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Design of a structurally constrained suboptimal controller using an LMI method

Seog-Joo Kim, Young-Hyun Moon, Soonman Kwon, and Kook-Hun Kim

Abstract— **This paper is concerned with an iterative linear matrix inequality (LMI) approach to the design of a structurally constrained output feedback controller such as decentralized control. The structured synthesis is formulated as a novel rank-constrained LMI optimization problem, where the controller parameters are explicitly described so as to impose structural constraints on the parameter matrices. An iterative penalty method is discussed to solve the rank-constrained LMI problem. Numerical experiments and comparisons with previous works are performed to illustrate the practicality of the proposed method.**

I. INTRODUCTION

Structurally constrained control is the problem of designing a linear time-invariant controller that has structural constraints on its parameter matrices. It is sometimes called fixed-structure control or structured synthesis. Structured control problems have long been recognized as practically important control problems because structural constraints are inevitable in many fields of control and system engineering. Decentralized control is a typical example. In industrial plants, the design of a proportional-integral-derivative (PID) controller is the most frequent application. Moreover, static output feedback (SOF) stabilization belongs to this class of synthesis problems since it can be regarded as an unstructured fixed-order control problem.

Despite its usefulness in practical control applications, structured controller synthesis is a challenging task due to its inherent non-convexity [1]. No complete solution to this synthesis is found yet. Nonetheless, SOF controller design is a well-studied field in the linear matrix inequality (LMI) framework [2], [3]. The coupled LMI formulation by using the celebrated elimination lemma (see, for example, [4]) leads the SOF synthesis to the well-known rank-constrained LMI problem, which can be solved in a relatively efficient manner by numerous iterative methods [5], [6], [7], [8], [9]. However, when it comes to general structured control, we need a different LMI formulation other than the one using the elimination lemma so that the controller parameters can be directly manipulated. Several such LMI formulations along with numerical methods were proposed in the literature [10], [11], [12], [13], and this paper also discusses another heuristic algorithm.

In this paper, motivated by [10] and [12], we first propose a unified LMI representation for the design of structurally constrained controllers. The synthesis is characterized as a rank-constrained LMI optimization problem that is different from the coupled LMI formulation. Then, the problem is solved iteratively by using a penalty function method [14]. Our method can find a suboptimal solution without using a less efficient bisection method. Furthermore, no specific initialization procedure is required. The solution of our algorithm moves towards the region satisfying the rank condition as the penalty parameter increases.

In Section II, a basic lemma for the new LMI formulation is presented. Section III describes an LMI representation for structured synthesis by using the lemma in Section II. Section IV states the penalty function method for rank-constrained LMI problems, and practical implementation of the algorithm is given. Section V shows numerical experiments to demonstrate the performance of the proposed method.

We use the following notation. I_n denotes the $n \times n$ identity matrix, but the dimension n may be omitted if it can be inferred from the context. For a matrix A, its transpose, trace, and rank are denoted by A^T, $\mathbf{tr}(A)$, and $\mathbf{rank}(A)$, respectively. If A is a symmetric matrix, $A \succ 0$ (respectively, $A \succeq 0$) means that A is positive definite (respectively, semidefinite). For a rectangular matrix A, $A^\perp$ stands for an orthogonal complement of A, i.e., $A^\perp A = 0, A^\perp A^{\perp T} \succ 0$. For long matrix expressions, $(\star)^T AX$ means $X^T AX$. Finally, T_{zw} means the transfer function from w to z.

II. PRELIMINARIES

This preliminary section describes a basic tool for fixed-structure controller problems. In this framework, all LMI variables are explicitly expressed in the LMIs at the expense of an increase in the decision variables and a non-convex rank condition. The following lemma establishes the basis of our method. Similar formulations can be found in [10], [15].

Lemma 1: For a symmetric matrix $Q \in \mathbb{R}^{n \times n}$ and a matrix $B \in \mathbb{R}^{n \times m}$ with $\mathbf{rank}(B) < n$, the following statements are equivalent.

