(\Sigma_{c,2}')$

$$\left\{\begin{array}{l} \dot{\hat{\zeta}}(t) = A\hat{\zeta}(t) + Bu(t) + \widetilde{L}(\widetilde{C}\hat{\zeta}(t) - y_2(t)) \\ u(t) = F\hat{\zeta}(t) \end{array}\right. , \tag{15}$$

where $\widetilde{L} = e^{Ah}L$ and $\widetilde{C} = Ce^{-Ah}$. $\hat{\zeta}$ is an estimate of ζ, so that $(\Sigma_{c,2}')$ is decomposed in an observer part and in a static feedback part.
Then, it is easy to verify that the transfer function from w and z_2 in the closed loop (Σ_2')-$(\Sigma_{c,2}')$ coincides precisely with $T_2(s)$, i.e. (14)-(15) is a state-space realization of $T_2(s)$, which is minimal under minimality conditions of the state-space realization of (Σ). □

From Lemma 3.3 and Remark 3.2, since

$$T_2(s) = Ge^{Ah}\left[I - \Psi_{\Lambda}(s)LC\right]\Psi_{A^L}(s)E,$$

where $\Lambda = A + e^{-Ah}BFe^{Ah}$, another state-space realization (Σ_2'')-$(\Sigma_{c,2}'')$ of $T_2(s)$ can be described by

$$(\Sigma_2''): \left\{\begin{array}{l} \dot{\xi}(t) = A\xi(t) + e^{-Ah}Bu(t) + Ew(t) \\ y_2(t) = C\xi(t) \\ z_2(t) = Ge^{Ah}\xi(t) \end{array}\right. , \tag{16}$$

and $(\Sigma_{c,2}'')$ such that

$$\left\{\begin{array}{l} \dot{\hat{\xi}}(t) = A\hat{\xi}(t) + e^{-Ah}Bu(t) + L(C\hat{\xi}(t) - y_2(t)) \\ u(t) = Fe^{Ah}\hat{\xi}(t) \end{array}\right. . \tag{17}$$

These two realizations are obtained from (14)-(15) by a change of basis, i.e. $\zeta(t) = e^{Ah}\xi(t)$.
The problem of attenuating a disturbance on z by output feedback is reduced to the analysis of an equivalent linear system without delay (14), with the compensator (15), i.e. we analyze a classical problem of disturbance attenuation by output feedback on a linear time-invariant system without delay. In the next section, we describe more precisely the relation between the initial problem and the disturbance decoupling problem by output feedback on (Σ_2').

IV. Interpretation in the time-domain

The decomposition of the input–output transfer function described in Section III has an easy interpretation in the time domain. This section is devoted to describe it.
Consider a generalized function $f(t) \in \mathcal{A}$, of the form

$$f(t) = \left\{\begin{array}{ll} 0 & , \ t < 0 \\ f_a(t) + \sum\limits_{i=0}^{\infty} f_i\delta(t - t_i) & , \ t \geq 0 \end{array}\right. , \tag{18}$$

where $f_a \in L_1$, i.e. $\|f_a\|_{L_1} = \int_0^\infty |f_a(t)|\,dt < \infty$, $f_i \in \mathbb{R}$ for $i \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots$, $\delta(t)$ stands for the Dirac delta function, and $\sum_{i=0}^{\infty}|f_i| < \infty$. The set $\mathcal{A}$ is closed under addition, multiplication, and convolution, and is a commutative Banach algebra, with unit $\delta(t)$, for the norm defined by

$$\|f\|_{\mathcal{A}} = \|f_a\|_{L_1} + \sum_{i=0}^{\infty}|f_i|.$$

Similarly, the set $\widehat{\mathcal{A}}$ of Laplace transforms of elements of $\mathcal{A}$ is a commutative Banach algebra, with unit 1, for the induced topology [5], [9].

