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Efficient convex design of robust feedforward controllers

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Abstract— **A practical method is proposed for the convex design of robust feedforward controllers, which ensure H_∞/L_2 performance in the face of LTI and arbitrarily time-varying model uncertainties. A technique which computes the global minimum of this difficult infinite dimensional optimization problem is proposed, as well as a suboptimal but computationally less involving algorithm. A missile example illustrates the efficiency of the scheme: a robust feedforward controller is designed, either on the continuum of linearised time invariant models (corresponding to trim points), or on a quasi-LPV model representing the non-linear one.**

I. INTRODUCTION

The issue of designing robust feedforward controllers has been rather underestimated in the litterature w.r.t. the feedback case, despite its interest in industrial problems: in practice the aim of many design specifications is indeed to shape the time- or frequency-domain responses of a closed loop to a reference input rather than to an unmeasured disturbance. Moreover, if the convex closed loop design of a feedback controller requires Youla parametrisation [1], so that the order of the controller is necessarily greater or equal than the one of the open loop plant, the design of a robust feedforward controller which simultaneously ensures time- or frequency-domain specifications on several closed loops (for a fixed feedback controller) is much simpler, since it directly reduces to the minimisation of a convex objective under convex constraints, corresponding to the design specifications [2], [3].

We consider in this paper a much more complex problem, namely the design of a robust feedforward controller either on a continuum of linearised time invariant models (corresponding to trim points) or on an LPV model. The issue is just to obtain a closed loop LFT model, either with only LTI model uncertainties, or with mixed LTI and arbitrarily time-varying ones. The design of a feedforward controller, which minimises an upper bound of the worst-case H_∞/L_2 performance in the face of these model uncertainties, remains convex [4]. This is nevertheless a difficult optimization problem, with an infinite number of optimization parameters and frequency domain constraints. The principle is to solve it first on a frequency gridding with an LMI solver, and then to validate the result with a μ frequency sweeping technique [5], [6], [7], even when time-varying uncertainties are accounted for: remember that the structured singular value (s.s.v.) μ is used to analyse the robustness properties in the face of LTI model uncertainties.

A suboptimal but computationally less involving solution to this infinite dimensional optimization problem is also proposed, in which the frequency-dependent D, G scaling matrices, corresponding to a μ upper bound, are computed with the routine $mu.m$ of the μ-Analysis and Synthesis Toolbox instead of an LMI solver. Note that the μ frequency sweeping technique also uses this routine.

Note that to a large extent the order and structure of the feedforward controller $H(s)$ are free. The sole constraint is to put it under the form $H(s) = \sum_{i=1}^{N} \theta_i H_i(s)$, where filters $H_i(s)$ are fixed while the θ_i are the design parameters. As for the choice of the filters, the orthonormal basis of [8] is used to reduce numerical problems. Moreover, with a suitable choice of the poles of the basis the whole set of asymptotically stable transfer matrices is covered by this infinite-dimensional basis.

A missile example illustrates the computational efficiency of the method. The non-linear missile model is extracted from [9], a robust feedback controller is synthesised with a quadratic stability method [10] and the missile model is finally put under an LFT form with the LFR Toolbox [11]. The feedforward controller is then designed, either on the continuum of linearised time invariant models, or on a quasi-LPV model representing the non-linear one.

The paper is organised as follows. Section II states the problem while section III presents the method. Section IV then details the missile application and concluding remarks end the paper.

II. PROBLEM STATEMENT

With reference to Fig. 1.a, the issue is to design a feedforward controller $H(s)$ which minimises the L_2 induced norm of the transfer function $T_{y_r \to y}$ between the reference input y_r and y despite model uncertainties in $\Delta = diag(\Delta_1, \Delta_2)$, where $\Delta_1 = diag(\delta_i^{TI} I_{r_i})$ contains LTI parametric uncertainties δ_i^{TI} and $\Delta_2 = diag(\delta_i^{TV} I_{q_i})$ contains arbitrarily time varying parametric uncertainties δ_i^{TV}. Neglected dynamics could also be accounted for, and each normalised time invariant or time varying parametric uncertainty satisfies $\delta_i \in [-1,\ 1]$. As a consequence let the unit ball $B\Delta = \{\Delta \mid \overline{\sigma}(\Delta) < 1\}$. The issue is to minimise (an upper bound of) γ under the induced L_2 norm constraint:

$$\|T_{y_r \to y}\|_{iL_2} \leq \gamma \ \ \forall \Delta \in B\Delta \tag{1}$$

Frequency domain templates can be included in P.

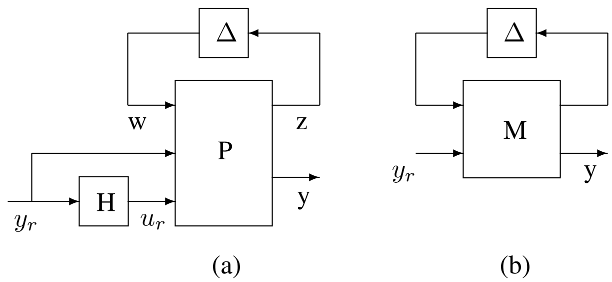

Fig. 1. Design scheme (a) and standard interconnection structure (b).

