Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On Infimum Quantization Density for Multiple-input Systems

Hernan Haimovich*† and María M. Seron*

Abstract— **This paper deals with quadratic stabilization of discrete-time linear time-invariant systems, when the control is based on a static (or memoryless) *quantized* measurement of the state. A measure of quantization density is utilized in accordance with previous definitions in the literature. Based on this quantization density measure, the paper finds, for multiple-input systems that can be stabilized using a one-dimensional subspace of the input space, the infimum quantization density over all state quantizers that are quadratically stabilizing with respect to a *given* control Lyapunov function. This result is shown to differ from a previously published result. This discrepancy is explored by means of a numerical example that shows that, whereas the previously published result yields an inconsistent value of the density, our result provides a suitable one. The paper thus corrects the previously published result on infimum quantization density for a given control Lyapunov function, for the class of multiple-input systems considered.**

I. INTRODUCTION

Systems involving quantization arise naturally in many areas of engineering, especially when digital implementations are involved. In recent years, especially motivated by control of systems over communication networks, different control schemes have been developed where the fact that controller and plant(s) may be connected via a communication channel is taken into account [1]–[6]. See also the special issue [7].

The new challenges to control design that arise from the introduction of a communication channel between controller and plant(s) are manifold. These challenges include the need to explicitly deal with quantization, nonuniform sampling, variable time delays and limited data-rate/bandwidth. Several lines of research exist that deal with different groups of these problems at a time.

The work in this paper is directly related with the line of research introduced in [6] and followed in [8]–[12], and deals mainly with quantization. The approach consists in defining a measure of density of a quantizer. Such a measure is intuitively related with the spacing between the quantization levels of a quantizer: the larger the spacing, the less the density. The goal is then to find (design) a quantizer that has the infimum density by searching over all quadratically stabilizing quantizers for a given system.

In [6], this infimum quantization density problem is solved for linear time-invariant single-input systems. It is shown that any quantizer that has the infimum density has logarithmically-spaced quantization levels. The authors then proceed to develop a comprehensive treatment of stabilization of this type of systems via quantized (state or output) feedback. In addition, [6] also details how to obtain a finite quantizer that achieves practical stability by truncating a logarithmic quantizer derived from an infimum density problem. This finite quantizer can then be practically implemented. In [10], the infimum quantization density problem is analyzed via a sector bound approach. The authors also deal with multiple-input systems by independently quantizing the different input channels. In [12], a geometric approach to quadratic stabilization with quantizers is developed and is utilized in [11] to reobtain some of the results of [6] and to design static output feedback strategies that employ infimum quantization density quantizers. In [9], different quantization schemes and their densities are analyzed for multiple-input systems. The first results regarding infimum quantization density for multiple-input systems appear in [8], where the problem is studied for two-input systems, providing (a) an exact solution for control Lyapunov functions (CLFs) of a specific type (which they name Type$_1$) and (b) a lower bound for another type. As evidenced by [8]–[10], the infimum quantization density problem is extremely hard for general multiple-input systems.

The current paper deals with linear multiple-input systems that can be stabilized using a one-dimensional subspace of the input space. This class of systems admits quadratic CLFs of Type$_1$ (see Section IV and [8]). For such a CLF, we search for the infimum density over all stabilizing (with respect to the given CLF) quantizers and obtain that the infimum density is also the infimum density for a single-input system derived from the original multiple-input system. This result, though conceptually similar, differs from [8, Theorem 1]. We analyze this discrepancy by means of a numerical example and show that the result in [8, Theorem 1] yields an inconsistent value of the infimum density. We stress that the main results of [8] are not invalidated by this fact. The main contribution of the current paper is then to obtain the infimum quantization density for the class of multiple-input systems considered, for a *given* CLF.

The remainder of this paper is organized as follows. Section II specifies the class of multiple-input systems that we deal with, as well as the problem to be solved and the assumptions made. In Section III, the problem is divided into two subproblems. The first of these subproblems is solved using existing results. The second subproblem constitutes the key contribution of the paper, obtaining a result on the infimum density for the aforementioned class of multiple-input systems, for a given CLF. In Section IV, we review the result of [8, Theorem 1] on the infimum quantization density for the class of multiple-input systems that we deal with. We then compare that result with the current one in

*The authors are with the ARC Centre for Complex Dynamic Systems and Control (CDSC), The University of Newcastle, Callaghan, NSW 2308, Australia

†email: hhaimo@ee.newcastle.edu.au

Section V by means of a numerical example. Conclusions are drawn in Section VI.

II. PROBLEM STATEMENT

We consider a nonscalar discrete-time linear time-invariant system, defined by

$$x^+ = Ax + Bu, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is unstable, $B \in \mathbb{R}^{n\times m}$ has full rank and x^+ denotes the successor state. We assume that system (1) can be stabilized using only a one-dimensional subspace of the input space. For this class of systems, we characterize all quantized feedbacks $u = q(x)$ that, in addition to having values in a one-dimensional subspace, render the closed-loop system $x^+ = Ax+Bq(x)$ quadratically stable with respect to a *given* Lyapunov function. Then, we find the infimum of the quantization density over all such feedbacks. In Section II-A, we specify the definition of quantizer employed and some basic facts on the quadratic stabilization approach. Section II-B defines the measure of quantization density that we employ and Section II-C formulates the problem of quantization density optimization.

