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Optimal Resource Partitioning in a Military Conflict based on Lanchester Attrition Models

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Abstract— **This paper develops a model for military conflicts where the defending forces have to determine an optimal partitioning of available resources to counter attacks from an adversary in two different fronts. The Lanchester attrition model is used to develop the dynamical equations governing the variation in force strength. Three different allocation schemes – Time-Zero-Allocation (TZA), Allocate-Assess-Reallocate (AAR), and Continuous Constant Allocation (CCA) – are considered and the optimal solutions are obtained in each case. Numerical examples are given to support the analytical results.**

I. Introduction

Force deployment and optimal resource allocation has been an area of considerable research interest in conventional warfare for a long time [1]-[8]. In the modern scenario, with significant advances in technology related to communication and computation, sophisticated decision-making in these situations has become feasible. This has generated renewed interest in formulating decision-making problems in these areas and seeking optimal solutions. This paper addresses one such problem in which the defending forces need to optimally partition their resources between two attacking forces of differing strengths. The basic model used is the Lanchester (2,1) model [9]. In [9] optimality of resource partitioning problems are not addressed. In this paper we address only the static case where the resource allocation is done in three different ways. The simplest is the case when allocation is done initially and no further action is taken (Time-Zero-Allocation (TZA)). The next is allocation followed by reallocation depending on certain decisive events (Allocate-Assess-Reallocate (AAR)). The last is the continuous allocation case where a constant allocation ratio is used continuously over time till the end of the conflict (Continuous Constant Allocation (CCA)).

II. Problem Formulation and Analytical Solution

Consider a military conflict between two opposing forces. Let Y denote the defending force and X denote the attacking force. It is assumed that the defending force consists of only one type of force and the attacking force consists of two types of forces. Let y denote the strength of the defending force and x_1 and x_2 denote the strength of each type of X. Let the initial values of y, x_1, x_2 be N, M_1 and M_2, respectively. The initial strength of y is divided into two parts, ηN and $(1-\eta)N$ so that ηN interacts with x_1 and $(1-\eta)N$ interacts with x_2 (Figure 1). This paper deals with the problem of optimally choosing η to maximize some objective of the defending forces. The model is developed for three different cases. Since this is a decision making problem for Y, we select an objective of maximizing a weighted evaluation of the surviving resource strength of Y force, and the annihilation of X force. Hence, the objective function is defined as,

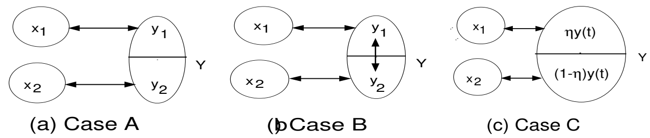

Fig. 1. *(a) Time-Zero-Allocation (TZA), (b) Allocate-Assess-Reallocate (AAR), (c) Continuous Constant Allocation (CCA)*

$$J = \gamma\big[\text{Surviving resources of Y}\big] + (1-\gamma)\big[\text{Destroyed resources of } X_1 + \text{Destroyed resources of } X_2\big] \quad (1)$$

where, $\gamma \in [0,1]$.
The classical Lanchester Square Law postulates that combat is described by the differential equations,

$$\dot{x}(t) = -\alpha y(t), \quad \dot{y}(t) = -\beta x(t) \quad (2)$$

where α and β are attrition constants independent of time. The classical Lanchester Linear Law is given by,

$$\dot{x}(t) = -\alpha x(t)y(t), \quad \dot{y}(t) = -\beta x(t)y(t) \quad (3)$$

In this paper we are dealing with the Lanchester (2,1) model instead of the classical (1,1) model. We will consider three cases of resource allocation.

A. Case A: Time-Zero-Allocation (TZA)

In this case, the initial strength of Y, $y(0)$ is partitioned into $y_1(0)$ and $y_2(0)$ using the decision parameter η. No redistribution of resources takes place when any of the resources is completely destroyed. Thus, allocation is done at the initial time (that is, at time zero) only.

Case A1: Lanchester Square Law
The attrition equations are given by,

$$\begin{aligned} \dot{x_i} &= -\alpha_i y_i, \quad x_i(0) = M_i \\ \dot{y_1} &= -\beta_1 x_1, \quad y_1(0) = \eta N \\ \dot{y_2} &= -\beta_2 x_2, \quad y_2(0) = (1-\eta)N \end{aligned} \quad (4)$$

where, x_1 and x_2 represent the resource strength of the forces X_1 and X_2 of the attacking force and y_1 and y_2 are the

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resource strengths of Y_1 and Y_2 which are obtained after partitioning the overall strength of the defending forces Y. Also, $\eta \in [0,1]$ and $(\alpha_i, \beta_i > 0,\ i = 1,2)$. Solutions to these equations are given by,

$$\begin{aligned} y_i(t) &= -\sqrt{\frac{\beta_i}{\alpha_i}}\, A_i\, e^{\sqrt{\alpha_i\beta_i}\, t} + \sqrt{\frac{\beta_i}{\alpha_i}} B_i\, e^{-\sqrt{\alpha_i\beta_i}\, t} \\ x_i(t) &= A_i\, e^{\sqrt{\alpha_i\beta_i}\, t} + B_i\, e^{-\sqrt{\alpha_i\beta_i}\, t}, \quad i = 1,2. \end{aligned} \quad (5)$$

where,

$$A_i = \left[-\frac{\eta_i N}{2}\sqrt{\frac{\alpha_i}{\beta_i}} + \frac{M_i}{2}\right],\ \ B_i = \left[\frac{\eta_i N}{2}\sqrt{\frac{\alpha_i}{\beta_i}} + \frac{M_i}{2}\right] \quad (6)$$

$i = 1,2$; $\eta_1 = \eta$ and $\eta_2 = 1-\eta$.