III. THE METHOD

As a preliminary the s.s.v. μ and its upper bound are introduced in section III-A, as well as the μ frequency sweeping technique. Section III-B then details how to convert the design specifications of section II into a convex infinite dimensional optimization problem. An efficient solution to this problem is proposed in section III-C, as well as a suboptimal but computationally less involving algorithm in section III-E. The aim of section III-D is to explain how the μ frequency sweeping technique is used in this context.

A. Preliminaries

The s.s.v. μ is first defined. The next lemma then introduces the μ upper bound of [12].

Definition 3.1: Let a structured model perturbation $\Delta = diag(\delta_1 I_{q_1}, \ldots, \delta_n I_{q_n}, \Delta_1^C, \ldots, \Delta_m^C)$, i.e. a block diagonal matrix with real parametric uncertainties $\delta_i \in [-1, 1]$ and full complex blocks Δ_j^C, i.e. complex matrices without specific structure satisfying $\overline{\sigma}(\Delta_j^C) < 1$. Δ thus belongs to the unit ball $B\Delta$. Let N a given complex matrix. The s.s.v. $\mu_\Delta(N)$ is defined as the inverse of the size of the smallest model perturbation Δ satisfying $det(I - \Delta N) = 0$.

Lemma 3.2: As a sufficient condition, $\mu_\Delta(N) \leq \beta$ if there exist scaling matrices $D = D^* > 0$ and $G = G^*$, whose structure fits the one of Δ (i.e. $D\Delta = \Delta D$ and $G\Delta = \Delta^* G$) satisfying:

$$\overline{\sigma}\left(\left(\frac{DND^{-1}}{\beta} - jG\right)\left(I + G^2\right)^{-1/2}\right) < 1 \quad (2)$$

If $\Delta = diag(\delta_1 I_{q_1}, \ldots, \delta_n I_{q_n}, \Delta_1^C, \ldots, \Delta_m^C)$, the structures of the matrices D and G are:

$$\begin{aligned} D &= diag(D_1, \ldots, D_n, d_1 I_{r_1}, \ldots, d_m I_{r_m}) \\ G &= diag(G_1, \ldots, G_n, 0, \ldots, 0) \end{aligned}$$

where the dimension of D_i and G_i is q_i and $\Delta_j^C \in C^{r_j, r_j}$.

The issue is to minimize the μ upper bound β w.r.t. scaling matrices D and G (i.e. w.r.t. submatrices D_i and G_i and scalars d_j). This can be done either with the $\overline{\sigma}$ formulation above [12], implemented in the routine $mu.m$ of the μ-Analysis and Synthesis Toolbox, or with the original LMI formulation.

Let us now state the general problem solved by the μ frequency sweeping technique [5], [6], [7].

μ test problem: Let a given transfer matrix $N(s)$ and Δ an LTI structured model perturbation. Do there exist scaling matrices $D(\omega)$ and $G(\omega)$ satisfying for $\omega \in [0, +\infty)$:

$$\overline{\sigma}\left(\left(D(\omega)N(j\omega)D^{-1}(\omega) - jG(\omega)\right)\left(I + G^2(\omega)\right)^{-1/2}\right) < 1$$

The issue is to test whether $\mu(N(j\omega)) \leq 1$ over the frequency domain with the μ upper bound. This can be performed in a very efficient way by computing the μ upper bound over a (small) finite subset of frequencies ω_i. If the μ upper bound is found to be greater than 1 at a frequency ω_i, stop since a worst-case frequency is found which does not satisfy the test. Otherwise let $D(\omega_i)$ and $G(\omega_i)$ the computed values of the scaling matrices at frequency ω_i. The issue is to compute the validity domain of these scaling matrices, i.e. the frequency intervals inside which:

$$\overline{\sigma}\left(\left(D(\omega_i)N(j\omega)D^{-1}(\omega_i) - jG(\omega_i)\right)\left(I + G^2(\omega_i)\right)^{-1/2}\right) < 1$$

so as to finally eliminate with a few frequency points ω_i the whole frequency domain. See [5], [6] for the detailed algorithm and [7] for an efficient implementation based on the routine $mu.m$ of the μ-Analysis and Synthesis Toolbox.

B. The infinite dimensional optimization problem

We now go back to the problem of section II. As a preliminary, Fig. 1.a is transformed into Fig. 1.b.

If $P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \end{bmatrix}$, then:

$$M = \begin{bmatrix} P_{11} & P_{12} + P_{13}H \\ P_{21} & P_{22} + P_{23}H \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad (3)$$

The following proposition is just an application of the robustness analysis theory.

