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The Complete Set of PID Controllers with Guaranteed Gain and Phase Margins

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Abstract— **The problem of determining the entire set of PID controllers that stabilize a linear time invariant(LTI) plant has been recently solved in [1], [2]. In this paper, we extend these results to the problem of obtaining the complete set of PID parameters that attains prescribed gain and phase margins. An example is given for illustration.**

I. INTRODUCTION

The basic approach in modern control theories is first to obtain the entire set of stabilizing controllers as in the Youla parameterization, and then to search over the set to get a controller that meets given performance criteria. Using this approach, Datta et al. [1] have recently reported very important results about computation of all stabilizing P, PI and PID controllers for a LTI system of arbitrary order. This complete solution is based on the generalized Hermite-Biehler theorem. They have shown that this problem is equivalent to the problem of linear programming. In [2], the result has been extended to an arbitrary order LTI plant with time delay.

In this paper, we consider a problem of computing PID controllers, $C(s) = (k_i + k_p\ s + k_d\ s^2)/s$, along with an arbitrary LTI plant, $P(s) = N(s)/D(s)e^{-T_d s}$, in the unit feedback configuration, that meet the stability margin specifications of gain and/or phase margins. We will show that this problem can be solved by extending the results from [1], [2]. Here let $\mathcal{A}_s$ denote the complete set of PID controllers which stabilizes a given LTI plant. If we wish to search a PID set over $\mathcal{A}_s$ which attains merely a phase margin requirement, it is straightforwardly solved by using the method in [2]. However, it will be shown that more conditions should be considered for the case where the gain margin specification is imposed. For this problem, a different computation method based on the characteristic polynomial with frequency sweeping has been proposed in [3]. Applying this method to PID controllers, we need to search three kinds of 2-D sets. Those shall be computed on (k_p, k_i), (k_p, k_d) and (k_i, k_d) planes respectively. The final set is determined by taking the intersections of those sets. As a similar result, a problem of determining the complete set of stabilizing first order controllers that guarantees prespecified gain and phase margins has been solved in [4].

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In the present paper, we start by finding the set of all stabilizing PID controllers, by means of the methods in [1], [2]. We also use the important property that at a fixed k_p, a subset of $\mathcal{A}_{s,k_p}$ on the (k_i, k_d) plane appears only in the form of a polygon. Due to the fact that there must exist at least one characteristic root on the imaginary axis if one chooses the PID gains at the boundary of the stabilizing region, the magnitude and phase conditions of the Nyquist plot derived at the point $-1 + j0$ are used to map the $\mathcal{A}_{s,k_p}$ onto the set $\mathcal{A}_{R,k_p}$ in the (k_i, k_d) plane. The $\mathcal{A}_{R,k_p}$ represents all (k_i, k_d) parameters that satisfy the prescribed stability margins at a fixed $k_p \in \mathcal{A}_s$. Then the complete set of PID controllers attaining the given gain and/or phase margin will be determined by sweeping over k_p. All these sets can be displayed using 2 and 3-D graphics. An example is given for illustration.

II. PROBLEM FORMULATION AND PRELIMINARIES

In this section, we present several definitions and preliminaries regarding the relationship between the Nyquist plot and the boundary of the stability region in controller parameter space. Consider the unit feedback system given by a LTI plant and PID controller:

$$P(s) := \frac{N(s)}{D(s)}e^{-T_d s},$$
$$C(s) := \frac{(k_i + k_p\ s + k_d\ s^2)}{s}$$

The open loop transfer function is written as

$$G_0(s) = C(s)P(s) = (k_i + k_p\ s + k_d\ s^2) \cdot G_s(s), \quad (1)$$

where

$$G_s(s) := \frac{N(s)}{sD(s)}e^{-T_d s}.$$

We first clarify the definitions of gain and phase margins. Let us replace the plant $P(s)$ by $KP(s)$ for gain margin. If the loop transfer function $G_0(s)$ has N unstable poles in the right half plane(RHP) of the s-plane, the Nyquist plot has to encircle the point $-1 + j0$ as much by N times in order for the overall system to be stable. Thus, there may exist two intervals of gain K for stability, which are $[K^-, 1]$ and $[1, K^+]$. K^- and K^+ are referred to as *lower gain margin* and *upper gain margin*, respectively. For simplicity, the prespecified gain margin is denoted by $[K^-, K^+]$. Similarly, let us replace the plant $P(s)$ by $e^{-j\theta}P(s)$ for the phase margin. In the case that $G_0(s)$ has zeros in RHP(or the plant is of nonminimum phase or time delay), the phase margin is also characterized by *lower phase margin* θ^- and *upper phase margin* θ^+, which are written simply as $[\theta^-, \theta^+]$.

Let $\mathcal{A}_s$ be the set of parameters of all PID controllers that stabilizes the given plant $P(s)$. $\mathcal{A}_s$ can be computed using the results from [1], [2]. It is important to note that the stabilizing region in the (k_i, k_d) plane for a fixed value k_p is a convex polygon as shown in Fig. 1. Let $\mathcal{A}_{s,k_p}$ denote a subset of $\mathcal{A}_s$ with k_p fixed at some value. $\partial\mathcal{A}_s$ is then the boundary of $\mathcal{A}_s$.