Now, consider the following situations that may occur during the progress of the conflict.

(a) Let $x_i = 0$ and $y_i > 0$ at time t=t_{x_i}, $i = 1,2$.

That is, at time t_{x_i}, X_i gets destroyed completely but some of the resource strength of Y still remains. Then,

$$t_{x_i} = \frac{1}{2\sqrt{\alpha_i\beta_i}} \ln\left(\frac{B_i}{-A_i}\right) \quad (7)$$

So, t_{x_i} is real and positive only when $\frac{B_i}{-A_i} \geq 1$.

Since B_i is positive always, the necessary condition for $\frac{B_i}{-A_i} \geq 1$ is $-A_i > 0$

Therefore, the condition for situation (a) to occur is,

$$0 < M_i < \eta_i N\sqrt{\frac{\alpha_i}{\beta_i}} \quad (8)$$

(b) Let y_i=0 and $x_i > 0$ at time t=t_{y_i}

That is, at time t_{y_i}, y_i gets destroyed completely but some of the resources of X_i still remains. Then,

$$t_{y_i} = \frac{1}{2\sqrt{\alpha_i\beta_i}} \ln\left(\frac{B_i}{A_i}\right) \quad (9)$$

So, t_{y_i} is real and finite only when, $\frac{B_i}{A_i} \geq 1$.

The necessary condition for $\frac{B_i}{A_i} \geq 1$ is $A_i > 0$

Therefore, the condition for situation (b) to occur is,

$$0 < \eta_i N\sqrt{\frac{\alpha_i}{\beta_i}} < M_i \quad (10)$$

The optimum value of η is obtained in the region where both the conditions $\eta > \frac{M_1}{N}\sqrt{\frac{\beta_1}{\alpha_1}}$ and $\eta < 1 - \frac{M2}{N}\sqrt{\frac{\beta_2}{\alpha_2}}$ are satisfied, when N is greater than the stronger X_i force; otherwise, the optimum value of η (say, η^*) is 0 or 1 depending on the strength of X_i force. The value of η^*, for the case when N is greater than the strength of X_i, occurs when, $\frac{dJ}{d\eta} = 0$, which gives ,

$$\eta^* = \frac{M_1\sqrt{\beta_1\alpha_2}}{M_1\sqrt{\beta_1\alpha_2} + M_2\sqrt{\beta_2\alpha_1}} \quad (11)$$

Case A2: Lanchester Linear Law

The attrition equations are given by,

$$\begin{aligned} \dot{x_i} &= -\alpha_i x_i y_i, \quad x_i(0) = M_i \\ \dot{y_1} &= -\beta_1 x_1 y_1, \quad y_1(0) = \eta N \\ \dot{y_2} &= -\beta_2 x_2 y_2, \quad y_2(0) = (1-\eta)N \end{aligned} \quad (12)$$

From (12), after some standard manipulations, we get,

$$\dot{x_i} = -\beta_i x_i^2 - k_i x_i, \quad \dot{y_i} = -\alpha_i y_i^2 + k_i y_i \quad (13)$$
$$\text{where,} \quad k_i = \Big[\beta_i x_i(0) - \alpha_i y_i(0)\Big], \quad i = 1,2.$$

Solutions to these equations (that are similar to Bernoulli's equation with time-invariant coefficients) when, $\beta_i x_i(0) \neq \alpha_i y_i(0)$ are given by,

$$\begin{aligned} x_i(t) &= \Big[x_i(0)^{-1} e^{-k_i t} + \beta_i k_i^{-1}(1 - e^{-k_i t})\Big]^{-1} \\ y_i(t) &= \Big[y_i(0)^{-1} e^{k_i t} + \alpha_i k_i^{-1}(e^{k_i t} - 1)\Big]^{-1} \end{aligned} \quad (14)$$

When $\beta_i x_i(0) = \alpha_i y_i(0)$, the solutions are given by,

$$x_i(t) = \frac{x_i(0)}{\beta_i x_i(0)t + 1}, \quad y_i(t) = \frac{y_i(0)}{\alpha_i x_i(0)t + 1} \quad (15)$$

With Lanchester Linear law, the termination time is always infinity. But the following properties hold:

(i) If $\alpha_i y_i(0) > \beta_i x_i(0)$, then as $t \to \infty$, $x_i(t) \to 0$ and $y_i(t) \to y_i(0) - (\frac{\beta_i}{\alpha_i})x_i(0)$.