(i) $x^T Q x < 0$ for all $x \in \mathbb{R}^n$ satisfying $B^T x = 0, x \neq 0$.
(ii) There exists a scalar σ such that $Q - \sigma BB^T \prec 0$.
(iii) $B^\perp Q B^{\perp T} \prec 0$.
(iv) There exist a matrix $W \in \mathbb{R}^{(n+m)\times(n+m)}, W \succeq 0$ and

S. J. Kim and S. Kwon are with I&C Group, Korea Electrotechnology Research Institute (KERI), 28-1, Sung Joo Dong, Changwon, Kyungnam 641-120 Korea (e-mail: sjkim@keri.re.kr; smkwon@keri.re.kr).

Y. H. Moon is with the Department of Electrical and Electronic Engineering, Yonsei University, Shinchon Dong, Seoul 120-749 Korea (e-mail: moon@yonsei.ac.kr).

K. H. Kim is with Korea Nuclear I&C System (KNICS) R&D Center, 28-1, Sung Joo Dong, Changwon, Kyungnam 641-120 Korea (e-mail: kimkook@keri.re.kr).

a scalar $\mu > 0$ such that

$$\begin{bmatrix} Q & B \\ B^T & \mu I_m \end{bmatrix} \prec W \tag{1}$$
$$\mathbf{rank}(W) = m. \tag{2}$$

Proof: Statements (i), (ii) and (iii) are equivalent to each other due to Finsler's Lemma (see, for example, [2]). By applying congruence transformation on (1), condition (iv) can be equivalently described as the existence of $\tilde{W} \succeq 0$ and $\mu > 0$ such that

$$T \begin{bmatrix} Q & B \\ B^T & \mu I \end{bmatrix} T^T = \begin{bmatrix} M & 0 \\ 0 & \mu I \end{bmatrix} \prec \tilde{W} \tag{3}$$
$$\mathbf{rank}(\tilde{W}) = m \tag{4}$$

where

$$T = \begin{bmatrix} I & -\mu^{-1}B \\ 0 & I \end{bmatrix}, \tilde{W} = TWT^T$$
$$M = Q - \mu^{-1}BB^T.$$

First, it is immediate that condition (ii) implies the existence of $\mu > 0$ and $\tilde{W} \succeq 0$ satisfying (3) and (4). If condition (ii) holds for some σ, there exists a $\mu > 0$ such that $M \prec 0$. Consequently, we can select $\tilde{W}$ in (3) as

$$\tilde{W} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mu} I \end{bmatrix}, \quad \mu < \tilde{\mu}.$$

Note that $\tilde{W} \succeq 0$ and $\mathbf{rank}(\tilde{W}) = m$. To prove the converse, suppose there exist $\mu > 0$ and $\tilde{W} \succeq 0$ such that (3) and (4) hold. Partition $\tilde{W}$

$$\begin{bmatrix} M & 0 \\ 0 & \mu I \end{bmatrix} \prec \tilde{W} = \begin{bmatrix} \tilde{W}_1 & \tilde{W}_2 \\ \tilde{W}_2^T & \tilde{W}_3 \end{bmatrix}. \tag{5}$$

From $\mu > 0$ and $\mathbf{rank}(\tilde{W}) = m$, the following hold

$$\mu I \prec \tilde{W}_3 \quad \Leftrightarrow \quad \mu^{-1} I - \tilde{W}_3^{-1} \succ 0$$
$$\tilde{W}_1 = \tilde{W}_2 \tilde{W}_3^{-1} \tilde{W}_2^T.$$

Also, it follows that

$$(\tilde{W}_3 - \mu I)^{-1} = \tilde{W}_3^{-1} + \tilde{W}_3^{-1}(\mu^{-1} I - \tilde{W}_3^{-1})^{-1} \tilde{W}_3^{-1}.$$

Now applying Schur complement to (5), we obtain

$$\begin{aligned} 0 &\succ M - \tilde{W}_1 + \tilde{W}_2 (\tilde{W}_3 - \mu I)^{-1} \tilde{W}_2^T \\ &= M + \tilde{W}_2 \tilde{W}_3^{-1} (\mu^{-1} I - \tilde{W}_3^{-1})^{-1} \tilde{W}_3^{-1} \tilde{W}_2^T \\ &\succeq M. \end{aligned}$$

This ends the proof. ■

In the next section we shall show that the structurally constrained controller synthesis can be described in the form of LMI (1) subject to the rank condition (2).

