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Local Stability Analysis of Saturating Systems via Polynomial Programming

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Abstract— **This paper considers the local stability analysis of feedback systems with saturation nonlinearities. First, considering the detailed structure of the saturation nonlinearity, we derive a local stability condition with the aid of *S*-Procedure of polynomial with higher degrees of freedom. Then, we obtain a Lyapunov function using Sum of Squares decomposition. Furthermore, it is shown that this method can be easily extended to the case where piecewise quadratic Lyapunov functions are adopted. Finally we demonstrate its effectiveness through numerical examples.**

I. INTRODUCTIONS

Most of physical systems are subject to input saturation. If we design a compensator without considering its saturation, performance degradation such as windup phenomena or destabilization occurs. In case of unstable plants, we are not able to find any globally stabilizing compensator. Therefore, many researchers have analyzed the local stability of saturating systems so far.

One of the standard approaches is to utilize the circle criterion (see e.g., [1]). In this approach, the saturation is treated as sector-bounded nonlinearity. Thus, the result tends to be too conservative since the detailed structure of saturation is not taken into account. Also, it is pointed out that the domain of attraction of the closed loop systems designed through the circle criterion is the same as those by linear analysis[2]. Therefore it was desired to exploit more detailed structure of saturation nonlinearity. Hu and co-workers [3], [4], [5] derived other types of local stability conditions and proposed a method to analyze the domain of attraction by solving LMI(Linear Matrix Inequality). Johansson [6], [7] proposed to adopt piecewise quadratic Lyapunov functions in order to obtain less conservative results. These two approaches do not take into account the case where anti-wind compensators are included. Takaba [8] presented local stability conditions of a wider class of systems including such a case. However, there is still some room to improve its conservativeness.

On the other hand, from the optimization point of view, Parrilo et.al [10], [11] presented the sum of square decomposition. This approach enables us to deal with polynomial programming easily. Since wider class of problems can be reduced to the polynomial programming, this approach is applied to various control analysis and synthesis problems[12], [13], [14]. Similarly, it is expected that we can describe the detailed structure of saturation via polynomials to obtain less conservative results on stability analysis.

The purpose of this paper is to give a local stability condition of saturation system using polynomial constraints, and to obtain less conservative domain of attraction based on the sum of squares decomposition.

Notation: Given $\boldsymbol{x} \in \mathbf{R}^n$, $\mathbf{R}[\boldsymbol{x}]$ denotes the set of scalar polynomials whose variables are entries of $\boldsymbol{x}$, and $\deg(f)$ be maximum degree of $f \in \mathbf{R}[\boldsymbol{x}]$. Also, $\Sigma[\boldsymbol{x}]$ denotes the set of sum of square polynomials in $\mathbf{R}[\boldsymbol{x}]$. In other words, $f(\boldsymbol{x}) \in \Sigma[\boldsymbol{x}]$ means that $f(\boldsymbol{x}) = \Sigma_i g_i^2(\boldsymbol{x})$ holds for some $g_i(\boldsymbol{x})$s. For a given matrix M, M^T represents its transpose, and $\mathrm{He}\{M\} := M + M^{\mathrm{T}}$. M_i is the i-th row of matrix M. I_k is the $k \times k$ identity matrix. $I_{\boldsymbol{x}}$:=diag$(I_n, 0_{(m+1)\times(m+1)})$. The level set of Lyapunov function $V(\boldsymbol{x})$ is defined by $\Xi(V,\gamma)$:=$\{\boldsymbol{x} \in \mathbf{R}^n | V(\boldsymbol{x}) \leq \gamma\}$.

II. PRELIMINARIES

In this section, we provide a result which plays a fundamental role to derive local stability conditions for saturating systems.

Theorem 1: Given polynomials $\{f_k\}_{k=0}^{r+u}$, if there exist scalar polynomials $\{s_k\}_{k=1}^{r} \in \Sigma[\bar{\boldsymbol{x}}]$ and $\{t_l\}_{l=1}^{u} \in \mathbf{R}[\bar{\boldsymbol{x}}]$ such that

$$f_0(\bar{\boldsymbol{x}}) - \sum_{k=1}^{r} s_k(\bar{\boldsymbol{x}}) f_k(\bar{\boldsymbol{x}}) + \sum_{l=1}^{u} t_l(\bar{\boldsymbol{x}}) f_{r+l}(\bar{\boldsymbol{x}}) \geq 0, \ \forall \bar{\boldsymbol{x}} \in \mathbf{R}^{\bar{n}} \tag{1}$$

