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Asymptotic tracking for state-constrained monotone systems

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Abstract— **This paper addresses tracking control of nonlinear discrete-time monotone systems subject to input and state constraints. Forcing saturation on a previously designed controller may, in general, lead to destabilization or, at least, result in constraint violation and performance losses. Hereby it is shown that for a certain class of nonlinear monotone systems it is possible to design a static nonlinear output feedback which, saturated among suitable state-dependent bounds, is able to guarantee constraint satisfaction and asymptotic tracking of piecewise constant references, with a moderate on-line computational burden.**

I. INTRODUCTION

Monotone systems [1]-[14] have recently attracted great attention in the control literature. So far most of the research efforts have been devoted to analysis issues while much less is known on specific control synthesis tools which could exploit system monotonicity in some respect. A crucial control problem is the design of an offset-free tracking controller for nonlinear systems subject to input and state constraints. In this respect, the ordering of trajectories in monotone systems could certainly help an efficient design of control laws that jointly ensure stability and constraint fulfilment. The present paper shows in fact that, for a certain class of nonlinear monotone systems, it is possible to design a stabilizing static output feedback in a straightforward way and then force saturation on the input taking into account state and input constraints. Based on this control design, it is possible to derive a tracking control algorithm which provides, with a moderate on-line computational burden, both constraint satisfaction and asymptotic tracking requirements. The paper is organized as follows. First a review of basic definitions and results on monotone systems is carried out in section 2. Section 3 first reviews previous results from [17] on tracking control of monotone systems subject to input constraints only, and then shows how for a monotone system it is possible to recast state constraints as appropriate state-dependent input constraints. Section 4 presents a control algorithm for monotone systems capable of handling state (in addition to input) constraints and analyzes its properties. The applicability of the method and its effectiveness are illustrated by means of simulation examples in section 5. Finally some conclusions are drawn in section 6.

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II. PROBLEM FORMULATION

Consider the following discrete-time SISO nonlinear system

$$\begin{array}{rcl} x(t+1) & = & f(x(t),u(t)) \\ y(t) & = & h(x(t)) \end{array} \tag{1}$$

where $t \in \mathbb{Z}_+ \triangleq \{0,1,\dots\}$ is the time index, $x(t) \in X \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}$, $y(t) \in \mathbb{R}$, the map $f(\cdot,\cdot)$ is continuous in (x,u) and the map $h(\cdot)$ is continuous in x. The solution of (1) for the initial state $x(0) = x_0 \in \mathbb{R}^n$ and the input signal $u(\cdot) \triangleq \{u(k) \ : \ k \geq 0\} \in \mathcal{U}$ will be denoted, for all $t \geq 0$, by $\varphi(t, x_0, u(\cdot))$. The constant unit signal will be denoted by $1(\cdot)$, defined as $1(k) = 1$ for all $k \geq 0$. In an Euclidean space $\mathbb{R}^\ell$ a partial order $\succeq$ induced by a positivity cone C is defined. Let $C \subseteq \mathbb{R}^\ell$ be a nonempty, closed, convex, pointed ($C \bigcap -C = \{0\}$) cone with nonempty interior, then $v_1 \succeq v_2$ ($v_1, v_2 \in \mathbb{R}^\ell$) means that $v_1 - v_2 \in C$. Strict ordering is denoted by $v_1 \succ v_2$, meaning that $v_1 \succeq v_2$ and $v_1 \neq v_2$. The partial order is extended to signals $v(\cdot) : \mathbb{Z}_+ \to \mathbb{R}^\ell$ in the sense that $v_1(\cdot) \succeq v_2(\cdot)$ if $v_1(t) \succeq v_1(t)$ for all $t \geq 0$. With reference to the system (1), let C_x, C_u and C_y denote the order cones for the state, input and, respectively, output. Without loss of generality (otherwise, it is always possible to consider $-u$ as an input or $-y$ as an output), the considered order on the input and output spaces is $C_u = C_y = \mathbb{R}_{\geq 0}$.
The system (1) is said monotone if the following property holds, with respect to the orders on the state and the inputs for all $x_1, x_2 \in X$ and input signals $u_1(\cdot),\ u_2(\cdot) \in \mathcal{U}$:

$$\begin{array}{l} x_1 \succeq x_2 \text{ and } u_1(\cdot) \succeq u_2(\cdot) \Rightarrow \\ \varphi(t, x_1, u_1(\cdot)) \succeq \varphi(t, x_2, u_2(\cdot)) \quad \forall t \geq 0 \end{array} \tag{2}$$

and the output map $h(\cdot)$ is monotone with respect to the partial order on the state and output space, i.e. $h(\varphi(t, x_1, u_1(\cdot))) \geq h(\varphi(t, x_2, u_2(\cdot)))$ for all $t \geq 0$. It is important to check monotonicity without having to compute the trajectories of (1). This amounts to checking monotonicity of the map $f(x,u)$ with respect to the partial order in X and U.
In this paper, the control objective is that