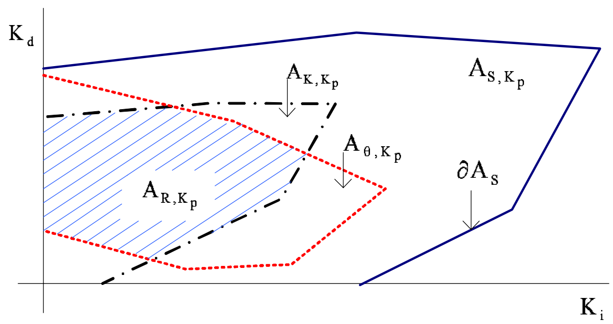


Fig. 1. Stabilizing PID set: $\mathcal{A}_{s,k_p}$, $\mathcal{A}_{K,k_p}$, $\mathcal{A}_{\theta,k_p}$ for a fixed k_p.

For any value of $\partial\mathcal{A}_s$, the Nyquist curve of the corresponding $G_0(s)$ should cross $(-1, j0)$ for some ω. At this point, we have $G_0(j\omega) = -1$. This, in turn, is equivalent to the following two conditions for $(k_p, k_i, k_d) \in \partial\mathcal{A}_s$:

$$arg[(k_i - k_d\omega^2 + jk_p\omega)G_s(j\omega)] = \pi, \quad (2)$$
$$|(k_i - k_d\omega^2 + jk_p\omega)G_s(j\omega)| = 1. \quad (3)$$

Here $arg(\cdot) \in [0, 2\pi)$. Obviously, every PID controller with $(k_p, k_i, k_d) \in \partial\mathcal{A}_s$ gives rise to the gain/phase margin of 0 $[dB]$ and 0^o. On the other hand, any values inside $\partial\mathcal{A}_s$ provide a larger stability margin than that of the stability boundary. Therefore it is natural to assume that there exists a subset of $\mathcal{A}_s$ attaining the given gain margin $[K^-, K^+]$ and/or phase margin $[\theta^-, \theta^+]$.

The problem of interest is to find the complete set of (k_p, k_i, k_d) from $\mathcal{A}_s$. The approach developed here to solve this problem proceeds as follows:

(i) Compute all stabilizing PID gains using the result of [1], [2].

(ii) Compute the set of PID parameters attaining the given gain margins $[K^-, K^+]$. The set is denoted by $\mathcal{A}_K$. To do this, replace the plant $P(s)$ in (1) by $KP(s)$. Compute the set $\mathcal{A}_K^c$:

$$\begin{aligned}\mathcal{A}_K^c := \{(k_p, k_i, k_d) \mid (k_p, k_i, k_d) \in \mathcal{A}_s \text{ and} \\ L(j\omega) := K(k_i - k_d\omega^2 + jk_p\omega)G_s(j\omega) \\ = -1, \quad \text{for} \quad K \in [K^-, K^+], \omega \in \Omega_K\}, \quad (4)\end{aligned}$$

where Ω_K will be defined in the next section. By the definition above, $\mathcal{A}_K^c$ is the subset of $\mathcal{A}_s$ which makes $KG_0(s)$ have a minimal destabilizing K. Then $\mathcal{A}_K$ is determined by excluding $\mathcal{A}_K^c$ from $\mathcal{A}_s$. That is, $\mathcal{A}_K = \mathcal{A}_s \setminus \mathcal{A}_K^c$. This is a PID design with guaranteed gain margin.

(iii) Compute the set of PID parameters attaining the given phase margins $[\theta^-, \theta^+]$. The set is denoted by $\mathcal{A}_\theta$. To do this, replace the plant $P(s)$ in (1) by $e^{-j\theta}P(s)$. Compute the set

$$\begin{aligned}\mathcal{A}_\theta^c := \{(k_p, k_i, k_d) \mid (k_p, k_i, k_d) \in \mathcal{A}_s \text{ and} \\ G(j\omega) := e^{-j\theta}(k_i - k_d\omega^2 + jk_p\omega)G_s(j\omega) \\ = -1, \quad \text{for} \quad \theta \in [\theta^-, \theta^+], \omega \in \Omega_\theta\}, \quad (5)\end{aligned}$$

where Ω_θ will be defined in section IV. From the definition above, $\mathcal{A}_\theta^c$ is the subset of $\mathcal{A}_s$ which makes $e^{-j\theta}G_0(s)$ have a minimal destabilizing θ. Then $\mathcal{A}_\theta$ is determined by excluding $\mathcal{A}_\theta^c$ from $\mathcal{A}_s$. That is, $\mathcal{A}_\theta = \mathcal{A}_s \setminus \mathcal{A}_\theta^c$. This is a PID design with guaranteed phase margin.

(iv) If $\mathcal{A}_K \cap \mathcal{A}_\theta \neq \emptyset$, it is clear that the set of PID controllers simultaneously satisfying both guaranteed gain and phase margins shall be obtained by $\mathcal{A}_R = \mathcal{A}_s \cap \mathcal{A}_K \cap \mathcal{A}_\theta$.

To proceed, we first investigate another equivalent condition for (2) as follows.