A. Quadratic stabilization via quantized state feedback

System (1) is analyzed whenever it can be quadratically stabilized via a quantized state feedback $u = q(x)$. In accordance with the literature (see [6], [8], [10]), we define a quantizer as follows.

Definition 1 (Quantizer): A *quantizer* q is a function $q : \mathbb{R}^p \to \mathbb{R}^\ell$ of the form

$$q(z) = u_i \text{ if } z \in \mathcal{R}_i, \quad \text{for } i \in \mathbb{Z}.$$

The sets $\mathcal{R}_i$ are called the *quantization regions* of q and u_i is called the *level* or *value* of q corresponding to $\mathcal{R}_i$. The sets $\mathcal{R}_i$ satisfy $\bigcup_{i\in\mathbb{Z}} \mathcal{R}_i = \mathbb{R}^p$ and $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ whenever $i \neq j$. By *state* quantizer we mean a quantizer q whose domain is the state space, that is, $q : \mathbb{R}^n \to \mathbb{R}^\ell$. By *scalar* quantizer we refer to a quantizer $q : \mathbb{R} \to \mathbb{R}$.

Specifically, we are interested in state quantizers $q : \mathbb{R}^n \to \mathbb{R}^m$ that are able to quadratically stabilize system (1) and satisfy $q(x) = v\mathring{q}(d^T x)$, for some $v \in \mathbb{R}^m$, $d \in \mathbb{R}^n$ and where $\mathring{q} : \mathbb{R} \to \mathbb{R}$ is a scalar quantizer, as shown in Fig. 1. Note that the fact that such a quantizer stabilizes the system implies that the pair (A, B) is stabilizable. Moreover, there must exist a control $v \in \mathbb{R}^m$ such that the pair (A, Bv) be stabilizable. In other words, the multiple-input system (1) can be stabilized using only a one-dimensional subspace of the input space.

Let $V(x) \triangleq x^T Px$ be a given CLF, where $P = P^T > 0$ is an $n\times n$ matrix. Let $\Delta V(x, u)$ denote the increment of V along the trajectories of system (1) when the control applied is $u \in \mathbb{R}^m$ and the state is $x \in \mathbb{R}^n$, that is,

$$\begin{aligned}\Delta V(x, u) &\triangleq V(x^+) - V(x)\\ &= x^T Lx + 2x^T Mu + u^T B^T PBu, \end{aligned}\tag{2}$$

where $$L \triangleq A^T PA - P, \quad M \triangleq A^T PB. \tag{3}$$

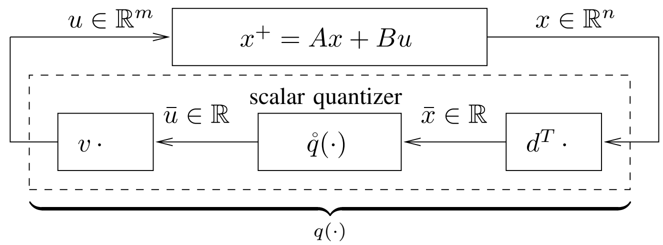

Fig. 1. The quantized feedback considered: $u = q(x) = v\mathring{q}(d^T x)$.

We are thus interested in quantizers q that satisfy $\Delta V(x, q(x)) < 0$ for all nonzero $x \in \mathbb{R}^n$. In the sequel, we assume that L is invertible. This assumption enables us to employ the results in [11] and can always be satisfied if A contains no eigenvalues with unit magnitude.

B. Quantization density

Given a quantizer $q : \mathbb{R}^p \to \mathbb{R}^\ell$, let $\mathcal{U}(q)$ denote the range of q, that is,

$$\mathcal{U}(q) \triangleq \{u \in \mathbb{R}^\ell : u = q(x) \text{ for some } x \in \mathbb{R}^p\}. \tag{4}$$

For $\epsilon \in (0, 1)$, let $C^\ell(\epsilon)$ be the following region in $\mathbb{R}^\ell$:

$$C^\ell(\epsilon) \triangleq \{u \in \mathbb{R}^\ell : \epsilon \le \|u\| \le 1/\epsilon\}. \tag{5}$$

We define the density of q, denoted $\eta(q)$, as follows:

$$\eta(q) \triangleq \limsup_{\epsilon\to 0} \frac{\#[\mathcal{U}(q) \cap C^\ell(\epsilon)]}{-2\ln\epsilon}, \tag{6}$$

where $\#S$ denotes the number of elements of the set S. This measure of density coincides with the one given in [6] when the quantizer output (control input) is a scalar (that is, $\ell = 1$) and q satisfies $q(x) = -q(-x)$. It is also equal to one half the density defined in [8] for two-input systems. According to (6), the density of a quantizer with a finite number of levels is zero, is infinite for a quantizer with radially uniformly spaced values and finite for radially logarithmically spaced values. Note that the density of a quantizer is always nonnegative.