1) the output $y(\cdot)$ track a piecewise constant reference $r(\cdot)$, i.e. a signal switching among different constant set-points;
2) the input $u(\cdot)$ satisfy the pointwise-in-time constraints
$$u(t) \in U \triangleq \{u \ : \ \underline{u} \leq u \leq \bar{u}\},\ \forall t \geq 0 \tag{3}$$
3) the state $x(t)$ satisfy the pointwise-in-time constraints
$$x(t) \in S \subset X,\ \forall t \geq 0 \tag{4}$$

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For the subsequent developments the following assumptions are made.

Assumption 1: For each constant set-point r there is associated a unique (state,input) equilibrium pair $(x_e(r), u_e(r))$ such that

$$f(x_e(r), u_e(r)) = x_e(r), \qquad r = h(x_e(r)) \tag{5}$$

Assumption 2: It is assumed that the set of admissible states

$$S \triangleq \{x \;:\; \underline{g}_i \leq g_i(x) \leq \overline{g}_i,\; i = 1, 2, \cdots, m\} \tag{6}$$

is described by monotone constraints i.e. $g_i(x)$ are monotone functions of x.

Clearly the constraints (3) and (4) restrict the statically admissible set-points r to the ones that belong to the set

$$R \;=\; \{r :\; u_e(r) \in U,\; x_e(r) \in S\}. \tag{7}$$

In order to ensure viability in finite time, the following assumption is made.

Assumption 3: It is assumed that

$$R \;=\; \{r : u_e(r) \in U\} \subset \{r :\; x_e(r) \in S_\delta\}. \tag{8}$$

where

$$S_\delta \triangleq \{x \in S \;:\; x + w \in S,\; \forall w \;:\; \|w\|_\infty \leq \delta\}. \tag{9}$$

and $\delta > 0$ is arbitrarily small.

III. SYSTEM STABILIZATION AND SOME RESULTS

In order to design a suitable tracking policy for (1) under the constraints (3) and (4), it is relevant to find how to stabilize such a system. A useful result on the stability of monotone systems is given hereafter.

Theorem 1: Suppose that:

(i) the dynamical system $x(t+1) = f(x(t))$ is monotone;
(ii) its trajectories are bounded in X;
(iii) X contains exactly one equilibrium point x_e;
(iv) for every compact subset S of X, both $\inf(S)$ and $\sup(S)$ belong to X (see [15] for a rigorous definition of $\inf(S)$ and $\sup(S)$).

Then x_e is asymptotically stable globally in X, i.e. it is stable and $\lim_{t\to\infty} \varphi(t, x_0) = x_e$ for all $x_0 \in X$. ∎

In [16] this result was proved for continuous-time systems, but the same argument can be applied to discrete time systems. However the open-loop system (1) need not have unique and stable equilibria. Its steady state behaviour will be useful in order to design a controller.

Definition 1: Under the assumption 1, the system (1) admits a (possibly multi-valued) input to state (I/S) steady-state characteristic defined as follows

$$k^X(u) \triangleq \{x \in X \;:\; f(x, u) = x\}\,. \tag{10}$$

If (1) admits an I/S characteristic, its input/output (I/O) characteristic is by definition the composition

$$k^Y(u) \triangleq \{y \in Y \;:\; y = h(x) \text{ and } f(x, u) = x\} \tag{11}$$

∎

Our interest will be in the design of a static nonlinear output feedback $u(\cdot) = \ell(y(\cdot), r)$ meeting the control objectives stated in the previous section. In order to design a static nonlinear output feedback guaranteeing input constraint satisfaction the subsequent theorem, presented in [17] for continuous-time systems and rewritten here for discrete-time systems, will be useful.