Proposition 1 *The phase condition (2) yields the following identity.*

$$k_i - k_d\omega^2 = -\frac{k_p\omega}{tan\{arg[G_s(j\omega)]\}}, \quad (6)$$

with

$$\begin{array}{lll}(a) & 0 < arg[G_s(j\omega)] < \pi & \text{if } k_p > 0, \\ (b) & arg[G_s(j\omega)] = 0 \; or \; \pi & \text{if } k_p = 0, \\ (c) & \pi < arg[G_s(j\omega)] < 2\pi & \text{if } k_p < 0.\end{array} \quad (7)$$

Proof: The following equation necessarily implies the phase condition (2).

$$arg[(k_i - k_d\omega^2 + jk_p\omega] = -arg[G_s(j\omega)] + i\pi, \; (i = 0, 1). \quad (8)$$

Expressing the left hand side of (8) in terms of arctangent,

$$tan^{-1}(\frac{k_p \; \omega}{k_i - k_d\omega^2}) \pm l\pi = -arg[G_s(j\omega)] + i\pi, \; (l = 0, 1). \quad (9)$$

Using $tan\{\theta + (i \pm l)\pi\} = tan(\theta)$ and $tan(-\theta) = -tan(\theta)$, (9) becomes

$$\begin{aligned}\frac{k_p\omega}{k_i - k_d\omega^2} &= tan\{-arg[G_s(j\omega)] + (i \pm l)\pi\} \\ &= -tan\{arg[G_s(j\omega)]\}. \quad (10)\end{aligned}$$

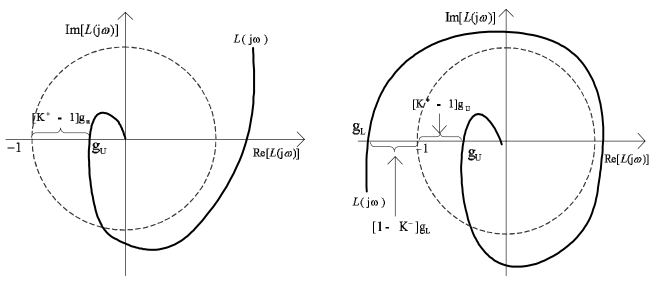
Equation (10) results in (6). This means that (6) is identical to two cases of (2); $arg[G_0(j\omega)] = 0$ and $arg[G_0(j\omega)] = \pi$. Thus, we need to impose other conditions on (6) to establish (2). Rewriting (8) with $i = 1$,

$$arg[G_s(j\omega)] = \pi - arg[k_i - k_d\omega^2 + jk_p\omega]. \quad (11)$$

Clearly, $0 < arg[k_i - k_d\omega^2 + jk_p\omega] < \pi$ if $k_p > 0$ and $\pi < arg[k_i - k_d\omega^2 + jk_p\omega] < 2\pi$ if $k_p < 0$. Furthermore, if $k_p = 0$, $arg[k_i - k_d\omega^2] = 0$ or π. Therefore, the conditions (a), (b) and (c) must hold. ■

III. ALL PID SET WITH GUARANTEED GAIN MARGIN

Since we have assumed that $\mathcal{A}_s$ is known, solving problem (i) described in the previous section is equivalent to the problem of finding $\mathcal{A}_K^c$ in (4). Consider a gain margin specification of $[1, K^+]$ for a stable plant and $[K^-, K^+]$ for unstable plant. According to the definition of stability margins, the Nyquist plot of $G_0(j\omega)$ should cross over the negative real axis at least $[1-K^-]g_L$ and $[K^+-1]g_U$ from $(-1, j0)$ in order for the PID controllers to guarantee the gain margin. This is shown in Fig. 2.



(a) $G_0(s)$ stable (b) $G_0(s)$ unstable

Fig. 2. Nyquist plots attaining the given gain margin

The Nyquist curve corresponding to the boundary $\partial\mathcal{A}_s$ shown in Fig. 1 must go through the point $(-1, j0)$ for some ω. For example, when the upper gain margin K^+ and the lower gain margin K^- are given, a PID controller that meets the stability margins should make the Nyquist curve of $L(j\omega)$ cross over the negative real axis outside the interval $[g_L, g_U]$ in Fig. 2. From these, we know that computing such all PID set can be achieved by finding all regions in $\mathcal{A}_s$ which provide gain margins $[K^-, 1]$ and $[1, K^+]$ reversely. The subset has been defined by $\mathcal{A}_K^c$. In the sequel, the solution to problem (i) will be obtained by extruding $\mathcal{A}_K^c$ from $\mathcal{A}_s$. Therefore it is sufficient to compute $\mathcal{A}_K^c$ for the guaranteed gain margin problem. Let us define (6) again as follows: For a fixed $k_p^* \in \mathcal{A}_s$,

$$M(\omega) := -\frac{k_p^*\ \omega}{tan\{arg\ [G_s(j\omega)]\}} = (k_i - k_d\ \omega^2). \quad (12)$$

From the magnitude condition $|KG_0(j\omega)| = 1$, we have

$$\begin{aligned} K(\omega) \quad &:= \quad \frac{1}{\sqrt{(k_i - k_d\ \omega^2)^2 + (k_p^*\ \omega)^2} \cdot |G_s(j\omega)|} \\ &= \quad \frac{1}{\sqrt{M(\omega)^2 + (k_p^*\ \omega)^2} \cdot |G_s(j\omega)|}. \quad (13) \end{aligned}$$