III. Rank-constrained LMI formulation of structured synthesis

Since general structured synthesis can be reduced to a structurally constrained SOF problem by using a system augmentation technique (for example, [6]), this section begins with discussing the SOF stabilization problem. Then, its extension to SOF optimal control satisfying generalized quadratic performance is described.

Consider the linear time-invariant (LTI) system represented by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{6}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^p$ is the output. A, B, and C are given system matrices with appropriate dimensions. Although we deal with continuous-time systems, our method is also applicable to discrete-time systems.

The goal of SOF stabilization is to find a static output control law

$$u = Ky \tag{7}$$

that places all the eigenvalues of the closed-loop system in a stability region described by

$$\mathcal{D}(p, q, r) = \{s \in \mathbb{C} : p + qs + q^* s^* + r|s|^2 < 0\}, \tag{8}$$

where $\mathbb{C}$ denotes the set of complex numbers and $*$ represents the complex conjugate of a complex number. For example, the region $\mathcal{D}(0, 1, 0)$ corresponds to the open left-half complex plane for continuous-time systems, and $\mathcal{D}(-1, 0, 1)$ to the open unit circle for discrete-time systems. The following lemma provides a necessary and sufficient condition for the existence of an SOF controller.

Lemma 2: The following statements are equivalent.

(i) The closed-loop poles of system (6) for the static output control (7) are located in the complex region (8).

(ii) There exist $P \in \mathbb{R}^{n \times n}, P \succ 0$, $W \in \mathbb{R}^{3n \times 3n}, W \succeq 0$, and a scalar $\mu > 0$ such that

$$\begin{bmatrix} pP & qP & (A + BKC)^T \\ q^* P & rP & -I_n \\ A + BKC & -I_n & \mu I_n \end{bmatrix} \prec W \tag{9}$$
$$\mathbf{rank}(W) = n. \tag{10}$$

Proof: From Lyapunov stability theory [16], statement (i) is equivalent to the existence of a positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that

$$(\star)^T \begin{bmatrix} pP & qP \\ q^* P & rP \end{bmatrix} \begin{bmatrix} I_n \\ A + BKC \end{bmatrix} \prec 0. \tag{11}$$

Applying Lemma 1 to (11) yields (9) and (10). ■

Since (9) is affine in the variables μ, P, K and W, we can easily impose structural constraints on K.

Now, let us consider the generalized quadratic performance optimization problem for the following LTI system with state-space representation:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w, \end{aligned} \tag{12}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^{n_w}$ is the exogenous input, $u \in \mathbb{R}^{n_u}$ is the control input, $z \in \mathbb{R}^{n_z}$ is the output to be regulated, and $y \in \mathbb{R}^{n_y}$ is the measured output. The system matrices are $A, B_1, B_2, C_1, C_2, D_{11}, D_{12}$, and D_{21}.

Our aim is to design an SOF controller such that the closed-loop poles are placed within (8) and the closed-loop system satisfies the performance specification for all $T > 0$,

$$\int_0^T \begin{bmatrix} w \\ z \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} dt < 0\,. \tag{13}$$

For instance, the performance matrices for H_∞ optimization are given by $Q = -\gamma^2 I, S = 0, R = I$. Similarly, we can formulate this problem in terms of rank-constrained LMIs by the following lemma.