(ii) If $\beta_i x_i(0) = \alpha_i y_i(0)$, then as $t \to \infty$, $x_i(t) \to 0$ and $y_i(t) \to 0$.

(iii) If $\alpha_i y_i(0) < \beta_i x_i(0)$, then as $t \to \infty$, $y_i(t) \to 0$ and $x_i(t) \to x_i(0) - (\frac{\alpha_i}{\beta_i})y_i(0)$.

For a range of values of η both x_1 and x_2 get destroyed completely as $t \to \infty$. The range of optimum values of η are given by,

$$\frac{\beta_1 M_1}{\alpha_1 N} < \eta < 1 - \frac{\beta_2 M_2}{\alpha_2 N} \quad (16)$$

When t is small, the objective function is given by,

$$\begin{aligned} J = \gamma\Bigg[&\Big\{(\eta N)^{-1} e^{k_1 t} + \alpha_1 k_1^{-1}(e^{k_1 t} - 1)\Big\}^{-1} + \\ &\Big\{((1-\eta)N)^{-1} e^{k_2 t} + \alpha_2 k_2^{-1}(e^{k_2 t} - 1)\Big\}^{-1}\Bigg] + \\ (1-\gamma)\Bigg[&M_1 - \Big\{M_1^{-1} e^{-k_1 t} + \beta_1 k_1^{-1}(1 - e^{-k_1 t})\Big\}^{-1} \\ &+ M_2 - \Big\{M_2^{-1} e^{-k_2 t} + \beta_2 k_2^{-1}(1 - e^{-k_2 t})\Big\}^{-1}\Bigg] \end{aligned} \quad (17)$$

The solution when $\frac{dJ}{d\eta} = 0$, gives the optimum value of η for smaller values of t. This equation is complicated and difficult to solve. Hence only numerical solutions are possible.

B. Case B: Allocate-Assess-Reallocate (AAR)

In this case, if x_i gets exhausted at time t_i and $x_j(t_i)$, $y_j(t_i)$, $y_i(t_i) > 0$, then the surviving $y_i(t_i)$ rejoins y_j at t_i $(i \neq j)$. Hence, in this case, redistribution of resources for Y takes place when any of the resources of X becomes zero. This case should not be considered for linear law, since the resources becomes zero only at $t = \infty$.

Case B1: Lanchester Square Law

Consider the following situations that may occur during the progress of the conflict.

(a) If $M_1 < \eta N\sqrt{\frac{\alpha_1}{\beta_1}}$ and $M_2 < (1-\eta)N\sqrt{\frac{\alpha_2}{\beta_2}}$ and if $t_{x_i} < t_{x_j}$, $i \neq j$, then x_i becomes zero before x_j.
Surviving defence strength is,

$$y_i(t_{x_i}) = \sqrt{\frac{\beta_i}{\alpha_i}}[2\sqrt{-A_i}\sqrt{B_i}] \quad (18)$$

$$y_j(t_{x_i}) = -\sqrt{\frac{\beta_j}{\alpha_j}}\, A_j\, e^{\sqrt{\alpha_j\beta_j}\ t_{x_i}} + \sqrt{\frac{\beta_j}{\alpha_j}}\, B_j\, e^{-\sqrt{\alpha_j\beta_j}\ t_{x_i}}$$

Now, since x_i is zero, y_i rejoins y_j at t_{x_i}.
Let, t_I=$t - t_{x_i}$, then

$$\begin{aligned} y_j(t_I) &= -\sqrt{\frac{\beta_j}{\alpha_j}}\, A_j\, e^{\sqrt{\alpha_j\beta_j}\ t_{x_i}} + \sqrt{\frac{\beta_j}{\alpha_j}}\, B_j\, e^{-\sqrt{\alpha_j\beta_j}\ t_{x_i}} \\ &\quad + \sqrt{\frac{\beta_i}{\alpha_i}}[2\sqrt{-A_i}\sqrt{B_i}] \\ x_j(t_I) &= A_j\, e^{\sqrt{\alpha_j\beta_j}\ t_{x_i}} + B_j\, e^{-\sqrt{\alpha_j\beta_j}\ t_{x_i}} \end{aligned} \quad (19)$$

Therefore,

$$\begin{aligned} y_j(t) &= -\sqrt{\frac{\beta_j}{\alpha_j}}\, C_j\, e^{\sqrt{\alpha_j\beta_j}\ t} + \sqrt{\frac{\beta_j}{\alpha_j}}\, D_j\, e^{-\sqrt{\alpha_j\beta_j}\ t} \\ x_j(t) &= C_j\, e^{\sqrt{\alpha_j\beta_j}\ t} + D_j\, e^{-\sqrt{\alpha_j\beta_j}\ t} \end{aligned} \quad (20)$$

The time at which x_j becomes zero, $(t_{x_{jf}})$ is given by,

$$t_{x_{jf}} = \frac{1}{2\sqrt{\alpha_j\beta_j}} \ln\left(\frac{-D_j}{C_j}\right) \quad (21)$$

$$\begin{aligned} \text{where, } C_j &= \left[-\frac{y_j(t_I)}{2}\sqrt{\frac{\alpha_j}{\beta_j}} + \frac{x_j(t_I)}{2}\right] \\ D_j &= \left[\frac{y_j(t_I)}{2}\sqrt{\frac{\alpha_j}{\beta_j}} + \frac{x_j(t_I)}{2}\right] \end{aligned} \quad (22)$$