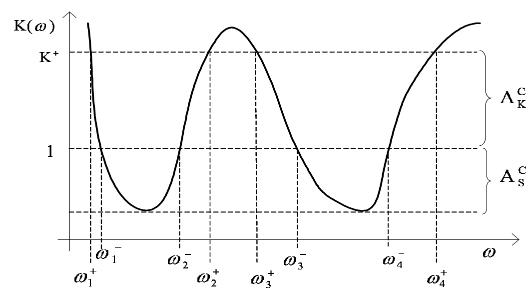
The frequency plot of $K(\omega)$ for $\omega \in [0, \infty)$ as shown in Fig. 3. It is worth comparing the regions $[1, K^+]$ and $[K^-, K^+]$ in Fig. 2 with those in Fig. 3. Here we define a set of ω as follows:

$$\Omega_K := \{\omega | \omega \in \{[\omega_1^-, \omega_1^+] \cup \cdots \cup [\omega_m^-, \omega_m^+]\}\} \quad (14)$$

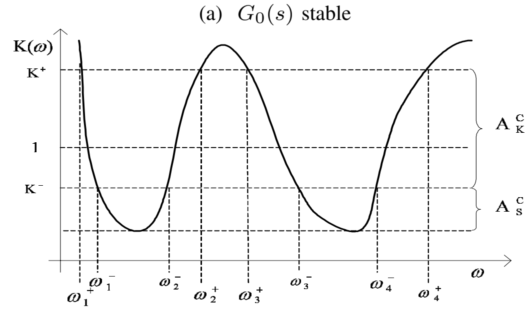
where

$$\begin{aligned} \omega_i^+ &:= \{\omega \mid K(\omega) = K^+\}, \text{ for a stable or unstable } P(s) \\ \omega_i^- &:= \{\omega \mid K(\omega) = 1\} \qquad \text{for a stable } P(s), \\ &:= \{\omega \mid K(\omega) = K^-\} \quad \text{for an unstable } P(s). \end{aligned}$$

We see that it is sufficient to search over $\omega \in \Omega_K$ when we



(a) $G_0(s)$ stable



(b) $G_0(s)$ unstable

Fig. 3. $K(\omega)$ vs. ω for a fixed $k_p^* \neq 0$

find the set $\mathcal{A}_K^c$ from $\mathcal{A}_s$ with a fixed k_p. Note that we need not consider the case of $k_i = 0$ because the corresponding point does not exist in the Nyquist plane [1]. However, we need to deal with two cases, $k_p \neq 0$ and $k_p = 0$, separately.

A. $k_p \neq 0$

Since $K(\omega)$ is the function for which the phase is π (see Proposition 1), (4) can be represented by a set of straight lines in the (k_i, k_d) plane. That is, for a fixed k_p^*, using (12), we have

$$\begin{aligned} \mathcal{A}_{K,k_p}^c := \{(k_i, k_d) \mid (k_i, k_d) \in \mathcal{A}_s \text{ and} \\ k_i - k_d\ \omega_j^2 = M(\omega_j), \quad \text{for} \ \ \omega_j \in \Omega_K\}. \quad (15) \end{aligned}$$

$\mathcal{A}_{K,k_p}^c$ denotes the subset of $\mathcal{A}_K^c$ with a fixed k_p^*. Once we determine the set $\mathcal{A}_{K,k_p}^c$ at a k_p^*, we can compute $\mathcal{A}_{K,k_p} = \mathcal{A}_{s,k_p} \setminus \mathcal{A}_{K,k_p}^c$.

B. $k_p = 0$

If $k_p = 0$, then, according to condition (b) in Proposition 1, (6) can not be occupied in (15). The plant $P(s)$ can be written in terms of its even and odd parts.

$$\begin{aligned} N(s) &= N_e(s^2) + sN_o(s^2), \\ D(s) &= D_e(s^2) + sD_o(s^2). \end{aligned}$$

Condition (b) is equivalent to the condition that the imaginary part of $G_s(j\omega)$ equals zero. This is derived from the definition of $G_s(s)$ and substituting $s = j\omega$ as follows:

$$(N_e(-\omega^2)\ D_e(-\omega^2) + \omega^2\ N_o(-\omega^2)\ D_o(-\omega^2))cos\omega T_d + \omega(N_o(-\omega^2)D_e(-\omega^2) - N_e(-\omega^2)D_o(-\omega^2))sin\omega T_d = 0 \quad (16)$$

We solve the identity (16) for both cases of $arg[G_s(j\omega)] = 0\ and\ \pi$. Let the solution be $\bar{\omega}_i$ for $i = 1, 2, \cdots$. Let's define

$$\Omega_0 := \{\bar{\omega}_i \mid arg[G_s(j\bar{\omega}_i)] = 0\},$$
$$\Omega_\pi := \{\bar{\omega}_i \mid arg[G_s(j\bar{\omega}_i)] = \pi\} \quad \text{for} \quad i = 1, 2, \cdots.$$

At these points, the magnitude condition (13) yields

$$k_i - k_d\ \bar{\omega}_k^2 = \pm\frac{1}{K\ |G_s(\bar{\omega}_k)|},\ \text{for}\ \bar{\omega}_k \in [\Omega_0 \cup \Omega_\pi]. \quad (17)$$

In the sequel, we can compute $\mathcal{A}^c_{K,k_p}$ with $k_p = 0$ as follows:

$$\mathcal{A}^c_{K,k_p} := \{(k_i, k_d) \mid (k_i, k_d) \in \mathcal{A}_s \text{ and}$$
$$k_i - k_d\ \bar{\omega}_j^2 = -\tfrac{1}{K\ |G_s(\bar{\omega}_j)|},\ \text{for}\ \bar{\omega}_j \in \Omega_0,$$
$$k_i - k_d\ \bar{\omega}_k^2 = \tfrac{1}{K\ |G_s(\bar{\omega}_k)|},\ \text{for}\ \bar{\omega}_k \in \Omega_\pi\}, \quad (18)$$

where $K \in [K^-, K^+]$.