Theorem 2: Suppose that the system (1) is monotone with respect to C_x in X, with $C_u = C_y = \mathbb{R}_{\geq 0}$, and that it has an I/O characteristic $k^Y(u)$. Moreover assume that $x(t+1) = f(x(t), \underline{u})$ and $x(t+1) = f(x(t), \overline{u})$ admit a unique asymptotically stable equilibrium point in $x \in X$. Design an output feedback $u = \ell(y, r)$ with the following properties.

1) It admits, for each fixed r, only one intersection point in the plane (y, u) with the I/O characteristic $k^Y(u)$.
2) It is such that the closed-loop system

$$x(t+1) = f(x(t), \ell(h(x(t)), r)) \tag{12}$$

is monotone with respect to the same partial order and has bounded trajectories.

Then the saturated control law

$$\text{sat}(\ell(h(x(t)), r)) = \begin{cases} \underline{u} & \text{if } \;\ell(h(x(t)), r) < \underline{u} \\ \ell(h(x(t)), r) & \text{if } \;\underline{u} \leq \ell(h(x(t)), r) \leq \overline{u} \\ \overline{u} & \text{if } \;\ell(h(x(t)), r) > \overline{u} \end{cases} \tag{13}$$

is such that the output asymptotically tracks any constant reference $r \in R$ globally in X, i.e. for all initial states $x_0 \in X$. ∎

The proof of the theorem for discrete time systems is obtained following the same reasoning carried out for continuous-time systems in [17] and exploiting theorem 1. An important issue concerns the ability of handling state constraints. In order to tackle this problem the following theorem will be exploited for the subsequent developments.

Theorem 3: Suppose that the system (1) is monotone with respect to C_x in X, with $C_u = \mathbb{R}_{\geq 0}$ and its trajectories are bounded in X. Moreover assume that the set of state constraints S satisfies assumption 2. Given $x \in S \subset X$, if the input satisfies the constraints

$$u(\cdot) \in \mathcal{U}_x \triangleq \{u(\cdot) : \underline{u}_x \leq u(t) \leq \overline{u}_x,\; \forall t \geq 0\}. \tag{14}$$

where $\underline{u}_x$ and $\overline{u}_x$ are constant inputs such that

$$\begin{aligned} \underline{u}_x &= \min\{u \;:\; \underline{g}_i \leq g_i(\varphi(t, x_0, u\,1(\cdot))),\; \forall i,\; \forall t \geq 0\} \\ \overline{u}_x &= \max\{u \;:\; g_i(\varphi(t, x_0, u\,1(\cdot))) \leq \overline{g}_i,\; \forall i,\; \forall t \geq 0\} \end{aligned} \tag{15}$$

then the constraints $\varphi(t, x_0, u(\cdot)) \in S$ are satisfied for all $t \geq 0$.

Proof - From the definition of monotone systems, for any $x \in S$ the following relation of order is met

$$u_1(\cdot) > u_2(\cdot) \Rightarrow \varphi(t, x, u_1(\cdot)) \succeq \varphi(t, x, u_2(\cdot)) \quad \forall t \geq 0 \tag{16}$$

Then, considering relation (16) under assumption of monotone constraints, there exist constant $\underline{u}_x$ and $\overline{u}_x$, with $\overline{u}_x > \underline{u}_x$, given by (15), such that

$$\begin{array}{l} \overline{g}_i \geq g_i(\varphi(t, x, \overline{u}_x 1(\cdot))) \geq g_i(\varphi(t, x, \underline{u}_x 1(\cdot))) \geq \underline{g}_i, \\ \forall i \text{ and } \forall\, t \geq 0 \end{array} \tag{17}$$

Since for any signal $u(\cdot)$ such that $\underline{u}_x \leq u(\cdot) \leq \overline{u}_x$

$$g_i(\varphi(t, x, \overline{u}_x 1(\cdot))) \geq g_i(\varphi(t, x, u(\cdot))) \geq g_i(\varphi(t, x, \underline{u}_x 1(\cdot))) \tag{18}$$

the state constraints $x \in S$ can be replaced with $\underline{u}_x \leq u(\cdot) \leq \overline{u}_x$ ■

This result will allow to face the tracking control problem considering appropriate state-dependent input constraints.

IV. Tracking control algorithm

Based on the previous results, in particular theorem 2 and theorem 3, it is now possible to formulate the tracking control algorithm described below.