(b) If $M_1 > \eta N\sqrt{\frac{\alpha_1}{\beta_1}}$ and $M_2 < (1-\eta)N\sqrt{\frac{\alpha_2}{\beta_2}}$ and if $t_{y_1} < t_{x_2}$, then y_1 becomes zero before x_2. Then, surviving offensive force strengths are,

$$x_i(t_{y_1}) = A_i\, e^{\sqrt{\alpha_i\beta_i}\ t_{y_1}} + B_i\, e^{-\sqrt{\alpha_i\beta_i}\ t_{y_1}} \quad (23)$$

Surviving defensive force strength is,

$$y_2(t_{y_1}) = -\sqrt{\frac{\beta_2}{\alpha_2}}\, A_2\, e^{\sqrt{\alpha_2\beta_2}\ t_{y_1}} + \sqrt{\frac{\beta_2}{\alpha_2}}\, B_2\, e^{-\sqrt{\alpha_2\beta_2}\ t_{y_1}} \quad (24)$$

Surviving Y force when X_2 was destroyed is,

$$y_2(t_{x_2}) = -\sqrt{\frac{\beta_2}{\alpha_2}}\, A_2\, e^{\sqrt{\alpha_2\beta_2}\ t_{x_2}} + \sqrt{\frac{\beta_2}{\alpha_2}}\, B_2\, e^{-\sqrt{\alpha_2\beta_2}\ t_{x_2}} \quad (25)$$

Now, surviving y_2 rejoins y_1 at time t_{x_2} and engages with X_1.
Let, t_I=$t - t_{x_2}$

$$x_1(t_I) = x_1(t_{y_1}), \quad y_1(t_I) = y_2(t_{x_2}) \quad (26)$$

Therefore,

$$\begin{aligned} y_1(t) &= -\sqrt{\frac{\beta_1}{\alpha_1}}\, C_1\, e^{\sqrt{\alpha_1\beta_1}\ t} + \sqrt{\frac{\beta_1}{\alpha_1}}\, D_1\, e^{-\sqrt{\alpha_1\beta_1}\ t} \\ x_1(t) &= C_1 e^{\sqrt{\alpha_1\beta_1}\ t} + D_1 e^{-\sqrt{\alpha_1\beta_1}\ t} \end{aligned} \quad (27)$$

(c) If $M_1 < \eta N\sqrt{\frac{\alpha_1}{\beta_1}}$ and $M_2 > (1-\eta)N\sqrt{\frac{\alpha_2}{\beta_2}}$ and if $t_{y_2} < t_{x_1}$, then y_2 becomes zero before x_1.
Surviving attacking force strengths are,

$$x_i(t_{y_2}) = A_i\, e^{\sqrt{\alpha_i\beta_i}\ t_{y_2}} + B_i\, e^{-\sqrt{\alpha_i\beta_i}\ t_{y_2}} \quad (28)$$

Surviving defensive force strength is,

$$y_1(t_{y_2}) = -\sqrt{\frac{\beta_1}{\alpha_1}}\, A_1\, e^{\sqrt{\alpha_1\beta_1}\ t_{y_2}} + \sqrt{\frac{\beta_1}{\alpha_1}}\, B_1\, e^{-\sqrt{\alpha_1\beta_1}\ t_{y_2}} \quad (29)$$

Surviving Y force when X_1 was destroyed is,

$$y_1(t_{x_1}) = -\sqrt{\frac{\beta_1}{\alpha_1}}\, A_1 e^{\sqrt{\alpha_1\beta_1}\ t_{x_1}} + \sqrt{\frac{\beta_1}{\alpha_1}}\, B_1\, e^{-\sqrt{\alpha_1\beta_1}\ t_{x_1}} \quad (30)$$

Now, surviving y_1 rejoins y_2 at time t_{x_1} and engages with X_2.
Let, t_I=$t - t_{x_1}$

$$x_2(t_I) = x_2(t_{y_2}),\ y_2(t_I) = y_1(t_{x_1}) \quad (31)$$

Therefore,

$$\begin{aligned} y_2(t) &= -\sqrt{\frac{\beta_2}{\alpha_2}}\, C_2\, e^{\sqrt{\alpha_2\beta_2}\ t} + \sqrt{\frac{\beta_2}{\alpha_2}}\, D_2\ e^{-\sqrt{\alpha_2\beta_2}\ t} \\ x_2(t) &= C_2\, e^{\sqrt{\alpha_2\beta_2}\ t} + D_2\, e^{-\sqrt{\alpha_2\beta_2}\ t} \end{aligned} \quad (32)$$