Proposition 3.3: Assume that $M(s)$ is asymptotically stable. Let $D_1(\omega) = D_1^*(\omega) > 0$ and $G_1(\omega) = G_1^*(\omega)$ some frequency-dependent scaling matrices whose structure fits the one of Δ_1. Let then $D_2 = D_2^* > 0$ and $G_2 = G_2^*$ some constant scaling matrices whose structure fits the one of Δ_2, assuming moreover that D_2 is a real matrix and G_2 an imaginary matrix. Let $\mathcal{D}(\omega) = diag(D_1(\omega), D_2, I)$ and $\mathcal{G}(\omega) = diag(G_1(\omega), G_2, 0)$. As a sufficient condition (1) is satisfied if:

$$\begin{aligned} & M_\gamma(j\omega)\mathcal{D}(\omega)M_\gamma^*(j\omega) + \\ & \quad j\left(\mathcal{G}(\omega)M_\gamma^*(j\omega) - M_\gamma(j\omega)\mathcal{G}(\omega)\right) < \mathcal{D}(\omega) \quad (4) \end{aligned}$$

for $\omega \in [0, +\infty)$ and with $M_\gamma = M \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\gamma} \end{bmatrix}$.

The next proposition then states the main result, namely the convex infinite dimensional optimization problem.

Proposition 3.4: With reference to proposition 3.3, let $D(\omega) = diag(D_1(\omega), D_2)$ and $G(\omega) = diag(G_1(\omega), G_2)$. Let then $H(s) = \sum_{i=1}^{N} \theta_i H_i(s)$, where filters $H_i(s)$ are fixed while the θ_i are the design parameters. As a sufficient condition (1) is satisfied if there exist θ_i and frequency dependent scaling matrices $D(\omega)$ and $G(\omega)$ satisfying $\forall \omega \in [0, +\infty)$ (the ω dependence is dropped out to alleviate the notations):

$$\left[\begin{array}{ccc} D - P_{11}DP_{11}^* + \\ j\left(P_{11}G - GP_{11}^*\right) & -(P_{11}D + jG)P_{21}^* & M_{12} \\ -P_{21}(DP_{11}^* - jG) & \gamma I - P_{21}DP_{21}^* & M_{22} \\ M_{12}^* & M_{22}^* & \gamma I \end{array}\right] > 0 \quad (5)$$

with $M_{12} = P_{12} + P_{13}\sum_{i=1}^{N}\theta_i H_i$ and $M_{22} = P_{22} + P_{23}\sum_{i=1}^{N}\theta_i H_i$.

Proof of Proposition 3.4: First assume that $\gamma = 1$ to alleviate the notations. (4) can be equivalently rewritten:

$$\left[\begin{array}{c} M_{11} \\ M_{21} \end{array}\right] D \left[\begin{array}{cc} M_{11}^* & M_{21}^* \end{array}\right] + j\left[\begin{array}{cc} GM_{11}^* - M_{11}G & GM_{21}^* \\ -M_{21}G & 0 \end{array}\right]$$

$$-\left[\begin{array}{cc} D & 0 \\ 0 & I \end{array}\right] + \left[\begin{array}{c} M_{12} \\ M_{22} \end{array}\right]\left[\begin{array}{cc} M_{12}^* & M_{22}^* \end{array}\right] < 0$$

i.e. $-Q + SS^* < 0$, with $S = \left[\begin{array}{c} M_{12} \\ M_{22} \end{array}\right]$. Just apply the Schur complement and finally note that:

$$M_\gamma = M\left[\begin{array}{cc} I & 0 \\ 0 & \frac{I}{\gamma} \end{array}\right] = \left[\begin{array}{cc} M_{11} & \frac{M_{12}}{\gamma} \\ M_{21} & \frac{M_{22}}{\gamma} \end{array}\right] \qquad \square$$

Inequality (5) corresponds to the minimisation of a linear objective γ under LMI constraints, which can be efficiently solved with an LMI solver. The aim of sections III-C to III-E is now to exploit this property to propose a practical solution to the problem of proposition 3.4.

C. Solutions to the infinite dimensional optimization problem

The problem of proposition 3.4 has an infinite number of frequency dependent constraints and also of optimization parameters, namely the scaling matrices $D_1(\omega)$ and $G_1(\omega)$. Let γ^* the minimal solution of this infinite dimensional problem. There are 2 classical ways to solve it.

The first one is to solve it on the continuum of frequencies $[0,\ +\infty)$ with the KYP Lemma, i.e. a state-space solution. This requires the use of bases for $D_1(s) = \sum_i \alpha_i D_i(s)$ and $G_1(s) = \sum_i \beta_i G_i(s)$, where filters $D_i(s)$ and $G_i(s)$ are fixed while the α_i and β_i are the optimization parameters. An augmented finite dimensional optimization problem is obtained. This approach has 3 drawbacks:

- Just an upper bound of γ^* is computed, since finite dimensional bases of $D_1(s)$ and $G_1(s)$ are used, which can not cover the whole set of possible scaling matrices $D_1(\omega)$ and $G_1(\omega)$.
- The choice of the basis is not obvious.
- The order of the state-space representation of the augmented plant, which is used in the KYP Lemma, can be very high, since it contains $P(s)$ and the bases of $H(s)$, $D_1(s)$ and $G_1(s)$. This may lead to an excessive computational burden for the LMI solver.