Finally, by sweeping over k_p, we will have the entire set of PID gains that guarantees the given gain margin:

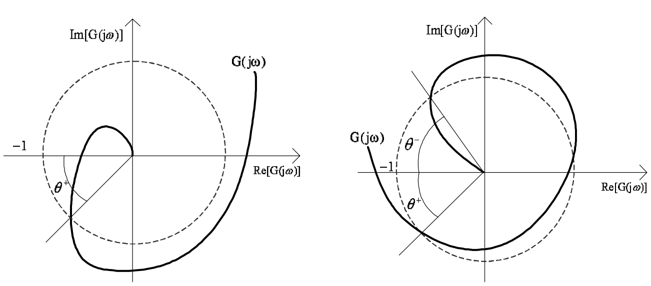
$$\mathcal{A}_K = \bigcup_{k_p} \mathcal{A}_{K,k_p}. \quad (19)$$

IV. ALL PID SET WITH GUARANTEED PHASE MARGIN

To implement problem (ii) formulated in section II, we first replace plant $P(s)$ by $e^{-j\theta}\ P(s)$. Then,

$$G(s) := e^{-j\theta}\ G_0(s) = e^{-j\theta}\ (k_i + k_p\ s + k_d\ s^2)\ G_s(s). \quad (20)$$

Similar to the guaranteed gain margin, we consider a phase margin specification of θ^+ for a stable plant and $[\theta^-, \theta^+]$ for a nonminimum phase plant. According to the definition of stability margins, the Nyquist plot of $G(j\omega)$ should cross over the unit circle at least θ^+ or $[\theta^-, \theta^+]$ from $(-1, j0)$ in order for the PID controllers to guarantee the phase margin. This is shown in Fig. 4.



(a) $G_0(s)$ stable (b) $G_0(s)$ nonminimum phase

Fig. 4. Nyquist plots attaining the given phase margin

It is clear that if one chooses PID gains on the boundary $\partial\mathcal{A}_s$, there exists at least one ω such that the Nyquist curve of $G(j\omega)$ goes through the point $(-1, j0)$. That is, those PID gains give a phase margin of 0. Fig. 4 shows that $\mathcal{A}^c_\theta$ in (5) is nothing but the subset of $\mathcal{A}_s$ that makes the phase margin either θ^+ or $[\theta^-, \theta^+]$. The gain and phase conditions for $G(j\omega)$ can be written as

$$arg[(k_i - k_d\omega^2 + jk_p\omega)G_s(j\omega)] - \theta = \pi, \quad (21)$$
$$|G(j\omega)| = |(k_i - k_d\omega^2 + jk_p\omega)G_s(j\omega)| = 1. \quad (22)$$

For a fixed k_p^*, (22) becomes

$$k_i - k_d\ \omega^2 = \pm\ \sqrt{\Pi(\omega)}, \quad (23)$$

where

$$\Pi(\omega) := \frac{1}{|G_s(j\omega)|^2} - (k_p^*\ \omega)^2. \quad (24)$$

Note that $\Pi(\omega)$ should be greater than or equal to zero since $k_i - k_d\ \omega^2 \in \mathcal{R}$. Furthermore, rewriting (21), we have

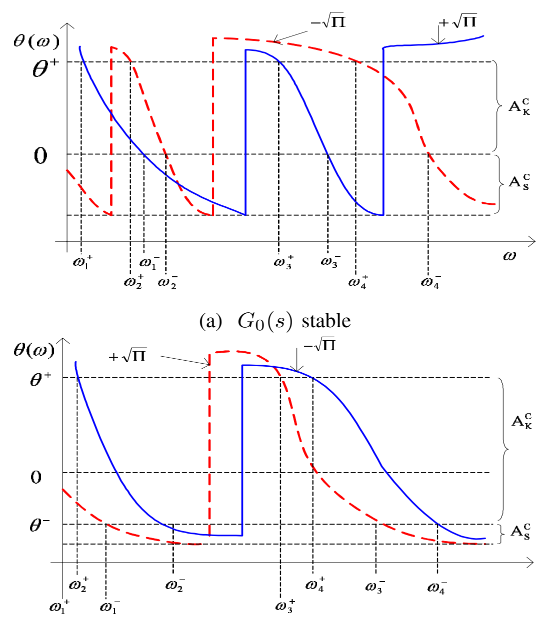
$$\theta(\omega) = arg[(\pm\ \sqrt{\Pi(\omega)} + j\omega\ k_p^*)\ G_s(j\omega)] - \pi. \quad (25)$$