We consider the class of causal linear systems described by a convolution

$$y(t) = \int_0^t f(t-\tau)u(\tau)\,\mathrm{d}\tau \doteq (f * u)(t), \qquad (19)$$

or equivalently $\hat{y}(s) = \hat{f}(s)\hat{u}(s)$, where the kernel f and the input u are assumed Laplace transformable, in the sense of distributions. One says that (19) is BIBO stable if $f \in \mathcal{A}$, or equivalently if $\hat{f} \in \widehat{\mathcal{A}}$, i.e. $\|f\|_{\mathcal{A}} < \infty$. The BIBO stability is also equivalent to an input-output stability: every bounded input $u \in L_\infty$ produces a bounded output $y \in L_\infty$.

Consider the closed loop transfer matrix $T_{wz}(s)$ from the disturbance w to the output z. The closed loop system is described by a convolution, as in (19). Then, denoting by $\gamma(t)$ the impulse response of the closed loop transfer matrix $T_{wz}(s)$, we have $T_{wz}(s) = \hat{\gamma}(s)$.

For $1 \le p \le \infty$, the L_p-induced norm of the L_p-norm of $T_{wz}(s)$, denoted by $\|T_{wz}\|_p$, is defined by [9]

$$\|T_{wz}\|_p = \sup_{w \in L_p, w \neq 0} \frac{\|\gamma * w\|_{L_p}}{\|w\|_{L_p}}.$$

It is well known that the following equality holds

$$\|T_{wz}\|_1 = \|T_{wz}\|_\infty = \|\gamma\|_{\mathcal{A}},$$

which is well defined if and only if the closed loop system is BIBO stable [9].

For all $1 < p < \infty$, an upper bound of $\|T_{wz}\|_p$ is also given by

$$\|T_{wz}\|_p \le \|\gamma\|_{\mathcal{A}}, \;\; 1 < p < \infty.$$

By the decomposition of the input-output transfer matrix $T_{wz}(s)$ established in Lemma 3.1, we have the following result.

Lemma 4.1: Let $T_{wz}(s) = \hat{\gamma}(s)$ be the closed–loop transfer matrix from w to z. Then,

$$\gamma(t) = \gamma_1(t) + \gamma_2(t), \qquad (20)$$

where γ_1 and γ_2 are generalized functions with non overlapping supports, and are respectively the impulse response of $T_1(s)$, with bounded support $[0, h]$, and the impulse response of $\mathrm{e}^{-sh}T_2(s)$ given in (5), with support contained in $[h, \infty[$. Moreover, if the matrices A_F and A^L are stable, then $\|\gamma\|_{\mathcal{A}} < \infty$, and

$$\|\gamma\|_{\mathcal{A}} = \|\gamma_1\|_{\mathcal{A}} + \|\gamma_2\|_{\mathcal{A}}. \qquad (21)$$

Proof. In (5), the transfer matrix $T_1(s)$ defined by

$$G\Psi_A(s)(I - \mathrm{e}^{-sh}\mathrm{e}^{Ah})E,$$

admits a finite impulse response $\gamma_1(t)$ given by

$$\gamma_1(t) = \begin{cases} G\mathrm{e}^{At}E & , \; t \in [0, h] \\ 0 & , \; t > h \end{cases}, \qquad (22)$$

which lies in $L_p^{c \times d}$, for all $1 \le p \le \infty$. In particular, $\gamma_1 \in \mathcal{A}$.

The impulse response $\gamma_2(t)$ of $\mathrm{e}^{-sh}T_2(s)$ has a support included in $[h, \infty[$. Under the assumption that the matrices A_F and A^L are stable, it is clear that $\gamma_2 \in \mathcal{A}$. Since γ_1 and γ_2 have non overlapping supports, the norm decomposition (20) directly follows. □

It is worth noting that γ_1 does not depend on the control law applied in closed loop. One can evaluate $\|\gamma_1\|_{\mathcal{A}}$ from the knowledge of A, E, G, by direct integration of (22).