Lemma 3: For system (12), there exits a static output control law (7) that makes the close-loop system satisfy both the pole constraints (8) and the performance specification (13) if and only if the following rank-constrained LMI problem is feasible in matrices $P \in \mathbb{R}^{n\times n}, P \succ 0$, $W \in \mathbb{R}^{(3n+2n_z+n_w)\times(3n+2n_z+n_w)}, W \succeq 0$, and a scalar $\mu > 0$

$$\begin{bmatrix} pP & 0 & qP & 0 & A_{cl}^T & C_{cl}^T \\ 0 & Q & 0 & S & B_{cl}^T & D_{cl}^T \\ q^*P & 0 & rP & 0 & -I_n & 0 \\ 0 & S^T & 0 & R & 0 & -I_{n_z} \\ A_{cl} & B_{cl} & -I_n & 0 & \mu I_n & 0 \\ C_{cl} & D_{cl} & 0 & -I_{n_z} & 0 & \mu I_{n_z} \end{bmatrix} \prec W \tag{14}$$

$$\mathbf{rank}(W) = n + n_z, \tag{15}$$

where

$$\left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] = \left[\begin{array}{c|c} A + B_2KC_2 & B_1 + B_2KD_{21} \\ \hline C_1 + D_{12}KC_2 & D_{11} + D_{12}KD_{21} \end{array}\right].$$

Proof: According to [17] and [18], satisfying the pole constraint (8) and the performance condition (13) is equivalent to the existence of a positive definite matrix P such that

$$(\star)^T \left[\begin{array}{cc|cc} pP & 0 & qP & 0 \\ 0 & Q & 0 & S \\ \hline q^*P & 0 & rP & 0 \\ 0 & S^T & 0 & R \end{array}\right] \left[\begin{array}{cc} I & 0 \\ 0 & I \\ \hline A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{array}\right] \prec 0,$$

from which we have (14) and (15). ∎

Finally, the following lemma states structured SOF synthesis for H_2 optimal control.

Lemma 4: For system (12) with $D_{11} = D_{21} = 0$, we can find an SOF control law such that the closed-loop system satisfies the H_2 performance, $||T_{zw}||_2 = ||C_{cl}(sI - A_{cl})^{-1}B_{cl}||_2 < \gamma$, and the pole constraints (8) as well if and only if there exist matrices $P \in \mathbb{R}^{n\times n}, P \succ 0$, $W \in \mathbb{R}^{(3n+2n_z)\times(3n+2n_z)}, W \succeq 0$, and a scalar $\mu > 0$ such that

$$\mathbf{tr}(B_1^T P B_1) \leq \gamma^2, \tag{16}$$

$$\begin{bmatrix} pP & qP & 0 & A_{cl}^T & C_{cl}^T \\ q^*P & rP & 0 & -I_n & 0 \\ 0 & 0 & I_{n_z} & 0 & -I_{n_z} \\ A_{cl} & -I_n & 0 & \mu I_n & 0 \\ C_{cl} & 0 & -I_{n_z} & 0 & \mu I_{n_z} \end{bmatrix} \prec W \tag{17}$$

$$\mathbf{rank}(W) = n + n_z, \tag{18}$$

where

$$A_{cl} = A + B_2KC_2, \quad C_{cl} = C_1 + D_{12}KC_2.$$

Proof: See [19] and [18]. The proof follows similar ideas as the proof of Lemmas 2 and 3. ∎

Consequently, fixed-structure synthesis results in finding a constant matrix with structural constraints. In the next section, we describe a computation method for the rank-constrained LMI problems.

IV. Penalty method for rank-constrained LMI problems

With a slight abuse of notation, the problems in the previous section can be written as the generic form:

$$\begin{array}{ll} \min_x & c^T x \\ \text{subject to} & W(x) \succeq 0, \quad L(x) \succ 0 \\ & \mathbf{rank}(W(x)) = r, \end{array} \tag{19}$$

where x is the decision vector, and $W(x) \in \mathbb{R}^{n\times n}$ and $L(x) \in \mathbb{R}^{m\times m}$ are matrices that are affine functions of x.