For this case, the optimum η for larger values of N are always 1 and 0. Since N is large, both X_1 and X_2 are destroyed completely. Hence, the second term in the objective function expression, that is the $(1-\gamma)$ term will be the same for both $\eta = 1$ and $\eta = 0$. Hence, if the first term, that is, the term involving the surviving Y resources, is the same for both $\eta = 1$ and $\eta = 0$, then the objective function value for both $\eta = 1$ and $\eta = 0$ will be the same and hence also the optimum value.
When $\eta = 0$, X_2 is destroyed by Y_2 at time $t = t_{x_2}$.

$$y_2(t_{x_2}) = \left[N^2 - \frac{\beta_2}{\alpha_2}M_2^2\right]^{1/2} \quad (33)$$

Now, $y_1(t_I) = y_2(t_{x_2})$, $x_1(t_I) = x_1(t_{x_2}) = M_1$

$$y_1(t_{x_1}) = \left(N^2 - \frac{\beta_2}{\alpha_2}M_2^2 - \frac{\beta_1}{\alpha_1}M_1^2\right)^{1/2} \quad (34)$$

When $\eta = 1$, X_1 is destroyed by Y_1 at time $t = t_{x_1}$.

$$y_1(t_{x_1}) = \left[N^2 - \frac{\beta_1}{\alpha_1}M_1^2\right]^{1/2} \quad (35)$$

Now, $y_2(t_I) = y_1(t_{x_1})$, $x_2(t_I) = x_2(t_{x_1}) = M_2$

$$y_2(t_{x_2}) = \left(N^2 - \frac{\beta_1}{\alpha_1}M_1^2 - \frac{\beta_2}{\alpha_2}M_2^2\right)^{1/2} \quad (36)$$

Since the expressions for both $y_1(t_{x_1})$ and $y_2(t_{x_2})$ are the same, the objective function value will be the same for both $\eta = 1$ and $\eta = 0$ and therefore η^*=0 and 1. For smaller values of N, the optimum η is either 1 or 0 depending on the strength of X_1 and X_2 force.

C. Case C: Continuous Constant Allocation (CCA)

In this case, continuous allocation of resources takes place with constant η. Thus, at any given time the strength y is partitioned into y_1 and y_2 by the constant factor η.

Case C1: Lanchester Square Law
The attrition equations for $\eta \in (0, 1)$ are given by,

$$\begin{aligned} \dot{x_1} &= -\alpha_1 \ \eta \ y, & x_1(0) = M_1 \\ \dot{x_2} &= -\alpha_2(1-\eta)y, & x_2(0) = M_2 \\ \dot{y} &= -\beta_1 x_1 - \beta_2 x_2, & y(0) = N \end{aligned} \quad (37)$$

Note that for $\eta = 0$ and 1, the analysis for the TZA case (Case A1) holds. The solutions to these equations are given by,

$$x_i(t) = M_i + \frac{\alpha_i \eta_i}{2\omega} \left[(k_1 - N)e^{\omega t} + (k_1 + N)e^{-\omega t} - 2k_1\right]$$

$$y(t) = -\frac{1}{2}(k_1 - N)e^{\omega t} + \frac{1}{2}(k_1 + N)e^{-\omega t} \quad (38)$$

where, $\eta_1 = \eta$, $\eta_2 = 1-\eta$, $k_1 = \frac{\beta_1 M_1 + \beta_2 M_2}{\omega}$ and $\omega^2 = \alpha_1\beta_1\eta + \alpha_2\beta_2(1-\eta)$.
Now, consider the following situations that may occur during the progress of the conflict.
(a) Let $x_i = 0$ and $y > 0$ at time t=t_{x_i}, $i = 1, 2$. That is, at time t_{x_i}, X_i gets destroyed completely but some of the resource strength of Y still remains. Then,

$$t_{x_i} = \frac{1}{\omega} \ln\left(\frac{A}{B}\right) \quad (39)$$

where,

$$\begin{aligned} B &= 2(k_1 - N), \\ A &= \left(\frac{-2M_i\omega}{\alpha_i\eta_i} + 2k_1\right) \pm \\ &\quad \sqrt{\left(\frac{-2M_i\omega}{\alpha_i\eta_i} + 2k_1\right)^2 - 4\left(k_1^2 - N^2\right)} \end{aligned} \quad (40)$$

For t_{x_i} to be real and positive, $\frac{A}{B} \geq 1$.
The necessary conditions for $\frac{A}{B} \geq 1$ are,

$$M_i \leq \frac{\alpha_i\eta_i}{\omega}\left[k_1 - \sqrt{k_1^2 - N^2}\right] \quad \text{and} \quad k_1 \neq N \quad (41)$$

When $x_i = 0$, the attrition equations become,

$$\dot{x_j} = -\alpha_j n_1 y, \quad \dot{y} = -\beta_j x_j \quad (42)$$

The solutions to these equations are given by,

$$y(t) = \left[\frac{y(t_{x_i}) - \beta_j x_j(t_{x_i})/a}{2}\right] e^{at} + \left[\frac{y(t_{x_i}) + \beta_j x_j(t_{x_i})/a}{2}\right] e^{-at} \quad (43)$$

$$x_j(t) = x_j(t_{x_i}) - \frac{\alpha_j n_1}{2a}\Big[(y(t_{x_i}) - \beta_j x_j(t_{x_i})/a)e^{at} - (y(t_{x_i}) + \beta_j x_j(t_{x_i})/a)e^{-at} + 2\beta_j x_j(t_{x_i})\Big] \quad (44)$$

where,

$$a = \sqrt{\beta_j\alpha_j n_1}, \quad n_1 = \begin{cases} 1-\eta & \text{if } x_1 = 0 \\ \eta & \text{if } x_2 = 0 \end{cases} \quad (45)$$