Fig. 5 shows a typical plot of $\theta(\omega)$. From these figures, we define

$$\Omega_\theta := \{\omega | \omega \in \{[\omega_1^-, \omega_1^+] \cup \cdots \cup [\omega_t^-, \omega_t^+]\}\} \quad (26)$$

where

$$\omega_i^+ := \{\omega \mid \theta(\omega) = \theta^+\}, \quad \text{for any } G_0(s)$$
$$\omega_i^- := \{\omega \mid \theta(\omega) = 0\}, \quad \text{for stable } G_0(s),$$
$$:= \{\omega \mid \theta(\omega) = \theta^-\}, \quad \text{for other } G_0(s).$$



(a) $G_0(s)$ stable

(b) $G_0(s)$ nonminimum phase or time delay

Fig. 5. $\theta(\omega)$ vs. ω for a fixed k_p^*

Therefore it is sufficient to search over only $\omega \in \Omega_\theta$ for the sake of finding $\mathcal{A}^c_\theta$ from $\mathcal{A}_s$. Combining this set and (23), we can compute $\mathcal{A}^c_\theta$ with fixed k_p. Let's define

$$\mathcal{A}^c_{\theta,k_p} := \{(k_i, k_d) \mid (k_i, k_d) \in \mathcal{A}_s \text{ and}$$
$$k_i - k_d\ \omega_j^2 = \pm\sqrt{\Pi(\omega)}(\omega_j), \quad \text{for} \quad \omega_j \in \Omega_\theta\}. \quad (27)$$

Here $\mathcal{A}^c_{\theta,k_p}$ denotes the subset of $\mathcal{A}^c_\theta$ for the fixed k_p, which is a set of straight lines. After determining $\mathcal{A}^c_{\theta,k_p}$ at a k_p^*, we compute $\mathcal{A}_{\theta,k_p} = \mathcal{A}_{s,k_p} \setminus \mathcal{A}^c_{\theta,k_p}$. Finally sweeping over k_p, we will have the complete set of PID gains that guarantees the given phase margin. That is,

$$\mathcal{A}_\theta = \bigcup_{k_p} \mathcal{A}_{\theta,k_p}. \tag{28}$$

This is the solution to problem (ii) in section II.

Remark 1 PI and PD controllers are special cases of a PID controller with either $k_d = 0$ or $k_i = 0$. Thus the above algorithm can be extended to these cases as well.

V. ILLUSTRATIVE EXAMPLES

In this section, we will give two examples; a time delayed plant and the same plant with delay free.

Example 1 Consider the following nonminimum phase plant in [2]:

$$P(s) = \frac{s^3 - 4s^2 + s + 2}{s^5 + 8s^4 + 32s^3 + 46s^2 + 46s + 17} e^{-T_d s} \tag{29}$$

with time delay $T_d = 1[sec]$.

We wish to find the entire set of PID controllers that guarantees the following gain and phase margins:

$$\text{Gain margin}: \quad K^+ = 2 \text{ (about 6 [dB])},$$
$$\text{Phase margin}: \quad [\theta^-, \theta^+] = [-10^o, 60^o].$$

To begin with, we use the method proposed in [1], [2] to obtain the complete set of PID controllers $\mathcal{A}_s$ that stabilize the closed loop system in the plant above. Fig. 6 shows the set.

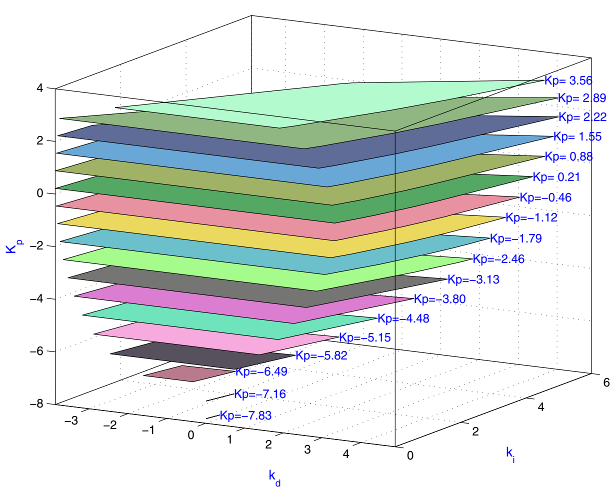


Fig. 6. All stabilizing PID controllers in the Example 1

A. Stabilizing Region for Guaranteed Gain Margin

Based on the discussion in section III, the following steps can be used for computing $\mathcal{A}_K$:

1) Choose a k_p from the results shown in Fig. 6.
2) Compute Ω_K using (13) after setting $K(\omega) = K^+$.
3) Compute $\mathcal{A}^c_{K,k_p}$ using (15) if $k_p \neq 0$ or (18) if $k_p = 0$.
4) Find $\mathcal{A}_{K,k_p}$ by extruding $\mathcal{A}^c_{K,k_p}$ from $\mathcal{A}_{s,k_p}$.