In the penalty method, the rank-constrained problem (19) is first converted to an LMI optimization problem without the rank condition by incorporating a penalty function into the objective function. Then, a sequence of convex LMI optimization problems are solved by an existing LMI solver. If the value of the penalty function becomes sufficiently small during the iteration process, we have obtained a locally optimal solution to the original problem (19).

To select a penalty function reflecting the violation of the rank condition, we note that the rank condition in (19) is satisfied if and only if the $n - r$ eigenvalues of W are zero, and that the following inequality with respect to the eigenvalues of the matrix W holds [20, p. 191]

$$\lambda_1 + \cdots + \lambda_{n-r} \leq \mathbf{tr}\ (V^T W V), \tag{20}$$

where $\lambda_1, \ldots, \lambda_{n-r}$ are the $n-r$ smallest eigenvalues of W, and $V \in \mathbb{R}^{n\times(n-r)}$ is an arbitrary matrix such that $V^T V = I_{n-r}$. Thus, we introduce the penalty function

$$p(x; V) = \mathbf{tr}(V^T W V), \tag{21}$$

where $V \in \mathbb{R}^{n\times(n-r)}$ such that $V^T V = I_{n-r}$. From (20), the proposed penalty function (21) is an upper bound on the sum of the $n - r$ smallest eigenvalues of W. Moreover, the equality in (20) holds when V consists of the eigenvectors corresponding to the $n - r$ smallest eigenvalues of W. The proposed penalty function (21) has the following properties:

- Its value can be zero only when $\mathbf{rank}(W) \leq r$. Otherwise, it is positive.
- For a given V, it is linear in x.

Therefore, our penalty function can be regarded as an exact penalty function over the set $\{x : W \succeq 0\}$.

Having clarified the penalty function, we define the penalized objective function as

$$\varphi(x; \rho, \mu, V) = \rho c^T x + \mathbf{tr}(W) + \mu\, p(x; V), \tag{22}$$

where μ is the positive penalty parameter, and ρ is the optimization weight. Note that the term $\mathbf{tr}(W)$ in (22) places relative weights on the eigenvalues of W since $\mathbf{tr}(W) = \Sigma_1^n \lambda_i$. Consequently, if we denote the convex set $\mathcal{C}$

$$\mathcal{C} = \{x : W(x) \succeq 0, \quad L(x) \succ 0\}, \tag{23}$$

then the sequential unconstrained form of problem (19) becomes

$$x_{k+1} = \arg\min_x \{\varphi(x; \rho_k, \mu_k, V_k) : x \in \mathcal{C}\}, \tag{24}$$

where V_k is constructed from the eigenvectors of $W(x_k)$ since the eigenvectors of W are orthonormal to each other.

Now let us take a look at the convergence properties of the algorithm. The convergence properties of the method can be summarized as follows. More detailed discussion is described in [14].

- For fixed ρ and μ, the sequence $\{\varphi(x_k; \rho, \mu, V_k)\}$ of (24) is always non-increasing and convergent; accordingly the penalty function converges. This can be seen from

$$\varphi(x_{k+1}; \rho, \mu, V_k) \le \varphi(x_k; \rho, \mu, V_k) \le \varphi(x_k; \rho, \mu, V_{k-1}).$$

- Increasing μ makes the value of the penalty function decrease. This is evident from rewriting (22) in the form

$$\varphi(x; \rho, \mu, V) = \rho c^T x + \sum_{i=1}^{n} \lambda_i + \mu \sum_{i=1}^{n-r} \lambda_i.$$

  Thus, the convergence of the solution sequence to (24) is guaranteed with an increasing sequence of μ, which however, does not mean that we can always find a solution to (19) by the penalty method.
- Like other local algorithms [6], [7], the global convergence of the penalty function method is not guaranteed; the convergence properties of the method are yet to be studied.

We may now proceed to the implementation of the penalty function method (PFM) for rank-constrained LMI optimization problems. The PFM first tries to find a feasible solution satisfying the rank constraint. Once a feasible solution is obtained, the algorithm computes a locally optimal solution while maintaining the feasibility condition by adjusting μ and ρ alternately.