The second approach, that is further investigated in this paper, is to solve a finite dimensional optimization problem corresponding to a frequency gridding, as usually done in μ analysis. Nevertheless the computational time is higher in our problem, since because of the design parameters in $H(s) = \sum_i \theta_i H_i(s)$ and of the constant D_2 and G_2 scaling matrices, it is impossible to independently solve the problem at each frequency ω_i (with just optimization parameters $D_1(\omega_i)$ and $G_1(\omega_i)$): LMI (5) at each point of the gridding must be stacked into a single one. Moreover, just a lower bound of γ^* is obtained since the optimization problem is less constrained than on a continuum of frequencies.

The algorithm below is now introduced in this context to tackle the aforementioned drawbacks. It computes an interval inside which γ^* is guaranteed to lie with a precision of ϵ.

1) Let $(\omega_i)_{i\in[1,\ N]}$ an initial (small size) frequency gridding.
2) Solve the optimization problem of proposition 3.4 on the gridding. Let $\gamma_{LB,N}$ the minimised value, and $H(s)$, D_2 and G_2 the associated values of the feedforward controller and of the constant scaling matrices.
3) For these values let $\gamma = (1+\epsilon)\gamma_{LB,N}$ with $\epsilon > 0$, and check (4) with the μ frequency sweeping technique (see section III-D). If this is satisfied γ is an upper bound of γ^* and the global minimum is computed with a satisfactory accuracy ϵ. Otherwise let $\tilde{\omega}$ a worst-case value of the frequency, where (4) is not satisfied. Include $\tilde{\omega}$ in the gridding and go back to step 2.

To some extent, this algorithm minimises the necessary size of the gridding to minimise the computational time. A parameter $\alpha > 0$ is also introduced to release the constraints and objectives between optimization at step 2 and validation at step 3, which are performed for $\Delta \in (1+\alpha)B\Delta$ and $\Delta \in B\Delta$ respectively. This allows to guarantee convergence and to reduce the number of iterations. The following proposition now states the key result to prove the finite-time convergence of the algorithm. For the sake of conciseness, only the case of zero G scaling matrices is presented and the proof is omitted.

Proposition 3.5: Let $M_{\gamma,\alpha} = M\left(\begin{array}{cc} (1+\alpha)I & 0 \\ 0 & \frac{I}{\gamma} \end{array}\right)$. Let a frequency gridding $(\omega_i)_{i\in[1,N]}$ of $[0,\ \overline{\omega}]$, with $\omega_1 = 0$ and $\omega_N = \overline{\omega}$. Assume that there exist scaling matrices $\mathcal{D}(\omega_i)$ satisfying $M_{\gamma_{LB},\alpha}(j\omega_i)\mathcal{D}(\omega_i)M^*_{\gamma_{LB},\alpha}(j\omega_i) \le \mathcal{D}(\omega_i)$ and $d_1 I \le \mathcal{D}(\omega_i) \le d_2 I$. Let $\gamma = \gamma_{LB}(1+\epsilon)$ and $\nu = min(\epsilon, \alpha) > 0$. Assume that the degree of stability of $M_{\gamma,0}(s)$ is ρ (i.e. all poles s_i satisfy $\Re(s_i) < -\rho$) and that its ρ shifted H_∞ norm is bounded by L, i.e. $\sup_{\Re(s)\ge-\rho} \overline{\sigma}(M_{\gamma,0}(s)) \le L$. If the frequency gridding satisfies:

$$\max_i |\omega_i - \omega_{i-1}| \le \frac{2\xi\rho}{\sqrt{4-\xi^2}} \quad \text{with} \quad \xi = \sqrt{\frac{d_1}{d_2}}\frac{\nu}{L(1+\nu)}$$

then $\forall\omega \in [\omega_{i-1},\ \omega_{i+1}]$:

$$M_{\gamma,0}(j\omega)\mathcal{D}(\omega_i)M^*_{\gamma,0}(j\omega) \le \mathcal{D}(\omega_i) \quad (6)$$

First note that $4 - \xi^2 > 0$ can be guaranteed by choosing a small enough value of ν, and thus of tolerances inside the algorithm. If relation (6) is satisfied, it means that there exists a frequency dependent scaling matrix $\mathcal{D}(\omega)$ satisfying $M_{\gamma,0}(j\omega)\mathcal{D}(\omega)M^*_{\gamma,0}(j\omega) \le \mathcal{D}(\omega)\ \ \forall\omega \in [0,\ \overline{\omega}]$, so that γ is a guaranteed worst-case performance level. It thus proves that if the optimization is performed on a suitable and fine enough frequency gridding, the validation at step 3 will necessarily be successful. On this basis, it is possible to prove the finite-time convergence of the algorithm. Finally note that $\overline{\omega}$ should be theoretically infinite. Nevertheless, as usually done in μ analysis, a large value is used in practice.