C. Infimum quantization density

Given the unstable system (1) and a function $V(x) = x^T Px$, where $P = P^T > 0$ is such that L in (3) is invertible, we want to solve the following problem.

Problem 1 (Infimum Density Problem):

$$\eta^\star = \inf \eta(q) \tag{7}$$

subject to the following constraints:

C1) There exist $v \in \mathbb{R}^m$, $d \in \mathbb{R}^n$ and a scalar quantizer $\mathring{q}$ such that $q(x) = v\mathring{q}(d^T x)$, for all $x \in \mathbb{R}^n$,
C2) $q(x) = -q(-x)$ for all $x \in \mathbb{R}^n$,
C3) V is a Lyapunov function for the closed-loop system $x^+ = Ax + Bq(x)$.

Constraint C1) insures that the quantizer q has the form shown in Fig. 1 for some $v \in \mathbb{R}^m$ and $d \in \mathbb{R}^n$. Constraint C2) is imposed in accordance with [6], since it incurs no loss of generality in regard to quantization density. C3) restricts the search to quantizers that are quadratically stabilizing with respect to the given CLF.

Before solving Problem 1, the following comment is in order. From C1) and C3), it follows that V is a Lyapunov function for the closed-loop system $x^+ = Ax + Bv\mathring{q}(d^T x)$, for some $v \in \mathbb{R}^m$ and $d \in \mathbb{R}^n$. Hence V is a CLF for the single-input system

$$x^+ = Ax + \bar{B}\bar{u}, \quad \text{where} \quad \bar{B} \triangleq Bv, \tag{8}$$

for some $v \in \mathbb{R}^m$. Note that V is a CLF for such a system if and only if $\Delta V(x, u)$ in (2), where $u = v\bar{u}$, is negative when $\bar{u} = K_{\mathrm{GD}} x$ (and $x \neq 0$), with

$$K_{\mathrm{GD}} = -(\bar{B}^T P \bar{B})^{-1} v^T M^T, \tag{9}$$

since this is the feedback that causes $V(x)$ to decrease the most along the trajectories of system (8) (see [6, Section II-B]). Thus, V is a CLF for (8) if and only if

$$L - \bar{M}(\bar{B}^T P \bar{B})^{-1} \bar{M}^T < 0, \text{ where } \bar{M} \triangleq Mv. \tag{10}$$

In summary, the function $V(x) = x^T P x$ must be such that there exists $v \in \mathbb{R}^m$ that satisfies (10). If this is not the case, then the constraint set of Problem 1 is empty and it is not meaningful to solve this problem for the given V.

The following result is used in the sequel.

Lemma 1: Let $V(x) = x^T P x$ be a CLF for the single-input system (8), let L, defined in (3), be invertible and assume that the system is nonscalar, that is, $n \geq 2$. Then, L has $n-1$ negative and 1 positive eigenvalues.

Proof: Note that (2) and (10) imply that

$$\begin{aligned} x^T[L - \bar{M}(\bar{B}^T P \bar{B})^{-1} \bar{M}^T]x = \\ \Delta V(x, vK_{\mathrm{GD}} x) < 0, \text{ for all } x \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \tag{11}$$

Since A is unstable and $\bar{u} = K_{\mathrm{GD}} x$ stabilizes (8), we know that $K_{\mathrm{GD}} \neq 0$, and since $K_{\mathrm{GD}} \in \mathbb{R}^{1 \times n}$, then $\dim(\ker K_{\mathrm{GD}}) = n - 1 \geq 1$. From (11), $\Delta V(\bar{x}, 0) < 0$ for all $\bar{x} \in \ker K_{\mathrm{GD}} \setminus \{0\}$, and using (2) and (3), then $\Delta V(\bar{x}, 0) = \bar{x}^T L \bar{x} < 0$ for all $\bar{x} \in \ker K_{\mathrm{GD}} \setminus \{0\}$. Since $\bar{x}^T L \bar{x} < 0$ for all nonzero vectors in a subspace of dimension $n-1$, L must have at least $n-1$ negative eigenvalues (see [13, Sec. 4.3.23, p. 192]). However, L cannot have n negative eigenvalues because this implies that L is negative definite, contradicting the assumption that A is unstable. Also, since L is invertible, then the remaining eigenvalue must be positive. ■

III. PROBLEM SOLUTION

Fig. 1 shows that, conceptually, there exists a single-input system between the fictitious input $\bar{u}$ and the state x. This observation motivates us to divide Problem 1 into the following two subproblems. First, we assume that some "feasible" $v \in \mathbb{R}^m$ is given and consider the single-input system (8). The existing results [6], [10], [11] can be used to design a quantizer having infimum density for this single-input system. The resulting infimum density is a function of the input matrix $\bar{B} = Bv$ and hence a function of v. Second, we optimize density over all "feasible" vectors $v \in \mathbb{R}^m$. The resulting density is the infimum density *for the single-input system (8)*. The following result shows that, fortunately, this is also the infimum density for the multiple-input system (1) when q satisfies constraint C1) and $\|v\| = 1$, or when $\bar{q}(x) \triangleq \mathring{q}(d^T x)$ is logarithmic.