Algorithm 1: The PFM for rank-constrained LMI optimization problems

1) *Initialization*. Find an initial x_0 by solving

$$x_0 = \{x : x \in \mathcal{C}\}. \tag{25}$$

   Set $x_k = x_0$. Choose $\mu_k = \mu_0 > 1$, $\rho_k = \rho_0 \gg 1$, $\alpha \in (0, 1)$, $\beta \ll 1$, $\tau > 1$, $\xi > 1$, $\varepsilon_1 \ll 1$, $\varepsilon_2 \ll 1$.
2) *Computation of* V. Compute V_k from $W(x_k)$ using eigenvalue decomposition.
3) *Convex optimization*. Compute x_{k+1} by solving the convex LMI optimization problem (24).
4) *Feasibility test*. If $p(x_{k+1}; V_k) \le \varepsilon_1$, then go to step 5. Otherwise, go to step 6.
5) *Optimality test*. If $|c^T x_{k+1} - c^T x_k| \le \varepsilon_2$, then a locally optimal solution x_{k+1} is obtained and stop. Otherwise, go to step 7.
6) *Penalty parameter update*. If $p(x_{k+1}; V_k) > \alpha p(x_k; V_k)$, then increase the penalty parameter by $\mu_{k+1} = \tau \mu_k$. Go to step 8
7) *Optimization weight update*. If $|c^T x_{k+1} - c^T x_k| < \beta$, then increase the optimization weight by $\rho_{k+1} = \xi \rho_k$.
8) *Next step*. Set $k \leftarrow k + 1$ and go to step 2.

Remark 1: Although the implementation code for the PFM is as simple as that of the cone complementarity linearization algorithm [6], our method can be applied to optimization problems without using a bisection approach.

Remark 2: In the PFM, the most important computation parameters that affect the convergence of the algorithm are μ_0, α, and τ. Although we need further studies on the selection of them, by using the adaptive approach in [9] for adjusting the penalty parameter, we can solve the examples in the next section without much difficulty. Moreover, notice that our algorithm does not require any specific initialization procedure such as those in [10], [12]. It suffices to initiate the algorithm with the most feasible point x_0 over the convex set $\{x : x \in \mathcal{C}\}$.

V. Numerical examples

To illustrate the proposed penalty-based method, we selected several examples from previous research. The algorithm was implemented in a MATLAB program using SeDuMi [21] and YALMIP [22]. All simulations were performed on a Pentium-IV 2 GHz PC with 512 MB memory.

Example 1: Consider the chemical reactor presented in [23], where the continuous-time state-space matrices for H_∞ optimization are given.

To see the performance of the PFM, an SOF H_∞ controller was designed by using the existing coupled LMI formulation in the form (see, for example, [2])

$$\min_{X,Y} \quad \gamma_\infty \quad \text{subject to} \tag{26}$$

$$(\star)^T \begin{bmatrix} A^T X + XA & XB_1 & C_1^T \\ B_1^T X & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} C_2^T \\ D_{21}^T \\ 0 \end{bmatrix}^{\perp T} \prec 0$$

$$(\star)^T \begin{bmatrix} AY + YA^T & YC_1^T & B_1 \\ C_1 Y & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} \begin{bmatrix} B_2 \\ D_{12} \\ 0 \end{bmatrix}^{\perp T} \prec 0$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$$

$$\mathbf{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} = n.$$

where X, Y are Lyapunov matrices, γ_∞ means the H_∞ norm of the close-loop system. Table I shows the results of the PFM compared with those of the cone complementarity linearization (CCL) algorithm [6], where we used a bisection method to get a suboptimal H_∞ norm. The CPU time of the CCL method in Table I means the computation time for the H_∞ norm bound 1.1691, while the CPU time of the

PFM denotes the total elapsed time to get the solution. For comparison, the result of [23] is included in Table I, where we realize that the PFM finds a suboptimal solution to this example efficiently.