Tracking control algorithm - At time t, given the state $x(t)$ and the desired reference $r_d(t) \in R$, perform the following steps

1) Estimate state-dependent bounds $\underline{u}_{x(t)}$ and $\overline{u}_{x(t)}$ satisfying (15).
2) Apply the control input $u(t) = \text{sat}(\ell(y(t), r_d(t))) =$

$$\begin{cases} \tilde{u}_{min}(t) & \text{if} \quad \ell(y(t), r_d(t)) < \tilde{u}_{min}(t) \\ \ell(y(t), r_d(t)) & \text{if} \quad \tilde{u}_{min}(t) \leq \ell(y(t), r_d(t)) \leq \tilde{u}_{max}(t) \\ \tilde{u}_{max}(t) & \text{if} \quad \ell(y(t), r_d(t)) > \tilde{u}_{max}(t) \end{cases} \tag{19}$$

where $\tilde{u}_{min}(t) = \max\{\underline{u}, \underline{u}_x(t)\}$ and $\tilde{u}_{max}(t) = \min\{\overline{u}, \overline{u}_x(t)\}$

The proposed algorithm enjoys the following property.

Theorem 4: Let us assume that, for all $t \geq 0$, $r_d(t) = r_d \in R$, where R satisfies assumption 3. Under the assumptions of theorems 2 and 3, if at time $t = 0$ $[\tilde{u}_{min}(0),\ \tilde{u}_{max}(0)] \neq \emptyset$, the tracking control algorithm guarantees that the input and state constraints are satisfied for all $t \geq 0$ and the system asymptotically reaches the desired equilibrium i.e. $\lim_{t\to\infty} x(t) = x_e(r_d)$, $\lim_{t\to\infty} u(t) = u_e(r_d)$ and $\lim_{t\to\infty} y(t) = r_d$.

Proof - Let us consider the situation in which the constraints due to the states are active. Given the state x_0, at time $t = 0$, the constraints $\varphi(t, x_0, u(\cdot)) \in S$ are satisfied for all $t \geq 0$ with $\underline{u}_{x_0} \leq u(\cdot) \leq \overline{u}_{x_0}$ by virtue of theorem 3. Moreover, due to the monotonicity of the system, $\varphi(t, x_0, u(\cdot)) \preceq \varphi(t, x_0, \overline{u}\ 1(\cdot))$ for any $u(\cdot)$ satisfying $\underline{u}_{x_0} \leq u(\cdot) \leq \overline{u}_{x_0}$ and for all $t > 0$. Considering the k steps ahead state prediction, computed applying a constant input $\overline{u}\ 1(\cdot)$, we have $\varphi(k, \varphi(t, x_0, u(\cdot)), \overline{u}\ 1(\cdot)) \preceq \varphi(k, \varphi(t, x_0, \overline{u}\ 1(\cdot)), \overline{u}\ 1(\cdot)) = \varphi(t + k, x_0, \overline{u}\ 1(\cdot))$. Under the assumption 3 and since the map $f(x, \overline{u})$ admits a unique asymptotically stable equilibrium point in X

$$\lim_{t\to\infty} \varphi(t + k, x_0, \overline{u}\ 1(\cdot)) \in S_\delta \tag{20}$$

Hence $\exists\ \overline{t} > 0$ such that $\varphi(\overline{t} + k, x_0, \overline{u}\ 1(\cdot)) \in S$ for all $k \geq 0$. This implies $\overline{u}_{x(t)} \geq \overline{u}$ for all $t > \overline{t}$. Similarly $\exists\ \hat{t} > 0$ such that $\underline{u}_{x(t)} \leq \underline{u}$ is satisfied for all $t \geq \hat{t}$. Then, since only input constraints are active for all $t > \max(\overline{t}, \hat{t})$, the result follows from theorem 2 proceeding along similar lines as the ones used in [17] for a continuous-time system and, here, suitably adapted to a discrete-time system. ■

Remark 1: Under monotonicity of the map ℓ with respect to r, the saturated control feedback in the second step is equivalent to the following one-step-ahead reference governor policy:

1) Solve:

$$\begin{array}{rl} r(t) = & \arg\min_{\overline{r}} (\overline{r} - r_d(t))^2 \\ & \text{subject to} \\ & \tilde{u}_{min}(t) \leq \ell(y(t), \overline{r}) \leq \tilde{u}_{max}(t) \end{array} \tag{21}$$