(b) Let $y = 0$ and $x_i > 0$, at time t=t_y, $i = 1, 2$. That is, at time t_y, Y get destroyed completely but some of the resource strength of X_i still remains. Then,

$$t_y = \frac{1}{2\omega} \ln\left\{\frac{(k_1 + N)}{(k_1 - N)}\right\} \quad (46)$$

For t_y to be real and positive,

$\frac{(k_1+N)}{(k_1-N)} \geq 1$ and $k_1 > N$

For $x_i > 0$,

$$M_i > \frac{\alpha_i\eta_i}{\omega}\left[k_1 - \sqrt{k_1^2 - N^2}\right] \quad (47)$$

where, $\eta_1 = \eta$ and $\eta_2 = 1 - \eta$.
Therefore, the conditions for situation (b) to occur are,

$$M_i > \frac{\alpha_i\eta_i}{\omega}\left[k_1 - \sqrt{k_1^2 - N^2}\right], \quad k_1 > N \quad (48)$$

For this case, the optimum value of η (say, η^*), when $k_1 < N$ and if X force is destroyed completely, occurs when, $\frac{dJ}{d\eta} = 0$, which gives,

$$\eta^* = \frac{M_1\sqrt{\beta_1\alpha_2}}{M_1\sqrt{\beta_1\alpha_2} + M_2\sqrt{\beta_2\alpha_1}} \quad (49)$$

The optimum value of η (η^*), when $k_1 < N$ and if Y force is destroyed completely, occurs when, $\frac{dJ}{d\eta} = 0$, which gives,

$$\eta^* = 1 + \frac{M_2\sqrt{\alpha_2\beta_2}}{-\alpha_2 N} \quad (50)$$

When $k_1 > N$ and $M_2 < \frac{\alpha_2(1-\eta)}{w}\left[k_1 - \sqrt{k_1^2 - N^2}\right]$, the optimum value of η is given by,

$$\eta^* = \frac{-b \pm \sqrt{b^2 - q}}{2\alpha_2^2 N^2} \quad (51)$$

where, $b = M_2^2\alpha_1\beta_1 + 2\alpha_2 M_1 M_2\beta_1 + M_2^2\alpha_2\beta_2 - 2\alpha_2^2 N^2$ and $q = 4\alpha_2^2 N^2(-2\beta_1\alpha_2 M_1 M_2 + \alpha_2^2 N^2 - \alpha_2\beta_2 M_2^2)$.
There are two solutions of which one is inside the above specified range and the other is outside the range. So, the optimum η is the value which is inside the range. For lower values of N, the optimum η is always 0.

Case C2: Lanchester Linear Law
The attrition equations for $\eta \in (0, 1)$ are given by,

$$\begin{aligned} \dot{x_1} &= -\alpha_1 \ x_1\eta \ y, & x_1(0) = M_1 \quad (52) \\ \dot{x_2} &= -\alpha_2 x_2(1-\eta)y, & x_2(0) = M_2 \quad (53) \\ \dot{y} &= -\beta_1 x_1\eta y - \beta_2 x_2(1-\eta)y, & y(0) = N \quad (54) \end{aligned}$$

Note that for $\eta = 0$ and 1, the analysis for the TZA case (Case A2) holds.
From (52) and (54),

$$y = aC_1 x_1^{\frac{1}{a}} + \frac{\beta_1}{\alpha_1}x_1 + D_1 \quad (55)$$

where, $a = \frac{\alpha_1 \eta}{\alpha_2(1-\eta)}, \quad C_1 = \frac{\beta_2(1-\eta)x_2(0)}{\alpha_1 \eta x_1(0)^{\frac{1}{a}}},$

$$D_1 = y(0) - \frac{\beta_1}{\alpha_1}x_1(0) - aC_1 x_1(0)^{\frac{1}{a}}$$

From (53) and (54),

$$y = \frac{C_2}{a}x_2^a + \frac{\beta_2}{\alpha_2}x_2 + D_2 \qquad (56)$$

where,

$$\begin{aligned} C_2 &= \frac{\beta_1 \eta x_1(0)}{\alpha_2(1-\eta)x_2(0)^a} \\ D_2 &= y(0) - \frac{C_2}{a}x_2(0)^a - \frac{\beta_2}{\alpha_2}x_2(0) \end{aligned} \qquad (57)$$

Substituting (55) and (56) in (52) and (53), respectively, we get,

$$\frac{dx_1}{x_1\left(aC_1 x_1^{\frac{1}{a}} + \frac{\beta_1}{\alpha_1}x_1 + D_1\right)} = -\alpha_1 \eta \, dt \qquad (58)$$

$$\frac{dx_2}{x_2\left(\frac{C_2}{a}x_2^a + \frac{\beta_2}{\alpha_2}x_2 + D_2\right)} = -\alpha_2(1-\eta) \, dt \qquad (59)$$