This norm gives a lower bound for the closed loop transfer between the disturbance w and the output z

$$\|T_{wz}\|_1 \ge \|\gamma_1\|_{\mathcal{A}}.$$

In the following, we shall be interested in the case where the lower bounds are reached, i.e.

$$\inf_{F,L} \|T_{wz}\|_1 = \|\gamma_1\|_{\mathcal{A}}, \qquad (23)$$

In this case, note that $\|\gamma_1\|_{\mathcal{A}}$ also provides an upper bound of $\|T_{wz}\|_p$ for the other values of p, and one has

$$\inf_{F,L} \|T_{wz}\|_p \le \|\gamma_1\|_{\mathcal{A}}, \; 1 \le p \le \infty.$$

V. Geometric characterizations

In this section, necessary and sufficient geometric conditions are given to solve various problems of disturbance attenuation by output feedback for the time-delay system (1), taking into account the stability of the closed-loop system or not.

For the case of linear systems without delays, conditions to solve this problem are given in the literature, the reader is referred to [16], [17], [19], [20], and references therein. In the geometric approach, controlled or (A, B) invariant subspaces, and conditioned or (C, A) invariant subspaces play a fundamental role. They describe the possibility under external actions (feedback or injection) to stay into a given subspace [22], [3].

Definition 5.1: A subspace $\mathcal{V}$ of $\mathcal{X}$ is called (A, B)–invariant if

$$A\mathcal{V} \subset \mathcal{V} + \operatorname{Im} B.$$

Definition 5.2: A subspace $\mathcal{S}$ of $\mathcal{X}$ is called (C, A)–invariant if

$$A(\mathcal{S} \cap \operatorname{Ker} C) \subset \mathcal{S}.$$

All properties of these subspaces and their computations can be found in references given above.

The subspaces $\mathcal{V}^*_{A,B,\operatorname{Ker} G}$, $\mathcal{R}^*_{A,B,\operatorname{Ker} G}$, and $\mathcal{S}^*_{C,A,\operatorname{Im} B}$, which are respectively the maximal (A, B)–invariant subspace contained in $\operatorname{Ker} G$, the maximal controllability subspace contained in $\operatorname{Ker} G$, and the smallest (C, A)–invariant subspace containing $\operatorname{Im} B$, play a fundamental role in geometric approach.

Any invariant subspace is associated to a spectrum, and the subspace is called stabilizing if the associated spectrum is stable. In the sequel, $\mathcal{V}^*_{g,A,B,\operatorname{Ker} G}$ and $\mathcal{S}^*_{g,A,B,\operatorname{Ker} G}$ respectively denote the maximal stabilizing (A, B)-invariant

subspace contained in $\operatorname{Ker} G$, and the smallest stabilizing (C, A)-invariant subspace containing $\operatorname{Im} B$.
As seen in Section IV, the problem of attenuating the effect of the disturbance is reduced to the analysis of an equivalent linear system without delay, and this theory can be applied on this new system.

Consider the disturbance attenuation problem described in Section I. By Lemma 3.1, the transfer in closed–loop from w to z is given by (5), where $T_2(s)$ is a transfer function of a linear system without delay. Since $T_1(s)$ is independent from any control action, the problem comes down to finding a static feedback F and an output injection L such that $T_2(s)$ is the zero transfer matrix. Note that in this section, we based our approach on the state-space representation of the transfer $T_2(s)$ as in the proof of Lemma 3.1. Note also that it can be done considering the state-space representation given in Remark 3.2.
Applying the classical results of the geometric approach leads to the following.

Theorem 5.3: Solvability conditions of the attenuation problem of (1) can be done in the following way.