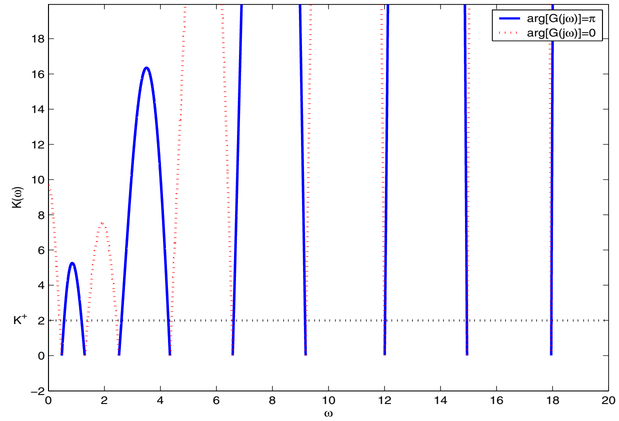


Fig. 7. $K(\omega)$ vs. ω for Example 1

From Fig. 6 $k_p = 0.88$ has been chosen. Fig. 7 shows the $K(\omega)$ plot over $\omega \in [0, 20][rad/sec]$. Solving (13) with $K(\omega) = K^+ = 2$, we obtain $\Omega_K = \{[0.5192, 0.5623] \cup [1.1860, 1.2430] \cup [2.5672, 2.61072] \cup [4.2830, 4.31220] \cup [6.5973, 6.61374] \cup [9.1653, 9.1750] \cup [12.0129, 12.0187] \cup [14.9370, 14.9428] \cup [17.9538, 17.9565]\}$. Next we compute a set of straight lines on the (k_i, k_d) plane, which is obtained by (15) at every $\omega = \{\omega_i^+, \omega_i^-\}$, as shown in Fig. 8. Then $\mathcal{A}_{K,k_p}$ has been determined, and appear in the same figure.

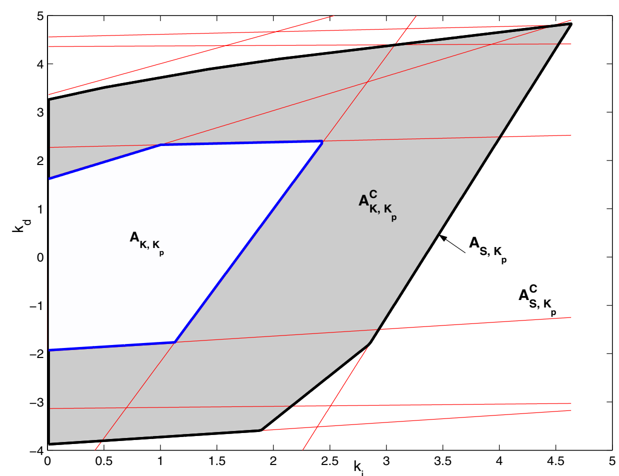


Fig. 8. The region for guaranteed gain margin at $k_p = 0.88$

B. Stabilizing Region for Guaranteed Phase Margin

The procedures computing $\mathcal{A}_\theta$ from $\mathcal{A}_s$ are as follows:

1) Choose a k_p from the result shown in Fig. 6.
2) Compute Ω_θ using (26) after setting $\theta(\omega) = \theta^-$ *and* θ^+.
3) Compute $\mathcal{A}^c_{\theta,k_p}$ using (27).
4) Find $\mathcal{A}_{\theta,k_p}$ by extruding $\mathcal{A}^c_{\theta,k_p}$ from $\mathcal{A}_{s,k_p}$.

Similar to the previous step, we choose $k_p = 0.88$ from Fig. 6. The $\theta(\omega)$ plot over $\omega \in [0, 20]$ $[rad/sec]$ is shown in Fig. 9. By solving (26) with $\theta^+ = 60^o$ and $\theta^- = -10^o$, we obtain $\Omega_\theta = \{[0.1963, 0.5714] \cup [2.0770, 2.6554] \cup [5.8075, 6.7332] \cup [11.0563, 12.1737] \cup [16.9474, 18.1220] \cup [0.9852, 1.2899] \cup [3.6546, 4.4262] \cup [8.2738, 9.3274] \cup [13.9533, 15.1084] \cup [19.9746, 20.0000]\}$.

By means of (27), we constitute a set of straight lines at every $\omega = \{\omega^+, \omega^-\}$. Fig. 10 shows $\mathcal{A}_{\theta,k_p}$, $\mathcal{A}^c_{\theta,k_p}$ and $\mathcal{A}_{s,k_p}$ respectively. Region $\mathcal{A}_{\theta,k_p}$ indicates the complete set of all PID parameters that guarantees the phase margin requirement when $k_p = 0.88$. Fig. 11 shows $\mathcal{A}_{K,k_p}$, $\mathcal{A}_{\theta,k_p}$, $\mathcal{A}_{s,k_p}$ and their intersection, $\mathcal{A}_{R,k_p}$. Thus any point $(k_i, k_d) \in \mathcal{A}_{R,k_p}$ constitutes a controller that guarantees the previously specified gain and phase margins. To verify this, let's pick four points out on the Fig. 11. The gain and phase margins corresponding to these PID gains are given in Table I.