Proposition 1: Let $\bar{q} : \mathbb{R}^n \to \mathbb{R}$ be a quantizer and let $q(x) = v\bar{q}(x)$, for some nonzero $v \in \mathbb{R}^m$. Then,

(i) If $\|v\| = 1$, then $\eta(q) = \eta(\bar{q})$,

(ii) If $\bar{q}$ is logarithmic, then $\eta(q) = \eta(\bar{q})$.

Proof: Let $\mathcal{U}(q)$ and $\mathcal{U}(\bar{q})$ denote the ranges of q and $\bar{q}$, respectively. Note that $\mathcal{U}(q) \subset \mathbb{R}^m$ and $\mathcal{U}(\bar{q}) \subset \mathbb{R}$. Let $C^m(\epsilon)$ and $C^1(\epsilon)$ be defined as in (5). We have $\mathcal{U}(q) = v\,\mathcal{U}(\bar{q})$ and

$$\begin{aligned} \#[\mathcal{U}(q) \cap C^m(\epsilon)] &= \#[v\mathcal{U}(\bar{q}) \cap C^m(\epsilon)] \\ &= \#[\|v\|\mathcal{U}(\bar{q}) \cap C^1(\epsilon)]. \end{aligned} \tag{12}$$

(i) From (12) and since $\|v\| = 1$, then $\#[\mathcal{U}(q) \cap C^m(\epsilon)] = \#[\mathcal{U}(\bar{q}) \cap C^1(\epsilon)]$, and hence

$$\limsup_{\epsilon \to 0} \frac{\#[\mathcal{U}(q) \cap C^m(\epsilon)]}{-2 \ln \epsilon} = \limsup_{\epsilon \to 0} \frac{\#[\mathcal{U}(\bar{q}) \cap C^1(\epsilon)]}{-2 \ln \epsilon},$$

proving that $\eta(q) = \eta(\bar{q})$.

(ii) Since $\bar{q}$ is logarithmic, then

$$\mathcal{U}(\bar{q}) = \{\pm \rho^i u_0 : i \in \mathbb{Z}\} \cup \{0\}, \tag{13}$$

for some $0 < \rho < 1$ and $u_0 \in \mathbb{R}$. We have

$$\|v\|\mathcal{U}(\bar{q}) = \{\pm \rho^i u_0 \|v\| : i \in \mathbb{Z}\} \cup \{0\}. \tag{14}$$

From (14), (13) and [6], it follows that

$$\limsup_{\epsilon \to 0} \frac{\#[\|v\|\,\mathcal{U}(\bar{q}) \cap C^1(\epsilon)]}{-2 \ln \epsilon} = -2/\ln \rho = \eta(\bar{q}),$$

and using (14) and (12), then $\eta(q) = \eta(\bar{q})$. ■

Fig. 2 illustrates equality (12) when $\|v\| = 1$. At the top left, the set $\mathcal{U}(q)$ is depicted by means of little circles. Note that $\mathcal{U}(q)$ is contained in the subspace of $\mathbb{R}^m$ generated by the vector v, also shown in the figure. The annular region $C^m(\epsilon)$ is the region between the circles with radii ϵ and $1/\epsilon$. At the top right, the intersection $\mathcal{U}(q) \cap C^m(\epsilon)$ is shown and consists of the four little circles that are contained in $C^m(\epsilon)$ in the top-left part of the figure. At the bottom left, the set $\mathcal{U}(\bar{q})$ and the region $C^1(\epsilon)$ are shown, and their intersection is shown at the bottom-right part of the figure. Note that the number of elements in the sets $\mathcal{U}(q) \cap C^m(\epsilon)$ and $\mathcal{U}(\bar{q}) \cap C^1(\epsilon)$ is the same.

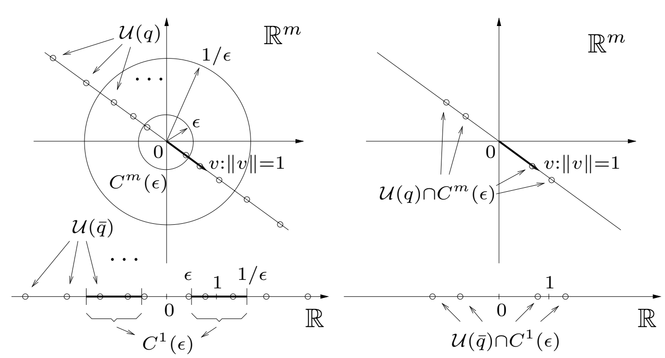

Fig. 2. The equality (12) for $\|v\| = 1$.

We now return to Problem 1. Considering constraint C1), defining $\bar{q}(x) \triangleq \mathring{q}(d^T x)$ and incorporating the additional constraint $\|v\| = 1$, we may use Proposition 1 to replace (7) by $\eta^\star = \inf \eta(\bar{q})$. We are now ready to divide Problem 1 into the following two subproblems.