2) Apply the control input $u(t) = \ell(y(t), r(t))$. ■

Remark 2: A feedback control law $u = \ell(y, r)$ satisfying the requirements that the two curves $(u, k^Y(u))$, i.e. the plant I/O characteristic, and $(\ell(y, r), y)$, i.e. the controller characteristic, intersect just at one point (u, y) for each r, and preserving the monotonicity of the open-loop system, can be easily designed with a graphical procedure provided that conditions of Proposition 1 in [17] are satisfied. In order to carry out easily a graphical choice of the feedback shape it is possible to re-parametrize the feedback as $\ell(y, \theta(r))$ for a suitable r-dependent parameter θ. Given the desired structure of $\ell(y, \theta(r))$ it is possible to determine $\theta(r)$ by solving the following equation

$$k^Y(\ell(r, \theta(r)) = r \tag{22}$$

The existence of a suitable $\ell(y, r)$ is ensured by assumption 1. Then, for each value of r, the computation of the matched value $\theta(r)$ is performed on-line, so that offset-free tracking is ensured. ■

At time t the estimation of bounds $\underline{u}_x(t)$ and $\overline{u}_x(t)$ satisfying (15) deserves further discussion. In fact, these bounds must be computed on-line and it is, therefore, desirable to keep low the required computational burden. In order to limit the horizon for which (15) needs to be evaluated it is fundamental to know a time $\overline{t}$ such that for all $t > \overline{t}$ the state constraints will be fulfilled. For linear systems it is possible to determine $\overline{t}$ and then to perform easily an on line estimation of $\underline{u}_x(t)$ and $\overline{u}_x(t)$ as shown in the following theorem.

Theorem 5: Assume that the system

$$x(t+1) = Ax(t) + Bu(t) \tag{23}$$

is asymptotically stable and monotone with respect to the partial orders induced by the orthants $C_x \subset \mathbb{R}^n$ and $C_u \subset R$. Let the sets of admissible states and inputs $S \triangleq \{x : Mx \leq N\}$ and, respectively, U be polytopes described by monotone constraints. Then there exists $\bar{t}$ such that for any $x_0 \in S$ and any $u \in U$, the following implication holds:

$$\begin{aligned} &\varphi(t, x_0, u\ 1(\cdot)) \in S \text{ for } t = 0, 1, \ldots, \bar{t} \Longrightarrow \\ &\varphi(t, x_0, u\ 1(\cdot)) \in S,\ \forall t \geq 0. \end{aligned} \tag{24}$$

Proof - Since the system (23) is asymptotically stable, given any compact set $K \subset \mathbb{R}^n$, it holds that

$$\forall \delta > 0,\ \exists\ \bar{t}_\delta \mid \forall t \geq \bar{t}_\delta, \forall x \in K :\ \|A^t x\|_\infty \leq \delta.$$

In order to obtain a $\bar{t}_\delta$ that is valid for any $x_0 \in S$ and for any $u \in U$, due to the system's monotonicity, it is sufficient to compute such $\bar{t}_\delta$ for the initial conditions $\underline{x} = \inf_{x \in S,\ u \in U} (x - H(1)u)$ and $\overline{x} = \sup_{x \in S,\ u \in U} (x - H(1)u)$ with respect to the partial order in X, where $H(1) = (I - A)^{-1}B$ and I denotes the identity matrix. More precisely, let

$$\bar{t}_\delta = \inf\{t : \|A^k \underline{x}\|_\infty \leq \delta \text{ and } \|A^k \overline{x}\|_\infty \leq \delta, \forall k \geq t\}$$

Then, for all $x_0 \in S$ and $u \in U$, $\varphi(t, x_0, u\ 1(\cdot)) \in S$ provided that $t \geq \bar{t}_\delta$ since, by assumption 3, $H(1)u \in S_\delta$. This proves the existence of $\bar{t} = \bar{t}_\delta$ in (24). ■