D. Validation with a μ frequency sweeping technique

The aim of this section is to show how the μ frequency sweeping technique presented in section III-A is used to validate on the whole frequency range the result computed on the frequency gridding at step 2. At the beginning of step 3, the feedforward controller $H(s)$ is fixed, as well as constant scaling matrices D_2 and G_2. $M(s)$ in (3) is thus fixed. The problem to be solved is the following :

Problem 1: Do there exist scaling matrices $D_1(\omega) = D_1^*(\omega) > 0$ and $G_1(\omega) = G_1^*(\omega)$ satisfying (4) for $\omega \in [0, +\infty)$, with $\mathcal{D}(\omega) = diag(D_1(\omega), D_2, I)$, $\mathcal{G}(\omega) = diag(G_1(\omega), G_2, 0)$ and $\gamma = (1+\epsilon)\gamma_{LB,N}$?

At the end of step 2, it is already known that there exist scaling matrices $D_1(\omega_i)$ and $G_1(\omega_i)$ satisfying inequality (4) at each point of a gridding $(\omega_i)_{i\in[1,N]}$ for $\gamma = \gamma_{LB,N}$. The following lemma explains how to transform the LMI formulation (4) into a $\overline{\sigma}$ one [12].

Lemma 3.6: There exist scaling matrices $\mathcal{D}(\omega)$ and $\mathcal{G}(\omega)$ satisfying (4) if and only if there exist scaling matrices $\tilde{\mathcal{D}}(\omega) = \mathcal{D}^{1/2}(\omega)$ and $\tilde{\mathcal{G}}(\omega) = \mathcal{D}^{-1/2}(\omega)\mathcal{G}(\omega)\mathcal{D}^{-1/2}(\omega)$ satisfying:

$$\overline{\sigma}\left(\left(\tilde{\mathcal{D}}(\omega)M_\gamma^*(j\omega)\tilde{\mathcal{D}}^{-1}(\omega) - j\tilde{\mathcal{G}}(\omega)\right)\left(I + \tilde{\mathcal{G}}^2(\omega)\right)^{-1/2}\right) < 1 \quad (7)$$

Proof: Following [12] first multiply inequality (4) by $\tilde{\mathcal{D}}^{-1}(\omega)$ on the left and on the right to obtain:

$$\tilde{M}(\omega)\tilde{M}^*(\omega) + j(\tilde{\mathcal{G}}(\omega)\tilde{M}^*(\omega) - \tilde{M}(\omega)\tilde{\mathcal{G}}(\omega)) < I$$

with $\tilde{M}(\omega) = \tilde{\mathcal{D}}^{-1}(\omega)M_\gamma(j\omega)\tilde{\mathcal{D}}(\omega)$. The above inequality can then be rewritten as:

$$\left(\tilde{M}(\omega) + j\tilde{\mathcal{G}}(\omega)\right)\left(\tilde{M}(\omega) + j\tilde{\mathcal{G}}(\omega)\right)^* < I + \tilde{\mathcal{G}}^2(\omega) \quad \square$$

Using the above lemma, problem 1 can be equivalently restated as follows:

Problem 2: Do there exist scaling matrices $\tilde{D}_1(\omega)$, $\tilde{G}_1(\omega)$ satisfying (7) for $\omega \in [0, +\infty)$, with $\gamma = (1+\epsilon)\gamma_{LB,N}$, $\tilde{\mathcal{D}}(\omega) = diag(\tilde{D}_1(\omega), \tilde{D}_2, I)$, $\tilde{\mathcal{G}}(\omega) = diag(\tilde{G}_1(\omega), \tilde{G}_2, 0)$, $\tilde{D}_i = D_i^{1/2}$ and $\tilde{G}_i = D_i^{-1/2}G_iD_i^{-1/2}$ for $i = 1, 2$?

Problem 2 can finally be equivalently transformed into problem 3. Let first:

$$H(j\omega) = \left[\begin{pmatrix} I & 0 & 0 \\ 0 & \tilde{D}_2 & 0 \\ 0 & 0 & I \end{pmatrix} M_\gamma^*(j\omega) \begin{pmatrix} I & 0 & 0 \\ 0 & \tilde{D}_2^{-1} & 0 \\ 0 & 0 & I \end{pmatrix} - j\begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{G}_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \begin{pmatrix} I & 0 & 0 \\ 0 & (I+\tilde{G}_2^2)^{-1/2} & 0 \\ 0 & 0 & I \end{pmatrix} \quad (8)$$

Problem 3: Do there exist scaling matrices $\tilde{D}_1(\omega)$ and $\tilde{G}_1(\omega)$ satisfying:

$$\overline{\sigma}\left(\left(\hat{\mathcal{D}}(\omega)H(j\omega)\hat{\mathcal{D}}^{-1}(\omega) - j\hat{\mathcal{G}}(\omega)\right)\left(I + \hat{\mathcal{G}}^2(\omega)\right)^{-1/2}\right) < 1$$

for $\omega \in [0, +\infty)$, with $\gamma = (1+\epsilon)\gamma_{LB,N}$, $\hat{\mathcal{D}}(\omega) = diag(\tilde{D}_1(\omega), I, I)$ and $\hat{\mathcal{G}}(\omega) = diag(\tilde{G}_1(\omega), 0, 0)$?