Subproblem a): For a given $v \in \mathbb{R}^m$,

$$\eta_v = \inf \eta(\bar{q})$$

subject to:

C1') There exists $d \in \mathbb{R}^n$ and a scalar quantizer $\mathring{q}$ such that $\bar{q}(x) = \mathring{q}(d^T x)$ for all $x \in \mathbb{R}^n$,

C2') $\bar{q}(x) = -\bar{q}(-x)$ for all $x \in \mathbb{R}^n$,

C3') V is a Lyapunov function for the closed-loop system $x^+ = Ax + (Bv)\bar{q}(x)$.

Subproblem b):

$$\eta^\star = \inf \eta_v,$$

where η_v is the solution to Subproblem a), subject to:

- $v \in \mathbb{R}^m$, $\|v\| = 1$ and v is such that there exists at least one feasible $\bar{q}$ for Subproblem a).

The solution to Problem 1, namely $\eta^\star$, is also the solution to Subproblem b).

A. Solution to Subproblem a)

Depending on the given value of $v \in \mathbb{R}^m$, the constraints of Subproblem a) will be satisfied or not. When the constraints are satisfied, we have, using the result in [11]:

$$\eta_v = -2/\ln \frac{\beta(v) - \gamma(v)}{\beta(v) + \gamma(v)}, \qquad (15)$$

where

$$\beta(v) \triangleq v^T M^T L^{-1} M v, \quad \gamma(v) \triangleq \sqrt{-v^T H v\, \beta(v)} \qquad (16)$$

$$H \triangleq B^T P B - M^T L^{-1} M, \qquad (17)$$

and L and M were defined in (3). As shown in [6], [10] and [11], a quantizer that has the infimum density for a single-input system is logarithmic and hence, by Proposition 1, the constraint $\|v\| = 1$ is not necessary in Subproblem b). Also, note that the expression η_v in (15)–(17) does not depend on the norm of v (provided it is nonzero).

B. Solution to Subproblem b)

It is only meaningful to evaluate η_v for vectors $v \in \mathbb{R}^m$ such that the constraint set of Subproblem a) is nonempty. Note that the only constraint of Subproblem a) that relates to v is C3'), which says that V is a CLF for the single-input system (8). We can equivalently characterize all such $v \in \mathbb{R}^m$ as follows:

- $v \in \mathbb{R}^m$ satisfies (10) for the given $V(x) = x^T P x$.

The following Lemma provides an equivalent algebraic characterization of all such $v \in \mathbb{R}^m$.

Lemma 2: Fix $P = P^T > 0$ such that L in (3) is invertible and there exists (at least one) $v \in \mathbb{R}^m$ satisfying (10). Then, for this fixed P, an arbitrary nonzero $v \in \mathbb{R}^m$ satisfies (10) if and only if $v^T H v < 0$, where H was defined in (17).

Proof: Necessity. The matrix on the left-hand side of (10) is negative definite if and only if its inverse also is. Using a matrix inversion formula, together with (17) and $\bar{B} = Bv$, it follows that (10) is true if and only if

$$L^{-1} + L^{-1}\bar{M}(v^T H v)^{-1}\bar{M}^T L^{-1} < 0, \qquad (18)$$

Since, from Lemma 1, L has one positive and $n-1$ negative eigenvalues, so does L^{-1}. Since $\bar{M} \in \mathbb{R}^{n\times 1}$, then the symmetric matrix $L^{-1}\bar{M}(v^T H v)^{-1}\bar{M}^T L^{-1}$ has rank at most one. In order that (18) be true, this matrix must then be nonzero and negative semidefinite. Thus, it follows that $v^T H v < 0$, concluding the necessity part of the proof.

Sufficiency. We now assume that $v^T H v < 0$ which, by (17) and (8), is equivalent to

$$\bar{M}^T L^{-1} \bar{M} > \bar{B}^T P \bar{B} > 0, \qquad (19)$$

where $\bar{M} = Mv$ and the last inequality follows from $P > 0$ and $\bar{B} = Bv$, where B has full rank. Eq. (10) is true if and only if all the eigenvalues of $L - \bar{M}(\bar{B}^T P \bar{B})^{-1}\bar{M}^T$ are negative. λ is an eigenvalue of this matrix if and only if there exists a nonzero $x \in \mathbb{R}^n$ satisfying

$$[\lambda \mathrm{I} - L + \bar{M}(\bar{B}^T P \bar{B})^{-1}\bar{M}^T]x = 0. \qquad (20)$$