Remark 3: If $[\tilde{u}_{min}(0),\ \tilde{u}_{max}(0)] \neq \emptyset$, at time k given the state $x(k) = x$, it is possible to compute on-line $\tilde{u}_{max}(k)$ and $\tilde{u}_{min}(k)$ such that the state constraints are satisfied at any time instant, by solving the following linear programming problem

$$\begin{aligned} &[\tilde{u}_{min}(k),\ \tilde{u}_{max}(k)] = [\min_{u \in U} u,\ \ \max_{u \in U} u] \\ &\text{subject to } x(t) = A^t x + \phi(t, 1)u \in S \quad \forall\ t \in [0,\ \bar{t}] \end{aligned} \tag{25}$$

where $\phi(t, 1)$ is the unit step response. ■

Remark 4: For nonlinear systems the situation is theoretically more complicated. However, it is possible to estimate empirically the desired $\bar{t}$ by considering that x and u belong to compact sets. Then, on-line, it is possible to apply a bisection algorithm to find some feasible bounds on u. In order to reduce the computational burden, it is possible to compute off-line the bounds for a suitable number of state values in S belonging, for instance, to the segment joining the points $\underline{x} = \inf(S)$ and $\overline{x} = \sup(S)$. Then, exploiting the system's monotonicity and given the state $x(t)$, it is possible to obtain on-line some conservative information on the bounds. Moreover, it is convenient to exploit the bounds obtained at the previous step since they will be feasible and, hopefully, they may be enlarged. ■

V. SIMULATION EXAMPLES

The effectiveness of the proposed control procedure is now illustrated by means of two different numerical examples of practical interest.

A. Example 1

Diffusion reaction processes are described by equations that present spatial and temporal dependence. In order to handle these models, they are usually approximated through a spatial discretization subdividing the reactor in a cascade of cells with a length depending on the accuracy required by the model. Whenever the cells are all equal, the following linear model is obtained

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n \tag{26}$$

$$A = \begin{bmatrix} \beta - k & \gamma & 0 & 0 & \cdots & 0 \\ \alpha & \beta & \gamma & 0 & \cdots & 0 \\ 0 & \alpha & \beta & \gamma & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \gamma \\ 0 & \cdots & \cdots & 0 & \alpha & \beta + \gamma \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & 0 & 0 & \cdots & 0 \end{bmatrix}' \tag{27}$$

for which the conservation relation $\alpha + \beta + \gamma = 0$ has been imposed and k denotes a dispersion coefficient acting on the first cell. For the simulation experiments we have used $n = 100$, $b_1 = 1$, $k = 0.1$, $\alpha = 2.3$, $\beta = -2.5$ and $\gamma = 0.2$. The structure of the system suggests the choice

$$y = x_1 \tag{28}$$

as output map. The considered system is positive and, hence, monotone with respect to the positive orthant The I/O characteristic of the system is the line

$$y = \kappa u \quad (\kappa = 0.41\overline{6}) \tag{29}$$

In order to apply the proposed procedure the system has been discretized with sampling time $T_s = 0.01$ and a suitable control structure satisfying the conditions of theorem 2 is

$$u = -\theta_1 y + \theta_2 \tag{30}$$

Chosen the desired reference $r \in R$, for a given θ_1, the corresponding $\theta_2(r)$ is easily computed as

$$\theta_2(r) = \left(\frac{1}{\kappa} + \theta_1\right) r \tag{31}$$

The monotonicity condition of the closed loop system imposes the constraint $\theta_1 < \frac{1 + T_s(\beta - k)}{T_s b_1} \simeq 97.39$ and $\theta_1 = 97$ has been selected. The obtained behaviour of the proposed tracking strategy is shown by simulation experiments choosing the following input and state constraints

$$0.1 \leq u \leq 1, \quad 0 \leq x_i \leq 1 \text{ for } i = 1, \ldots, n \tag{32}$$

The statically admissible set-points are $0.0417 \leq r \leq 0.4167$ The existence of a unique equilibrium point is guaranteed for all $r \in R$. The output response to a square wave set-point, applying the control law $u = \text{sat}(-97y + \theta_2(r))$, is shown in figure 1. The input and, respectively, state responses are reported in figures 2 and, respectively, 3. Finally figure 1 also displays the choice of the selected reference r.