These equations seem to be unsolvable in the closed form. But for specific value of the parameter they have solution. For example, consider the case when $a = 1$, we get,

$$x_1(t) = \frac{D_1}{e^{-[-\alpha_1 \eta t D_1 + E_1 D_1]} - (C_1 + \frac{\beta_1}{\alpha_1})} \qquad (60)$$

$$x_2(t) = \frac{D_2}{e^{-[-\alpha_2(1-\eta) t D_2 + E_2 D_2]} - (C_2 + \frac{\beta_2}{\alpha_2})} \qquad (61)$$

$$\begin{aligned} y(t) = \exp\Bigg\{ & \frac{\beta_1 \eta D_1}{C_1 + \frac{\beta_1}{\alpha_1}}\left[t - \frac{1}{\alpha_1 \eta D_1}\ln\left(e^{\alpha_1 \eta D_1 t - E_1 D_1} - (C_1 + \frac{\beta_1}{\alpha_1})\right)\right] + \frac{\beta_2(1-\eta)D_2}{C_2 + \frac{\beta_2}{\alpha_2}}\left[t - \frac{1}{\alpha_2(1-\eta)D_2} \ln\left(e^{\alpha_2(1-\eta)D_2 t - E_2 D_2} - (C_2 + \frac{\beta_2}{\alpha_2})\right)\right] + \ln y(0) - \\ & \frac{\beta_1 \eta D_1}{C_1 + \frac{\beta_1}{\alpha_1}}\left[-\frac{1}{\alpha_1 \eta D_1}\ln\left(e^{-E_1 D_1} - (C_1 + \frac{\beta_1}{\alpha_1})\right)\right] \\ & -\frac{\beta_2(1-\eta)D_2}{C_2 + \frac{\beta_2}{\alpha_2}}\left[-\frac{1}{\alpha_2(1-\eta)D_2}\ln\left(e^{-E_2 D_2} - (C_2 + \frac{\beta_2}{\alpha_2})\right)\right]\Bigg\} \end{aligned} \qquad (62)$$

where,

$$E_1 = \frac{1}{D_1}\ln\frac{x_1(0)}{(C_1 + \frac{\beta_1}{\alpha_1})x_1(0) + D_1} \qquad (63)$$

$$E_2 = \frac{1}{D_2}\ln\frac{x_2(0)}{(C_2 + \frac{\beta_2}{\alpha_2})x_2(0) + D_2} \qquad (64)$$

When $\eta = 1$, the state equations are given by,

$$\dot{x_1} = -\alpha_1 y x_1, \quad \dot{y} = -\beta_1 x_1 y \qquad (65)$$

When $\eta = 0$, the state equations are given by,

$$\dot{x_2} = -\alpha_2 y x_2, \quad \dot{y} = -\beta_2 x_2 y \qquad (66)$$

The solution to (65) is the same as (14), when $i = 1$ and solution to (66) is the same as (14), when $i = 2$. Hence the analysis for Case $A2$ holds for $\eta = 0$ and 1.

III. Some Numerical Results

For all the example cases, $M_1 = 150$, $M_2 = 200$, $\alpha_1 = 2$, $\alpha_2 = 3$, $\beta_1 = 1$, $\beta_2 = 2$ and $\gamma = 0.5$ are chosen.

Case A (TZA): The simulation results for Case $A1$ with different initial conditions are shown in Figure 2 (a). It can be seen that, for larger values of N, η^* is the value corresponding to Equation 11 and for lower values of N, $\eta^* = 0$ or 1. The variation of time instants, at which resources get destroyed completely, against η, for various values of N, are shown in Figure 3. It can be seen that, when N is very large, only the X forces are destroyed. As N is decreased, both Y and X forces are destroyed for different η but the simultaneous destruction of both the forces occurs only at infinity. When N is very small compared to the strength of X, only the Y force gets destroyed as shown in Figure 3 when $N = 100$.

In Figure 2 (b), the curve η_1 represents the values of η at which both x_1 and y_1 becomes zero simultaneously, when N is varied. The curve η_2 has similar information for x_2 and y_2. The region (x_1, y_2) represents the region where x_1 and y_2 are destroyed. Similarly, the other regions represent the regions of destruction of the corresponding resources. When $N < 165$, X_2 is not destroyed and when $N < 107$, X_1 is not destroyed and only Y gets destroyed.

Case B (AAR): The simulation results for Case B with different initial conditions are shown in Figure 4 (a). For larger values of N, η^* is 0 and 1. For smaller values of N, $\eta^* = 1$ or 0, similar to Case A. The change in values of x_1, x_2, y_1 and y_2 with time for Case B with constant η are shown in Figure 5. Figure 5 (a) represents the case when y_1 is destroyed first. Figure 5 (b) represents the case when x_2 is destroyed first and Figure 5 (c) represents the case when y_2 is destroyed first.