The equivalence of problems 2 and 3 is proved with:

$$\left(\tilde{\mathcal{D}}(\omega)M_\gamma^*(j\omega)\tilde{\mathcal{D}}^{-1}(\omega) - j\tilde{\mathcal{G}}(\omega)\right)\left(I + \tilde{\mathcal{G}}^2(\omega)\right)^{-1/2} =$$
$$\left(\hat{\mathcal{D}}(\omega)H(j\omega)\hat{\mathcal{D}}^{-1}(\omega) - j\hat{\mathcal{G}}(\omega)\right)\left(I + \hat{\mathcal{G}}^2(\omega)\right)^{-1/2}$$

Problem 3 is equivalent to a μ test problem (see section III-A), with $N(j\omega) = H(j\omega)$ and $\Delta = diag(\Delta_1, \Delta^c)$, where Δ^c is a full complex block, since the set of scaling matrices D and G whose structure fits the one of Δ is the set of scaling matrices $\hat{\mathcal{D}}$ and $\hat{\mathcal{G}}$ in problem 3. The μ frequency sweeping technique can thus be applied to solve this μ test problem.

E. A suboptimal design

The LMI optimization performed at step 2 of the optimal algorithm of section III-C can be computationally very demanding. The number of optimization parameters and thus the computational time indeed increase with the size of the frequency gridding, since the frequency dependent scalings $D_1(\omega)$ and $G_1(\omega)$ have to be determined at each point of the gridding. Thus, it appears worthwhile to resort to the routine $mu.m$ of the μ-Analysis and Synthesis Toolbox to determine these frequency dependent scalings, instead of determining all parameters in a single LMI step. A suboptimal algorithm is introduced in this context, allowing to optimize first w.r.t. the frequency dependent scalings $D_1(\omega)$ and $G_1(\omega)$ with the routine $mu.m$, and then w.r.t. the design parameters θ_i and constant scalings D_2, G_2 by solving inequality (5) with LMI tools. Step 2 of the optimal algorithm is just replaced by:

2a) For the current values $H(s)$, D_2, G_2 of the feedforward controller and constant scaling matrices (see *Remark* (i) below for comments about how to initialize these parameters), perform an optimization on the gridding w.r.t. $D_1(\omega_i)$ and $G_1(\omega_i)$ to minimize γ in inequality (4). This is done by first transforming the LMI formulation (4) into a $\overline{\sigma}$ one as described in section III-D and then performing a dichotomy search on γ.

2b) For the values $D_1(\omega_i)$ and $G_1(\omega_i)$ determined at step 2a, solve the optimization problem of proposition 3.4 w.r.t. θ_i, D_2 and G_2. Let $\gamma_{N_{min}}$ the minimized value of γ and $H(s)$, D_2, G_2 the corresponding values of the feedforward controller and constant scaling matrices. If $\gamma_{N_{min}} < (1-\eta)\,\gamma_N$, with γ_N the value of γ at the beginning of step 2a and $\eta > 0$ a given threshold, it means that it is worth going on with the optimization and γ minimization process. In this case, let $\gamma_N = \gamma_{N_{min}}$ and go back to step 2a. Otherwise, let $\gamma_N = \gamma_{N_{min}}$ and continue to step 3.

Remarks:
(i) In the first occurrence of step 2a, the initial values of

$H(s)$, D_2 and G_2 can be arbitrarily chosen, e.g. all design parameters equal to zero, $D_2 = I$ and $G_2 = 0$. An other solution is to slightly modify the algorithm by performing a first optimal iteration using the algorithm of section III-C and then switching to the aforementioned suboptimal algorithm. The delicate initialization of the parameters and scalings can thus be avoided without penalizing the computational cost.
(ii) The choice of η conditions the relevance of γ_N at the end of step 2b. A smaller value of η indeed increases the number of iterations between steps 2a and 2b and thus leads to a less conservative value of γ_N. This parameter as well as a tuning parameter of the routine $mu.m$ enable to achieve a trade-off between the computational time and the accuracy of the final upper bound of γ^*.

Note that unlike the optimal algorithm, it is not guaranteed that the value of γ_N determined at the end of iteration 2b is a lower bound of γ^*. Thus it is not possible to quantify the accuracy of the final upper bound of γ^*. Nevertheless, the following heuristics can be applied to combine both optimal and suboptimal algorithms to determine guaranteed lower and upper bounds of γ^* with sufficient accurary while keeping a reasonable computational time:

1) Perform the suboptimal algorithm to determine a guaranteed upper bound γ_{UB} of γ^*.
2) Perform the optimal algorithm to determine a guaranteed lower bound γ_{LB} of γ^* and stop as soon as $\gamma_{LB} > (1 - \xi)\ \gamma_{UB}$, with ξ the desired accuracy. The algorithm can be initialized with the frequency gridding and the values of $H(s)$, D_2, G_2, $D_1(\omega_i)$ and $G_1(\omega_i)$ obtained at the end of step 1.