We first prove that if λ satisfies (20) then λ is not the only positive eigenvalue of L. For a contradiction, assume that λ satisfies (20) and is the only positive eigenvalue of L. Then, the matrix $\lambda \mathrm{I} - L$ is positive semidefinite. Also, the rank-one matrix $\bar{M}(\bar{B}^T P \bar{B})^{-1}\bar{M}^T$ is positive semidefinite since, from (19), $\bar{B}^T P \bar{B} > 0$ and $\bar{M} \neq 0$. Premultiplying (20) by x^T yields

$$x^T(\lambda \mathrm{I} - L)x + x^T[\bar{M}(\bar{B}^T P \bar{B})^{-1}\bar{M}^T]x = 0,$$

which is therefore satisfied if and only if

$$x^T(\lambda \mathrm{I} - L)x = x^T[\bar{M}(\bar{B}^T P \bar{B})^{-1}\bar{M}^T]x = 0. \qquad (21)$$

Since the matrices in (21) are positive semidefinite and x is nonzero, it follows that

$$(\lambda \mathrm{I} - L)x = \bar{M}(\bar{B}^T P \bar{B})^{-1}\bar{M}^T x = 0. \qquad (22)$$

The following conditions follow from (22): x is an eigenvector corresponding to the only positive eigenvalue of L and $\bar{M}^T x = 0$. In addition, x is also an eigenvector corresponding to the only positive eigenvalue of L^{-1} because L is symmetric. Then, $\bar{M}^T L^{-1} \bar{M} < 0$ because $\bar{M}$ is orthogonal to the eigenvector corresponding to the only positive eigenvalue of L^{-1}. This contradicts (19). Therefore, we have proved that if λ satisfies (20), then λ is not the only positive eigenvalue of L.

We now prove that if $\lambda \geq 0$, then it is not an eigenvalue of the matrix on the left-hand side of (10). For a contradiction, assume $\lambda \geq 0$ is an eigenvalue of this matrix. Then, λ satisfies

$$\det(\lambda \mathrm{I} - L + \bar{M}(\bar{B}^T P \bar{B})^{-1}\bar{M}^T) = 0. \qquad (23)$$

Since we have already proved that λ is not the only positive eigenvalue of L, then (23) is equivalent to

$$(1 + \bar{M}^T(\lambda \mathrm{I} - L)^{-1}\bar{M}(\bar{B}^T P \bar{B})^{-1})\det(\lambda \mathrm{I} - L) = 0. \qquad (24)$$

Since $\det(\lambda \mathrm{I} - L) \neq 0$, it then follows from (24) that

$$\bar{M}^T (L - \lambda \mathrm{I})^{-1} \bar{M} = \bar{B}^T P \bar{B}.$$

Consider the function $f(\alpha) \triangleq \bar{M}^T (L - \alpha \mathrm{I})^{-1} \bar{M}$, which is defined, continuous and differentiable at any $\alpha \in \mathbb{R}$ that is not an eigenvalue of L. Note that $f(\lambda) = \bar{B}^T P \bar{B} > 0$ and $f(0) > \bar{B}^T P \bar{B} > 0$ [see (19)]. We have

$$\frac{df}{d\alpha}(\alpha) = \bar{M}^T (L - \alpha \mathrm{I})^{-2} \bar{M},$$

which satisfies $\frac{df}{d\alpha}(\alpha) > 0$ at any α that is not an eigenvalue of L. Hence, $f(\alpha)$ is increasing at any such point. Also, it is straightforward to check that $\lim_{\alpha \to \infty} f(\alpha) = 0$. Denoting the only positive eigenvalue of L by $\lambda_{\max}(L)$, we conclude that $f(\alpha)$ has the form sketched in Fig. 3 for any $\alpha \geq 0$ that is not $\lambda_{\max}(L)$. Hence, there exists no $\alpha \geq 0$ such that $f(\alpha) = \bar{B}^T P \bar{B}$. This is a contradiction since we had $\lambda \geq 0$ and $f(\lambda) = \bar{B}^T P \bar{B}$. We have thus shown that whenever λ is an eigenvalue of the matrix on the left-hand side of (10), then $\lambda < 0$, proving (10). This concludes the proof. ∎

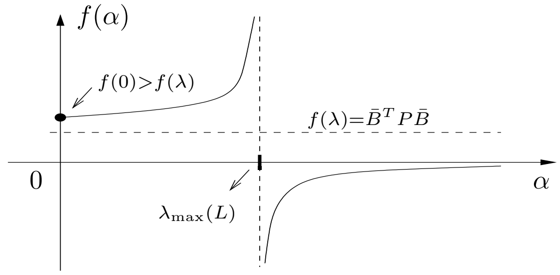

Fig. 3. The function $f(\alpha)$.

Using Lemma 2 we can recast Subproblem b) as:

$$\eta^\star = \inf \eta_v,$$

subject to:

- $v \in \mathbb{R}^m$ satisfies $v^T H v < 0$.

We are now ready to state the main result of the paper.