TABLE I
DESIGN RESULTS BASED ON FORMULATION (26) FOR EXAMPLE 1

Method	γ_∞	CPU(sec)	Gain	
CCL	1.1691	150.42	-34.428	-115.61
			-97.832	-348.98
PFM	1.1693	76.75	-36.389	-139.72
			-112.19	-453.78
QSDP [23]	1.202	-	-5.930	-9.251
			-4.464	-19.073

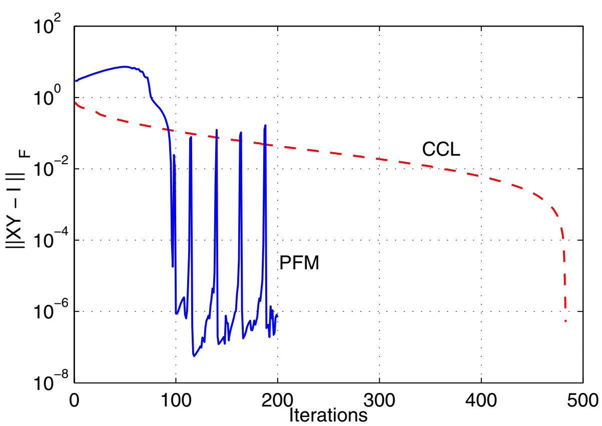

Fig. 1. Performance comparison of the PFM and the CCL method for Example 1 – formulation (26).

Fig. 1 shows a performance comparison of the PFM and the CCL method for the coupled LMI formulation. Here, the performance measure is selected as the Frobenius norm, $||XY - I||_F$ since the rank condition in (26) is equivalent to $XY = I$. In addition, the computational behavior of the PFM is shown in Fig. 2, where we see that the value of the penalty function decreases until the PFM finds a feasible solution in about 100 iterations. Since then, by adjusting the optimization weight and the penalty parameter alternately, a suboptimal solution is obtained.

Now, let us turn to the synthesis based on the new formulation (14) with (15). The results of the PFM are shown in Table II, where the first row corresponds to unstructured SOF synthesis. The second row is the result of decentralized controller design of the structure

$$K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

From Tables I and II, we can recognize that the new formulation requires high computational cost and produces conservative results for unconstrained SOF design. The proposed method, however, successfully yields a decentralized controller. Fig. 3 shows the behavior of the PFM for the decentralized controller design for Example 1.

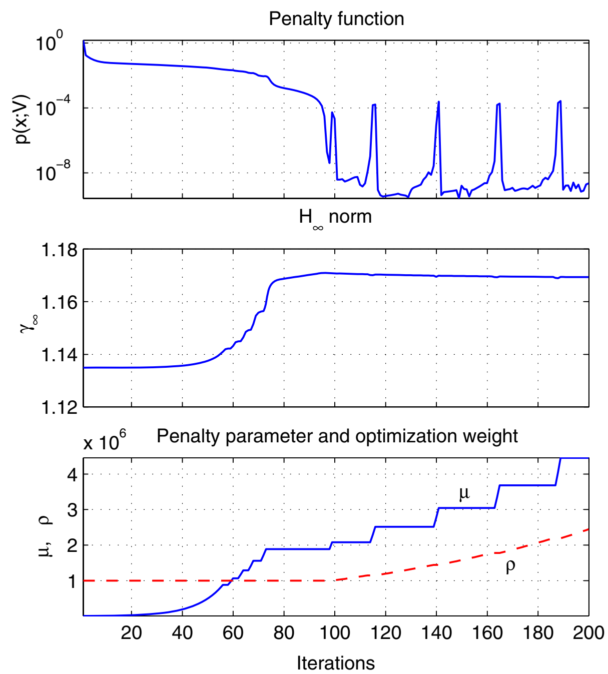

Fig. 2. Behavior of the PFM for Example 1 – formulation (26).