Case C (CCA): The simulation results for Case $C1$ with different initial conditions are shown in Figure 4 (b). When N is small compared to the strength of X, the optimum η is 0 as shown in Figure 4 (b) for $N = 100$ and $N = 200$. In Figure 4 (b), for $N = 250$, η^* is the value corresponding to Equation (51), for $N = 265$, η^* is the value corresponding to Equation (50) and for $N = 300$ and $N = 400$, η^* is the value corresponding to Equation (49).

The simulation results for Linear Law are shown in Figure 6. It can be seen that for large values of t, the objective function becomes almost independent of η for Case $C2$.

IV. Conclusions

In this paper we analyzed a battle between a defender with only one type of force and an attacker with two types of forces. The victory was assumed to be by the total destruction of the attacking force towards which the battle can progress

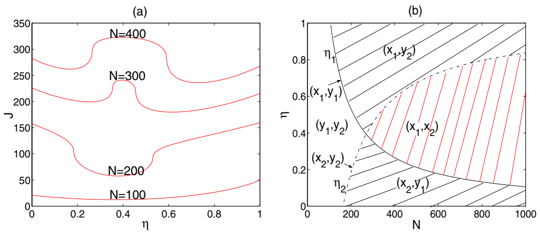

Fig. 2. *(a) Objective function value for Case A1 (TZA), (b) Event zones in the (N, η) space for Case A1 (TZA)*

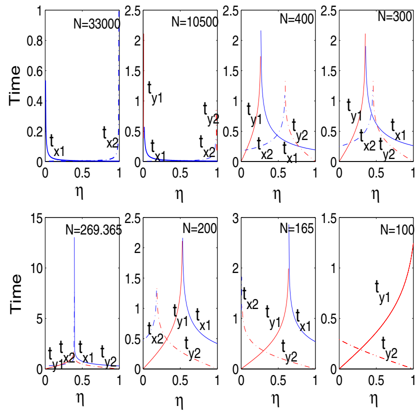

Fig. 3. *Resource destruction times for Case A1 (TZA)*

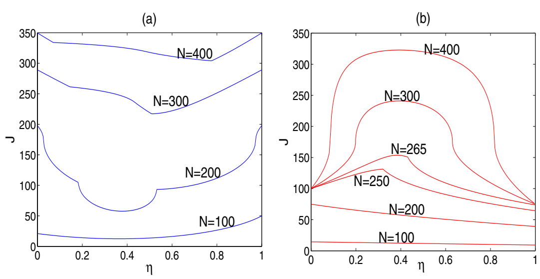

Fig. 4. *(a) Objective function value for Case B (AAR), (b) Objective function value for Case C1 (CCA)*

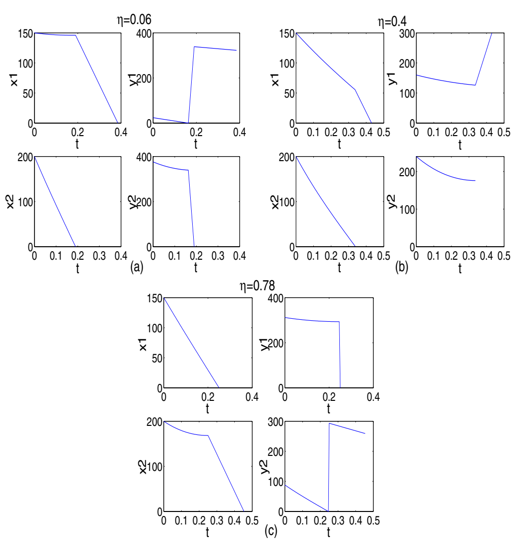

Fig. 5. *Example for Case B (AAR): (a) $\eta = 0.06$, (b) $\eta = 0.4$, (c) $\eta = 0.78$*

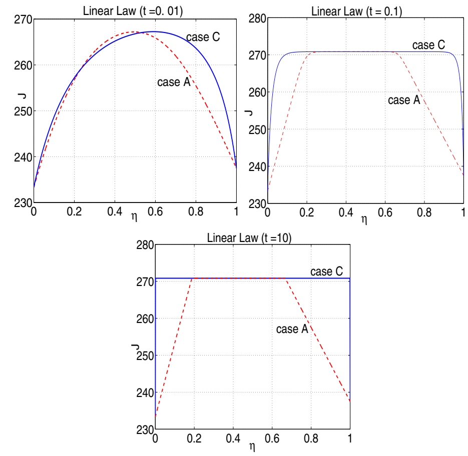

Fig. 6. *Comparison of Case A and Case C for Linear Law*

in any one of three ways. The results shows that for Square law, it is always better to allocate the whole of Y strength to the stronger X_i force alone when the strength of Y is small compared to the strength of X force. This shows that if the strength is less compared to the opposing force, instead of defending all the forces at a time, one should defend only one force at a time. For linear law, only Case A and Case C exists. If the battle progresses as in Case A, a range of η values will give the optimum performance. For Case C, the performance is independent of η when the duration of battle is large. The existence of the solutions for various cases of the (2,1) model considered in this paper provides a possibility of the solutions for the $(n, 1)$ model which will be a future extension of this work.

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