Theorem 1: The solution to Subproblem b), and hence to Problem 1, is given by

$$\eta^\star = -2 / \ln \frac{\beta(v^\star) - \gamma(v^\star)}{\beta(v^\star) + \gamma(v^\star)}, \text{ where} \tag{25}$$

$$v^\star = (B^T P B)^{-1/2} w^\star, \tag{26}$$

β and γ are defined in (16), and $w^\star$ is an eigenvector corresponding to the greatest eigenvalue of the matrix

$$(B^T P B)^{-1/2} M^T L^{-1} M (B^T P B)^{-1/2}. \tag{27}$$

Proof: Since the density of a quantizer is always nonnegative, we have $\eta_v \geq 0$ and thus any optimizer of Subproblem b) is also an optimizer of [see (15)]

$$\inf \frac{\beta(v) - \gamma(v)}{\beta(v) + \gamma(v)}, \tag{28}$$

subject to $v^T H v < 0$. From (17), $v^T H v < 0$ implies that $v^T M^T L^{-1} M v > 0$ and hence $\beta(v) > 0$ [see (16)]. Then, $\gamma(v) \neq 0$ and (28) is equivalent to

$$\inf \frac{\beta(v)/\gamma(v) - 1}{\beta(v)/\gamma(v) + 1}. \tag{29}$$

From (16) and (17), it follows that $\gamma^2(v) = -v^T H v \, \beta(v) = [\beta(v) - v^T B^T P B v] \beta(v)$ and then $\gamma^2(v) < \beta^2(v)$. Combining this last inequality with the fact that $\beta(v) > 0$ and $\gamma(v) > 0$ whenever $v^T H v < 0$, then $\beta(v) > \gamma(v) > 0$, and thus $\beta(v)/\gamma(v) > 1$ whenever $v^T H v < 0$. Since the expression to be optimized in (29), considered as a function of $\beta(v)/\gamma(v)$, is increasing for $\beta(v)/\gamma(v) > 1$, then any optimizer is also an optimizer of $\inf \beta(v)/\gamma(v)$, which in turn is an optimizer of

$$\inf \frac{\beta^2(v)}{\gamma^2(v)} = \inf \frac{v^T M^T L^{-1} M v}{v^T M^T L^{-1} M v - v^T B^T P B v}, \tag{30}$$

subject to $v^T H v < 0$. Since $v^T B^T P B v > 0$, (30) is equivalent to

$$\inf \frac{\frac{v^T M^T L^{-1} M v}{v^T B^T P B v}}{\frac{v^T M^T L^{-1} M v}{v^T B^T P B v} - 1}, \tag{31}$$

and since $v^T H v < 0$, then $\frac{v^T M^T L^{-1} M v}{v^T B^T P B v} > 1$ and any optimizer of (31) is also an optimizer of

$$\sup \frac{v^T M^T L^{-1} M v}{v^T B^T P B v}. \tag{32}$$

Let $w = (B^T P B)^{1/2} v$ and substitute into (32) to obtain

$$\sup \frac{w^T (B^T P B)^{-1/2} M^T L^{-1} M (B^T P B)^{-1/2} w}{w^T w}. \tag{33}$$

Note that any optimizer $w^\star$ of (33) is an eigenvector corresponding to the greatest eigenvalue of the matrix (27). Therefore, $v^\star = (B^T P B)^{-1/2} w^\star$ and the result follows. ∎

Remark 1: Theorem 1 solves a quantization density optimization problem when the CLF is *given*. For the class of multiple-input systems considered, namely those that can be stabilized using a one-dimensional subspace of the input space, optimization of the quantization density over all quadratic CLFs for the derived single-input system (8) yields the result in [6, Theorem 2.2]. This infimum density only depends on the unstable eigenvalues of A, that is, it is independent of the input matrix $\bar{B} = Bv$, so long as (A, Bv) is stabilizable.

IV. A RESULT IN THE LITERATURE

In this section, we review [8, Theorem 1] in order to be able to compare that result to the one above.

In [8], the authors define a CLF of the form $V(x) = x^T P x$ to be of Type$_J$ if the number of strictly positive eigenvalues of the matrix $L = A^T P A - P$, defined in (3), is J. Then, according to Lemma 1, the CLF considered in this paper is of Type$_1$. For simplicity, we now copy the statement of Theorem 1 in [8, p. 183] using the current notation.

Theorem 2 (Theorem 1 in p. 183 of [8]): If $V(x) = x^T P x$, $P > 0$, is a CLF of Type$_1$ for system (1). Then $V(x)$ is also a CLF for the single-input system (8) obtained by replacing B with $\bar{B} = B v^\natural$ where

$v^\natural = (B^TPB)^{-1}B^TPAw^\natural$, and $w^\natural$ denotes the eigenvector associated with the only positive eigenvalue of L. Moreover the coarsest (infimum density) quantizer for system (1), and such a V, is given by

$$q(x) = v^\natural \bar{q}(x)$$

where $\bar{q}(x)$ is the coarsest quantizer for system (8).

According to this theorem, the infimum density $\eta^\natural \triangleq \eta(\bar{q})$ can be obtained from [6] as:

$$\eta^\natural = -2/\ln\rho, \tag{34}$$

where

$$\rho = \frac{\sqrt{\frac{\bar{B}^TPAQ^{-1}A^TP\bar{B}}{\bar{B}^TP\bar{B}}} - 1}{\sqrt{\frac{\bar{B}^TPAQ^{-1}A^TP\bar{B}}{\bar{B}^TP\bar{B}}} + 1}, \tag{35}$$

and

$$Q = P - A^TPA + \frac{A^TP\bar{B}\bar{B}^TPA}{\bar{B}^TP\bar{B}}, \tag{36}$$

with $\bar{B} = Bv^\natural$.