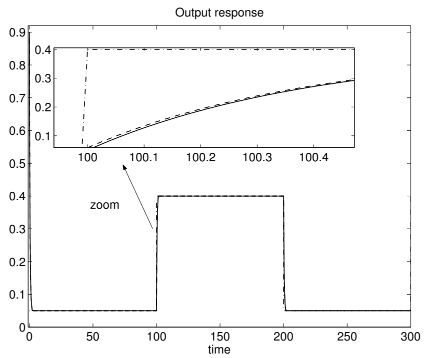

Fig. 1. Desired reference (dashed-dot), feasible reference (dashed) and output response (solid)

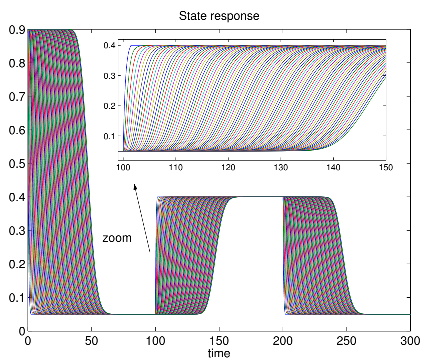

Fig. 2. State response for $1 \leq i \leq n = 100$

B. Example 2

An interesting application for the proposed approach is the n-dimensional cooperative system, which extends a model of the protein synthesis in the cell [14]

$$\begin{array}{rcll} \dot{x}_1 & = & -\alpha_1 x_1 + \gamma(x_n) + u & \\ \dot{x}_i & = & -\alpha_i x_i + x_{i-1} & i = 2, \ldots, n \end{array} \tag{33}$$

where $\alpha_i > 0$ for all i, and $\gamma(x_n) = x_n^2/(1+x_n^2)$. For $u > 0$, the equilibria satisfy the relationships

$$\begin{array}{rcl} x_i & = & (\alpha_{i+1} \ \cdots \ \alpha_n) x_n, \quad i = 1, 2, \ \cdots, \ n-1 \\ \alpha x_n & = & \gamma(x_n) + u \end{array} \tag{34}$$

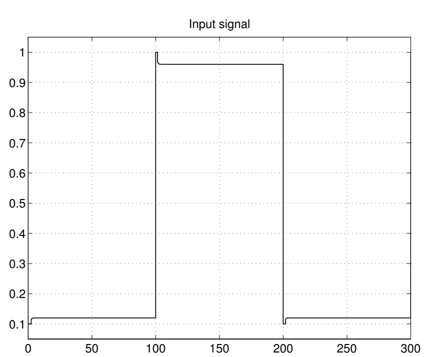

Fig. 3. Saturated input

where $\alpha = \alpha_1 \ \alpha_2 \ \cdots \ \alpha_n$. For the simulation experiments the following parameter values have been selected

$$n = 5, \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, \quad \alpha_5 = 0.55 \tag{35}$$

A possible choice for the output is

$$y = x_1 \tag{36}$$

Then the parametrized family of equilibrium points for (33) with respect to y is easily computed by (34). It is straightforward to check that the system (33) is monotone with respect to the order induced by the positivity cone $C_x = \mathbb{R}^n_{\geq 0}$ in $X = \mathbb{R}^n_{\geq 0}$. The I/O characteristic of the system (33), (36) is not well-defined. It is an hysteresis, as shown in figure 4, which presents multiple equilibria for some values of u. The system model has been discretized with a sampling time $T_s = 0.01$. In order to stabilize the branch of unstable equilibrium points in $\mathbb{R}_{\geq 0}$, it is straightforward to design a controller satisfying the conditions of theorem 2. The

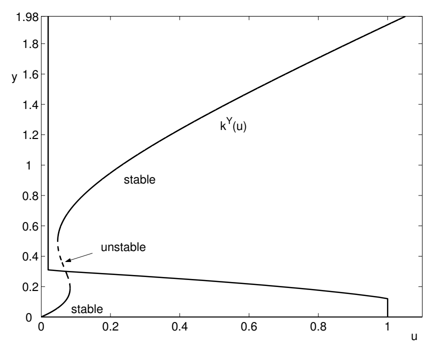

Fig. 4. The intersection between the plant I/O characteristic $k^Y(u)$ and $u = \text{sat}(\ell(y, r))$ for $r = 0.3$

simplest choice is

$$u = -\theta_1 y^2 + \theta_2 \tag{37}$$

Chosen the desired reference $r \in R$, for a given θ_1, the corresponding $\theta_2(r)$ is the following

$$\theta_2(r) = \alpha_1 r + \theta_1 r^2 - \frac{r^2}{(\frac{\alpha}{\alpha_1})^2 + r^2} \tag{38}$$