Comparison between optimal and suboptimal algorithms in section IV shows that conservatism of the suboptimal design remains very low and that the previous heuristics prove to be efficient, at least for this missile example.

IV. Missile application

The technique was implemented and applied to a missile example. Section IV-A describes the missile model, the robust feedback controller design and the resulting closed loop plant P as depicted on Fig. 1.a. Sections IV-B and IV-C then detail the feedforward design either on the continuum of linearised time invariant models or on the quasi-LPV model.

A. Linearised and non-linear missile models

The non-linear missile model is extracted from [9]. Its equations are:

$$\begin{aligned} \dot{\alpha} &= q + K_1\ M\ C_z(\alpha, M, \delta)\ cos(\alpha) \\ \dot{q} &= K_2\ M^2\ C_m(\alpha, M, \delta) \end{aligned} \tag{9}$$

where $cos(\alpha)$ is approximated in the following as $1 - \frac{\alpha^2}{2}$ and C_z, C_m are defined as:

$$\begin{aligned} C_z &= z_3\ \alpha^3 + z_2\ \alpha^2 + z_1\ (2 - (1/3)M)\ \alpha + z_0\ \delta \\ C_m &= m_3\ \alpha^3 + m_2\ \alpha^2 + m_1\ (-7 + (8/3)M)\ \alpha + m_0\ \delta \end{aligned}$$

K_1, K_2, z_i and m_j are fixed. $u = \delta$ is the control input, while $y = x = [\alpha\ q]^T$ is the output and state vector. δ is the tail fin deflection, q the rotational rate and α the angle of attack. This open loop missile model, which also depends on the Mach number M, is put under an LFT form $y = F_u(M(s), \Delta)u$ (see Fig. 1.b) with the LFR Toolbox [11].

If the continuum of linearised time invariant models is to be described, the first point is to linearise the state-equations above w.r.t. x and u, and to introduce the trim point condition $\dot{\alpha} = \dot{q} = 0$. A trim point is uniquely parametrised by values of $\alpha = 10 + 10\delta_1$ and $M = 3 + \delta_2$, so that when $\delta_i \in [-1, 1]$, $\alpha \in [0, 20\ deg]$ and $M \in [2, 4]$, i.e. the validity domain of the model. Thus the first open loop missile LFT model corresponds to an LTI structured model perturbation $\Delta = diag(\delta_1 I_4, \delta_2 I_6)$ containing variations of α and Mach.

The quasi-LPV model is directly obtained from the non-linear equations (9): this state-space model $\dot{x} = f(x, u)$ and $y = x$ is equivalently rewritten as $y = F_u(M(s), \Delta)u$, where Δ now contains time-varying signals α and M. More precisely, here again $\alpha = 10 + 10\delta_1$ and $M = 3 + \delta_2$ are injected in equations (9), so that $\Delta = diag(\delta_1 I_4, \delta_2 I_6)$. It is also possible to introduce LTI parametric uncertainties δ_3 to δ_{10} in the coefficients z_i and m_j of the polynomials, e.g. $z_3 = z_{30}\ (1 + 0.05\delta_3)$, so that $\Delta = diag(\delta_1 I_4, \delta_2 I_6, \delta_3, \delta_4 I_2, \delta_5 I_2, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10})$ contains time-varying and time-invariant parametric uncertainties. If e.g. $\delta_3 \in [-1, 1]$, the resulting uncertainty on z_3 is $\pm 5\%$. z_{30} represents the nominal value.

An integrator is added to the output α, and a robust feedback controller $u = K \left[\alpha\ \ q\ \int\alpha\right]^T$ is synthesised with a quadratic stability method [10]. This feedback controller is finally validated on the missile model with an actuator model, i.e. a second order transfer function $\frac{\omega_v^2}{s^2 + 2\xi_v\omega_v s + \omega_v^2}$ with $\omega_v = 150\ rad/s$ and $\xi_v = 0.7$.

Fig. 2 shows the resulting closed loop plant P as depicted on Fig. 1.a. The missile model can be either the continuum of LTI linearised models, or the quasi-LPV model. Its desired behaviour is represented by a reference model, namely a second order transfer function $\frac{\omega_0^2}{s^2 + 2\xi_0\omega_0 s + \omega_0^2}$ with $\omega_0 = 4\ rad/s$ and $\xi_0 = 0.6$. Frequency domain templates are also included via the first order low-pass filters $W_1 = \frac{s+1}{s+0.01}$ and $W_2 = \frac{0.5}{s+100}$. Thus the total order of the plant P is 9.