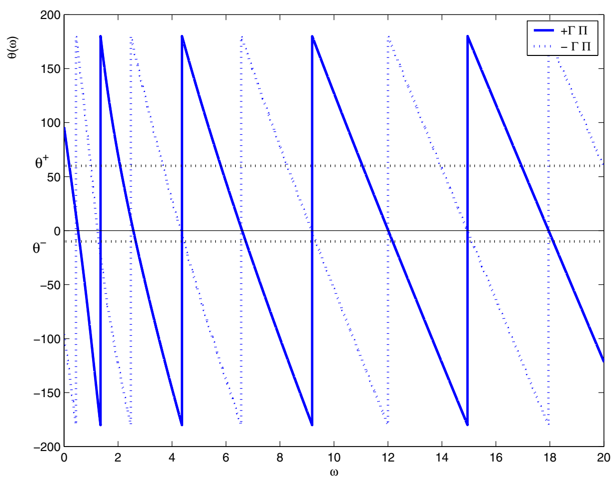


Fig. 9. $\theta(\omega)$ vs. ω at $k_p = 0.88$

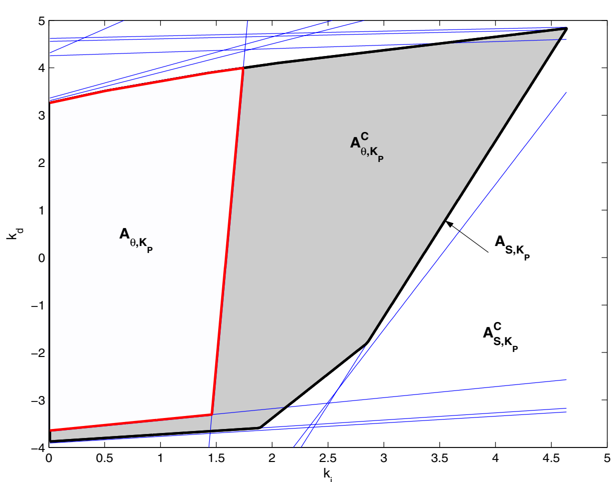


Fig. 10. The region for guaranteed phase margin at $k_p = 0.88$

TABLE I

GAIN AND PHASE MARGINS OF THE PID GAINS SELECTED IN FIG. 11

No.	k_p, k_i, k_d	Gm[dB]	Pm[deg]	Remark
#1	0.88, 0.50, 1.98	6.0	85.1	$\partial\mathcal{A}_{K,k_p}$
#2	0.88, 1.63, 1.01	8.3	60.0	$\partial\mathcal{A}_{\theta,k_p}$
#3	0.88, 1.71, 2.44	6.0	60.0	$\partial\mathcal{A}_{K,k_p} \cap \partial\mathcal{A}_{\theta,k_p}$
#4	0.88, 1.19, 0.44	10.0	69.7	inside $\mathcal{A}_{R,k_p}$

If we repeat this procedure over the whole range of $k_p \in \mathcal{A}_s$, we obtain the complete set $\mathcal{A}_R$ of PID controllers which guarantees both gain and phase margins simultaneously, as shown in Fig. 12.

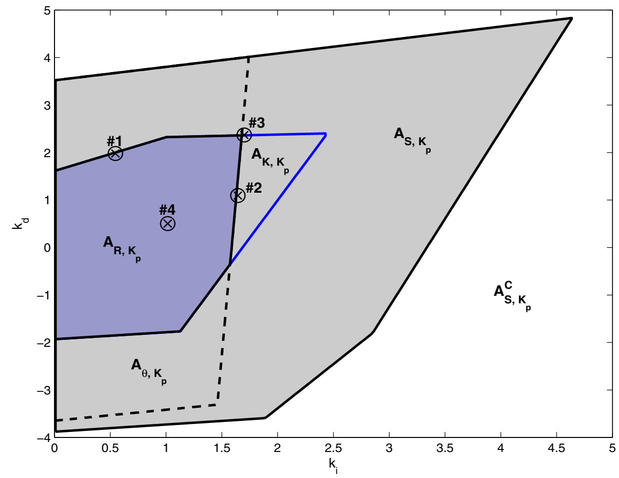


Fig. 11. The region for guaranteed gain and phase margins at $k_p = 0.88$

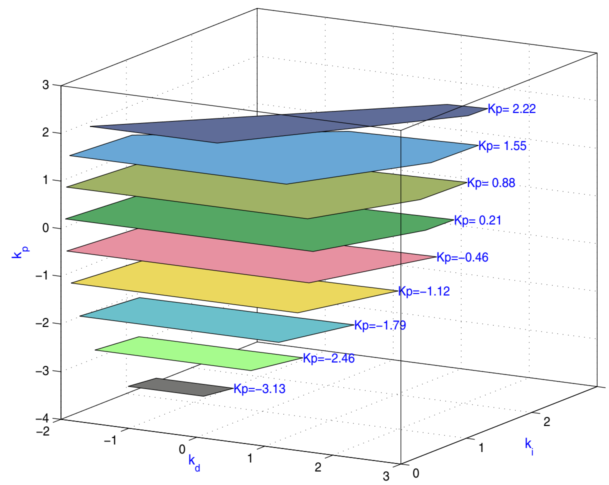


Fig. 12. The complete set of PID gains with guaranteed gain and phase margins in Example 1

VI. CONCLUDING REMARKS

In this paper, we have determined for a given LTI plant, the complete set of stabilizing PID controllers that attains the given gain and phase margins. This computing method extends the results of [1], [2] wherein all stabilizing PID controllers for a LTI system were found and also for a LTI plant with time delay. The procedures are simple and constitutional. 2-D and 3-D graphics of the PID set will be very useful for computer-aided design because they allow for much better designs than those for obtaining a single controller, as mentioned in [4].

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