The monotonicity condition of the closed loop system imposes the constraint $\theta_1 \leq \frac{1-\alpha_1 T_s}{T_s 8} \simeq 12.3750$. In order to get good performance, $\theta_1 = 12.37$ has been selected. The obtained behaviour of the proposed tracking strategy is shown by simulation experiments with the following input and state constraints selected as

$$0.02 \leq u \leq 1, \quad 0 \leq x_i \leq 4 \text{ for } i = 1, \ldots, 5 \tag{39}$$

The statically admissible set-points, induced by the input constraints, are $0.022 \leq r \leq 1.925$. The existence of a unique equilibrium point is guaranteed for all $r \in R = [0.022, \ 1.925]$. The output response to a square wave set-point of amplitude ± 0.8 and offset 1.2, applying the control law $u = \text{sat}(-\theta_1 y^2 + \theta_2(r))$, is shown in figure 5. The applied input reported in figure 7 guarantees convergence to the desired reference and state constraint satisfaction as illustrated in figure 6. Notice how, at the beginning, the feasible inputs are restricted (figure 7) in order to guarantee state constraint fulfilment. It has been empirically estimated that $\overline{t} = 400$ is a time window length sufficient for checking state constraints satisfaction. Moreover, since the positive orthant is invariant, if the interval $[\underline{u}_{x_0},\ \overline{u}_{x_0}]$ is not empty, then u_{min} is feasible and turns out to be a good initial point in the optimization problems (15).

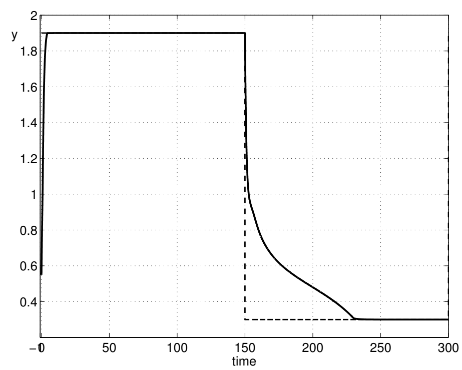

Fig. 5. Desired output (dashed) and output response (solid)

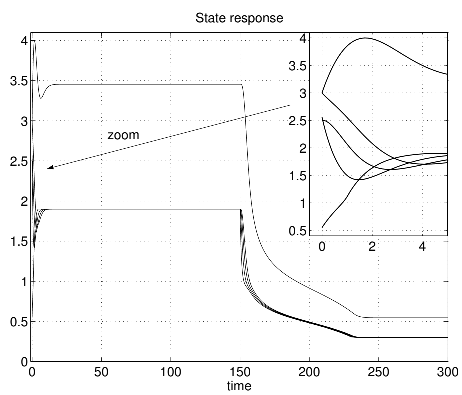

Fig. 6. State response

VI. CONCLUSIONS

The paper has addressed tracking control of monotone nonlinear system in the presence of input and state constraints. It has been shown that for a certain class of nonlinear monotone systems, it is possible to design off-line a stabilizing static output controller in a straightforward way and then take into account on-line state and input constraints by saturation. The proposed controller operates, therefore, in two steps. In the first step, it computes state-dependent bounds on the input that guarantee state constraint feasibility at any future time instant. In the second step, it saturates the off-line designed control law among the on-line computed bounds thus implicitly acting as a reference governor. It has been proved that this jointly guarantees asymptotic tracking of a constant feasible setpoint and pointwise in time constraint fulfilment provided that an initial feasibility condition

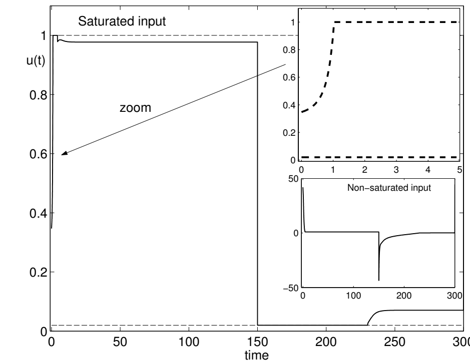

Fig. 7. Saturated input (solid), $\tilde{u}_{max}(t)$ and $\tilde{u}_{min}(t)$ behaviour (dashed) in the main picture and the non-saturated input (solid) in the minor one

holds. Further, the implementation of the proposed control strategy is simple and exhibits a mild on-line computational burden. The effectiveness of the proposed procedure has been illustrated by means of two numerical examples of practical interest, concerning a linear and a nonlinear system.

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