TABLE II
RESULTS OF THE PFM BASED ON LEMMA 3 FOR EXAMPLE 1

Control	γ_∞	CPU(sec)	Gain	
Unstructured SOF	1.2475	779.67	0.9101	-3.8446
			-0.36439	28.537
Decentralized SOF	1.8559	242.55	0.97648	0.0
			0.0	6.3382

Example 2: This example is a discrete-time H_2 optimal control problem discussed in [10] and [24]. A decentralized H_2 suboptimal controller was designed by using Lemma 4 with the PFM. The obtained controller was

$$K = \begin{bmatrix} -0.4104 & -0.3536 & 0 & 0 \\ 0 & 0 & -0.3492 & -0.1648 \end{bmatrix}.$$

Table III displays the design result of the PFM in comparison with those of previous research. Here, we see that the PFM produced the smallest H_2 norm of the closed-loop system.

TABLE III
COMPARISON OF THE H_2 NORM FOR EXAMPLE 2

Method	[24]	[10]	PFM
$\|\|T_{zw}\|\|_2$	0.33	0.27331	0.27296

Example 3: The last example is an H_2 optimal PID control problem taken from [25]. For practical purposes, we seek a PID controller of the following structure

$$C(s) = k_p + k_i \frac{1}{s + \tau_i} + k_d \frac{s}{\tau_d s + 1} \tag{27}$$

rather than the idealized PID controller where $\tau_i = 0$ and $\tau_d = 0$. A state-space representation of the PID controller

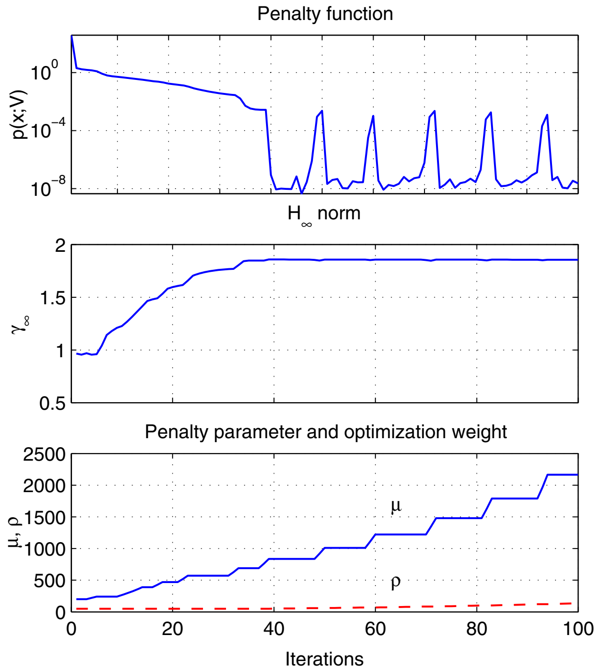

Fig. 3. Behavior of the PFM for decentralized control of Example 1.

(27) is

$$K = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{cc|c} -\tau_i & 0 & 1 \\ 0 & -\frac{1}{\tau_d} & \frac{1}{\tau_d} \\ \hline k_i & -\frac{k_d}{\tau_d} & k_p + \frac{k_d}{\tau_d} \end{array}\right]. \quad (28)$$

After augmenting the controller state and by solving the structured SOF problem for K of (28) using Lemma 4, we obtained the result of Table IV with $\tau_i = 0.0996$ and $\tau_d = 0.0991$. From Table IV, we see that the designed controller is almost the same as the optimal controller.

TABLE IV
RESULTS OF THE PID CONTROL DESIGN FOR EXAMPLE 3

Method	$\|\|T_{zw}\|\|_2$	k_p	k_i	k_d
[25]	0.9723	-0.2726	0.0	-0.2751
PFM	0.9769	-0.27135	-0.0004	-0.27295

VI. CONCLUSION

We have described a practical procedure for finding structurally constrained H_2 and H_∞ suboptimal output feedback controllers for linear systems. The synthesis is formulated as a new type of rank-constrained LMI optimization problem, which was solved by an iterative penalty function method. Our method is applicable to both continuous- and discrete-time systems. Although the proposed algorithm does not have global convergence property, numerical experiments on some examples from previous research showed promising results.

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