V. EXAMPLE

In this section, we compare the solution to Problem 1 obtained in Theorem 1 with the result in [8, Theorem 1] by means of a numerical example. Let system (1) be defined with matrices

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and consider the CLF $V(x) = x^TPx$, where

$$P = \begin{bmatrix} 1744 & 3901 & -4574 \\ 3901 & 8809 & -10356 \\ -4574 & -10356 & 12187 \end{bmatrix}.$$

Note that $P = P^T$, and $P > 0$ since the eigenvalues of P are approximately 1, 25 and 22714. The eigenvalues of the matrix L defined in (3) are approximately -253, -1 and 146756, showing that L has only one positive eigenvalue and is invertible. According to [8], then $V(x)$ is a CLF of Type$_1$.

Evaluating $\eta^\star$ according to Theorem 1, that is, according to (25) and (26), gives

$$\eta^\star = -2/\ln 0.9318 \approx 28.3, \text{ with}$$
$$v^\star = \begin{bmatrix} 0.712 & 0.7022 \end{bmatrix}^T.$$

Also, the same result is obtained by means of (34)–(36), with $\bar{B} = Bv^\star$. If we evaluate the density using Theorem 2, that is, according to [8, Theorem 1], we obtain

$$v^\natural = \begin{bmatrix} -0.8509 & 0.5254 \end{bmatrix}^T,$$

and the argument of the square root in (35) is negative, approximately equal to -7.2773, which results in an inconsistent (complex) value of the density. Note also that in this case, $V(x) = x^TPx$ is *not* a CLF for the single-input system (8) with $\bar{B} = Bv^\natural$, since (10) is not satisfied or, equivalently, Q in (36) is not positive definite. The reason for this inconsistency seems to stem from the fact that, in the proof of [8, Theorem 1], the authors study $\Delta V(x, u)$ [see (2)] only along the direction of the eigenvector corresponding to the only positive eigenvalue of $L = A^TPA - P$.

VI. CONCLUSIONS

We have derived a new result on the infimum quantization density for linear time-invariant multiple-input systems that can be stabilized using a one-dimensional subspace of the input space. This result yields the infimum density over all quantized feedbacks that are quadratically stabilizing with respect to a *given* control Lyapunov function. The infimum density derived was shown to differ from a previously published result. This discrepancy was explored by means of a numerical example that shows that, whereas the previously published result yields an inconsistent value of the density, our result provides a suitable one. We have thus corrected the previously published result on infimum quantization density for a given control Lyapunov function, for the class of multiple-input systems considered.

REFERENCES

[1] D. Hristu and K. Morgansen, "Limited communication control," *Systems and Control Letters*, vol. 37, pp. 193–205, 1999.

[2] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints—II: stabilization with limited information feedback," *IEEE Trans. on Automatic Control*, vol. 44, no. 5, pp. 1049–1053, 1999.

[3] S. C. Tatikonda, "Control under communication constraints," Ph.D. dissertation, Massachussets Institute of Technology, 2000.

[4] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. on Automatic Control*, vol. 45, no. 7, pp. 1279–1289, 2000.

[5] G. N. Nair and R. J. Evans, "Stabilization with data-rate-limited feedback: tightest attainable bounds," *Systems and Control Letters*, vol. 41, pp. 49–56, 2000.

[6] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. on Automatic Control*, vol. 46, no. 9, pp. 1384–1400, 2001.

[7] P. Antsaklis and J. Ballieul, Guest Eds., "Special issue on networked control systems," *IEEE Trans. on Automatic Control*, vol. 49, no. 9, 2004.

[8] N. Elia and E. Frazzoli, "Quantized stabilization of two-input linear systems: a lower bound on the minimal quantization density," in *Hybrid Systems and Control, HSCC*, ser. Lecture Notes in Computer Science, C. J. Tomlin and M. R. Greenstreet, Eds. Springer, 2002, no. 2289, pp. 179–193.

[9] C.-Y. Kao and S. R. Venkatesh, "Stabilization of linear systems with limited information – multiple input case," in *Proc. American Control Conference*, Anchorage, AK, USA, 2002, pp. 2406–2411.

[10] M. Fu and L. Xie, "On control of linear systems using quantized feedback," in *Proc. American Control Conference*, 2003, pp. 4567–4572.

[11] H. Haimovich, "Stabilizing static output feedback via coarsest quantizers," in *Proc. 16th IFAC World Congress, Prague, Czech Republic*, 2005.

[12] H. Haimovich and M. M. Serón, "A geometric approach to quadratic stabilization of discrete-time systems with quantizers," The University of Newcastle, Australia, Tech. Rep. EE05002, 2005, available at http://www.eecs.newcastle.edu.au/cgi-bin/report_db/new_search.cgi.

[13] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.