(i) There exists an output injection L and an estimate feedback F such that the closed loop system (1)-(2)-(3)-(4) is so that $T_2(s) = 0$ in (5) if and only if

$$\mathcal{S}^*_{C\mathrm{e}^{-Ah},A,\operatorname{Im}(\mathrm{e}^{Ah}E)} \subset \mathcal{V}^*_{A,B,\operatorname{Ker} G}. \tag{24}$$

(ii) There exists an output injection L and an estimate feedback F such that the closed loop system (1)-(2)-(3)-(4) is internally stable and so that $T_2(s) = 0$ if and only if

$$\mathcal{S}^*_{g,C\mathrm{e}^{-Ah},A,\operatorname{Im}(\mathrm{e}^{Ah}E)} \subset \mathcal{V}^*_{g,A,B,\operatorname{Ker} G}. \tag{25}$$

Proof. This theorem is a direct consequence of Lemmas 3.1 and 3.3. In fact, in closed loop, the first part $T_1(s)$ is independent from any control action, so that this part can not be reduced by output injection or estimate feedback. Indeed, the transfer function $T_2(s)$ is the closed loop transfer function of a linear system without delay (Σ_2') with the compensator $(\Sigma_{c,2}')$ described in (14) and (15) respectively. Then, the problem is to minimize the disturbance effect on $z_2(t) = G\zeta(t)$, i.e. we want to ensure the disturbance decoupling by output feedback on (Σ_2') to obtain $\frac{\hat{z}_2(s)}{\hat{w}(s)} = 0$. Conditions (i) and (ii) are then a direct consequence of the classical works [16], [20]. □

Theorem 5.3 gives necessary and sufficient conditions for an exact disturbance decoupling problem with static output feedback on (Σ_2'), with eventually internal stability. On (Σ), and under conditions of Theorem 5.3, we obtain

$$\min_{F,L} \|T_{zw}\|_1 = \|T_1\|_1 = \|\gamma_1\|_{L_1}.$$

It should be clear that, in general (see Section IV)

$$\inf_{F,L} \|T_{zw}\|_1 \geq \|T_1\|_1,$$

where the equality is reached under the conditions of Theorem 5.3. We can get further conditions using the concept of almost invariance and the associated subspaces.
Consider the problem of almost disturbance decoupling by output feedback, that is to obtain in closed loop for (14), an impulse response $\gamma_2(t)$ of the transfer matrix from w to z_2 such that

$$\forall \varepsilon > 0, \exists F \text{ s.t. } \|\gamma_2\|_{L_1} \leq \varepsilon,$$

i.e. $\forall \varepsilon > 0$, $\|z_2\|_{L_1} \leq \epsilon \|w\|_{L_1}$ in closed loop. Then, applying the results of [21], [19], [17], to the system (Σ_2') without delay associated to our system with input delay, we obtain the following.

Theorem 5.4: Solvability conditions of the attenuation problem of (1) can be done in the following way.

(i) There exists an output injection L and an estimate feedback F such that the closed-loop system (1)-(2)-(3)-(4) is so that

$$\inf_{F,L} \|T_{zw}\|_1 = \|T_1\|_1,$$

if and only if

$$\begin{gathered} \operatorname{Im}(\mathrm{e}^{Ah}E) \subset \mathcal{V}^*_{A,B,\operatorname{Ker} G} + \mathcal{S}^*_{G,A,\operatorname{Im} B} \\ \mathcal{V}^*_{A,\mathrm{e}^{-Ah}E,\operatorname{Ker}(C\mathrm{e}^{-Ah})} \cap \mathcal{S}^*_{C\mathrm{e}^{-Ah},A,\operatorname{Im}(\mathrm{e}^{Ah}E)} \subset \operatorname{Ker}(G) \end{gathered} \tag{26}$$

In that case, for every $\epsilon > 0$, there exists a pair (F, L) such that $\|T_2\|_p < \epsilon$, for all $1 \leq p \leq \infty$.