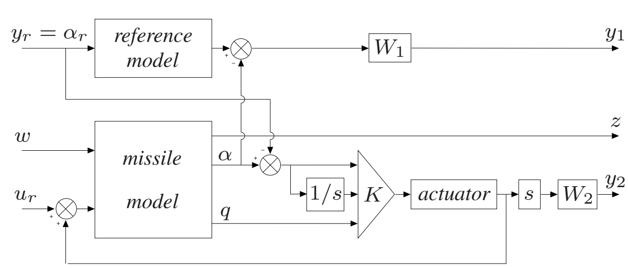

Fig. 2. Detailed constitution of the closed loop missile.

Robust performance is defined through the transfer function $T_{y_r \to y}$, whose L_2 induced norm has to be minimised. y_r corresponds to the reference angle of attack α_r and y is a two-dimensional weighted signal $[y_1\ \ y_2]^T$, where y_1

represents the difference between the real and desired angle of attack (to ensure reference model tracking) and y_2 the control surface deflection rate (to limit actuator efforts). Thus the feedforward $H(s)$ has to be designed to convert the reference angle of attack α_r into an appropriate control surface deflection u_r, i.e. $u_r = H(s)\alpha_r$.

B. Design on the continuum of LTI linearised models

Several parameters have first to be defined according to our knowledge of the physical behaviour of the missile: the considered frequency range is $[0, 10^4\ rad/s]$, while the poles of the feedforward filters $H_i(s)$ are -1, -5, -10 and -15. The initial gridding is just $5\ rad/s$.

The optimal algorithm of section III-C is first applied. The lower bound of γ^* calculated on the frequency gridding is $\gamma_{LB} = 0.881$ whereas the guaranteed upper bound on the whole frequency range is $\gamma_{UB} = 0.904$. The gap between the 2 bounds is about 2.5% and computations are achieved in 450 s on a SunBlade 1500 Workstation.

Then the suboptimal algorithm of section III-E is applied. As expected, it is computationally far more attractive (79 s) and the guaranteed upper bound is $\gamma_{UB} = 0.917$. This result is almost non conservative, since the difference with the optimal upper bound is less than 2%.

Finally the heuristics proposed in section III-E are applied. The suboptimal algorithm is first performed as before. Then a single iteration of the optimal algorithm (172 s) computes a lower bound $\gamma_{LB} = 0.873$. The heuristics prove here to be very conclusive, since the difference between the 2 bounds of γ^* is less than 5% for a total computational time of 251 s. Results are summarized in table I:

TABLE I

RESULTS FOR THE CONTINUUM OF LTI LINEARISED MODELS

Algorithm	optimal	suboptimal	heuristics
Guaranteed γ_{LB}	0.881	-	0.873
Guaranteed γ_{UB}	0.904	0.917	0.917
Computational time	450 s	79 s	79 s + 172 s

The value of γ_{UB} corresponding to performance analysis, i.e. computed with $H(s) = 0$, is 1.894. The introduction of a feedforward controller composed of only four filters $H_i(s)$ thus allows to reduce this value by more than 50%, which demonstrates the efficiency of the design tools.

C. Design on the quasi-LPV model

The optimal and suboptimal algorithms as well as the heuristics are successively performed, following the same steps as in section IV-B. Results are summarized in table II:

TABLE II

RESULTS FOR THE QUASI-LPV MODEL

Algorithm	optimal	suboptimal	heuristics
Guaranteed γ_{LB}	1.629	-	1.627
Guaranteed γ_{UB}	1.651	1.664	1.664
Computational time	730 s	469 s	469 s + 169 s

As in section IV-B, the introduction of a feedforward controller composed of only four filters $H_i(s)$ allows to reduce the value of γ_{UB} by almost 50%.

V. CONCLUSION

The aim of this paper is to synthesise robust feedforward controllers in the face of LTI and arbitrarily time-varying model uncertainties. Noting that the closed loop LFT model describes a set of time-invariant or time-varying linearised models, it would be possible to introduce additional time- or frequency-domain specifications on a finite subset of this continuum, i.e. on a finite subset of frozen-time linearised models, and especially on the nominal one corresponding to $\Delta = 0$. These specifications can be stronger than the robust performance one but the problem remains convex [2], [3].

The main contribution of this paper is to propose a practical solution to the infinite-dimensional optimization problem in [4] with an infinite number of frequency-domain constraints and also an infinite number of optimization parameters. Besides a more complex issue is addressed than in [4], since LTI and arbitrarily time varying model uncertainties are simultaneously considered. We first propose an optimal algorithm combining a frequency gridding with a μ frequency sweeping technique, and then a suboptimal algorithm which is faster. Moreover, thanks to the use of the frequency sweeping technique, our technique provides a reliable result even in the case of flexible systems, i.e. it is impossible to miss a critical peak between two points of the gridding. The missile example finally illustrates the computational efficiency of the suboptimal algorithm, as well as the possibility to efficiently combine the optimal and suboptimal algorithms to compute the minimum of the infinite dimensional optimization problem.

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