(ii) There exists an output injection L and an estimate feedback F such that the closed-loop system (1)-(2)-(3)-(4) is stable and so that

$$\inf_{F,L} \|T_{zw}\|_1 = \|T_1\|_1\,,$$

if and only if

$$\begin{array}{rcl} \operatorname{Im}(\mathrm{e}^{Ah}E) & \subset & \mathcal{V}^*_{b,g} \\ \mathcal{S}^*_{b,g} & \subset & \operatorname{Ker}(G) \\ \mathcal{S}^*_{b,g} & \subset & \mathcal{V}^*_{b,g} \end{array}, \tag{27}$$

where

$$\begin{array}{rcl} \mathcal{V}^*_{b,g} & = & \mathcal{V}^*_{g,A,B,\operatorname{Ker} G} + \mathcal{S}^*_{G,A,\operatorname{Im} B}, \\ \mathcal{S}^*_{b,g} & = & \mathcal{V}^*_{A,\mathrm{e}^{-Ah}E,\operatorname{Ker}(C\mathrm{e}^{-Ah})} \cap \mathcal{S}^*_{g,C\mathrm{e}^{-Ah},A,\operatorname{Im}(\mathrm{e}^{Ah}E)} \end{array}$$

In that case, for every $\epsilon > 0$, there exists a pair (F, L) such that $\|T_2\|_p < \epsilon$ with $T_2(s)$ stable, for all $1 \leq p \leq \infty$.

Proof. The result immediately comes from (14) and [21] for the case (i) , and from [19] for the case (ii). □

Note that conditions (24), (25), (26), and (27) can be reformulated on the equivalent system without delay (Σ_2''). In fact, by (16)-(17), condition (i) of theorem 5.3 is equivalent to find a pair (F, L) such that

$$\mathcal{S}^*_{C,A,\operatorname{Im} E} \subset \mathcal{V}^*_{A,\mathrm{e}^{-Ah}B,\operatorname{Ker}(G\mathrm{e}^{Ah})}.$$

With the same idea, condition (26) is equivalent to

$$\begin{array}{c}\operatorname{Im} E \subset \mathcal{V}^*_{A,\mathrm{e}^{-Ah}B,\operatorname{Ker}(G\mathrm{e}^{Ah})} + \mathcal{S}^*_{G\mathrm{e}^{-Ah},A,\operatorname{Im}(\mathrm{e}^{-Ah}B)} \\ \mathcal{V}^*_{A,E,\operatorname{Ker} C} \cap \mathcal{S}^*_{C,A,\operatorname{Im} E} \subset \operatorname{Ker}(G\mathrm{e}^{Ah})\end{array} .$$

All these conditions are numerically computable, and easy to verify. In fact, algorithms (ISA) and (COSA) are described in [22] and [21], i.e. for any given subspaces $\mathcal{K}$, $\mathcal{B}$, $\mathcal{L}$, and $\mathcal{D}$ of $\mathcal{X}$, such that $\mathcal{K} = \operatorname{Ker} M$ and $\mathcal{D} = \operatorname{Ker} N$, where M, N are linear maps from $\mathbb{R}^n$ to $\mathbb{R}^p$,

$$(\mathrm{ISA}) : \begin{cases} \mathcal{V}_0 = \mathcal{K} \\ \mathcal{V}_{i+1} = \mathcal{K} \cap A^{-1}(\mathcal{V}_i + \mathcal{B}) \end{cases} ,$$

$$(\mathrm{COSA}) : \begin{cases} \mathcal{S}_0 = 0 \\ \mathcal{S}_{i+1} = \mathcal{L} + A(\mathcal{S}_i \cap \mathcal{D}) \end{cases} ,$$

converge in finite steps to $\mathcal{V}^*_{A,\mathcal{B},\operatorname{Ker} M}$ and $\mathcal{S}^*_{N,A,\mathcal{L}}$ respectively, subspaces which are uniquely determined. Then, all conditions of Theorem 5.3 and 5.4 are numerically computable.

VI. Conclusion

This paper addresses the disturbance attenuation problem by output feedback in linear multivariable systems with a delayed input. To solve this problem, a static predictive control law is used, that allow to work in closed–loop on an equivalent linear system without delay.

Then, geometric conditions are provided to solve various formulations of the disturbance decoupling.

Furthermore, it is shown that any retroaction will act on the system only after a determined time, which corresponds evidently to the initial delay.

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