holds, then we have

$$\bigcap_{k=1}^{r} \{\bar{\boldsymbol{x}} \in \mathbf{R}^{\bar{n}} \,|\, f_k(\bar{\boldsymbol{x}}) \geq 0\} \subseteq \{\bar{\boldsymbol{x}} \in \mathbf{R}^{\bar{n}} \,|\, f_0(\bar{\boldsymbol{x}}) \geq 0\}$$
$$f_{r+l}(\bar{\boldsymbol{x}}) = 0 \ (l = 1, \cdots, u) \tag{2}$$

□

This theorem has the structure of *S*-Procedure with higher degrees of freedom. To prove this theorem, the following lemma plays a key role.

Lemma 1: [11], [12](*Positivstellensatz; P-Satz*) Given polynomial sets $\{f_1, \cdots, f_r\}, \{g_1, \cdots, g_t\}, \{h_1, \cdots, h_u\} \subset \mathbf{R}[\bar{\boldsymbol{x}}], \bar{\boldsymbol{x}} \in \mathbf{R}^{\bar{n}}$, the following conditions $i)$ and $ii)$ are equivalent:

i) $$\left\{ \bar{\boldsymbol{x}} \in \mathbf{R}^{\bar{n}} \left| \begin{array}{l} f_1(\bar{\boldsymbol{x}}) \geq 0, \cdots, f_r(\bar{\boldsymbol{x}}) \geq 0 \\ g_1(\bar{\boldsymbol{x}}) \neq 0, \cdots, g_t(\bar{\boldsymbol{x}}) \neq 0 \\ h_1(\bar{\boldsymbol{x}}) = 0, \cdots, h_u(\bar{\boldsymbol{x}}) = 0 \end{array} \right. \right\} = \emptyset$$

ii) There exist polynomials $f \in \mathcal{P}(f_1, \cdots, f_r)$, $g \in \mathcal{M}(g_1, \cdots, g_t)$, $h \in \mathcal{I}(h_1, \cdots, h_u)$ such that $f + g^2 + h = 0$.

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Note that for the finite nonnegative integer set $\mathbf{Z}_+$,

$$\mathcal{M}(g_1,\cdots,g_t):=\begin{cases} 1, & t=0 \\ \prod_{k=1}^{t} g_k^{m(k)}(\bar{\boldsymbol{x}}), & t\neq 0, m(k)\in\mathbf{Z}_+ \end{cases}$$

is the Multiplicative Monoid generated by $\{g_1,\cdots,g_t\}\in\mathbf{R}[\bar{\boldsymbol{x}}]$,

$$\mathcal{P}(f_1,\cdots,f_r):=\left\{ s_0+\sum_{i=1}^{l} s_i b_i \;\middle|\; \begin{array}{c} l\in\mathbf{Z}_+, s_i\in\Sigma[\bar{\boldsymbol{x}}] \\ b_i\in\mathcal{M}(f_1,\cdots,f_r) \end{array} \right\}$$

is the Cone generated by $\{f_1,\cdots,f_r\}\in\mathbf{R}[\bar{\boldsymbol{x}}]$ and

$$\mathcal{I}(h_1,\cdots,h_u):=\left\{ \sum_{l=1}^{u} h_l p_l \;\middle|\; p_l\in\mathbf{R}[\bar{\boldsymbol{x}}] \right\}$$

is the Ideal generated by $\{h_1,\cdots,h_u\}\in\mathbf{R}[\bar{\boldsymbol{x}}]$ □

proof of theorem 1: Written in the form of i), (2) becomes

$$\left\{ \bar{\boldsymbol{x}}\in\mathbf{R}^{\bar{n}} \;\middle|\; \begin{array}{c} f_1(\bar{\boldsymbol{x}})\geq 0,\cdots,f_r(\bar{\boldsymbol{x}})\geq 0 \\ -f_0(\bar{\boldsymbol{x}})\geq 0, f_0(\bar{\boldsymbol{x}})\neq 0 \\ f_{r+l}(\bar{\boldsymbol{x}})=0,\cdots,f_{r+u}(\bar{\boldsymbol{x}})=0 \end{array} \right\}=\emptyset \quad (3)$$

For $s_k\in\Sigma[\bar{\boldsymbol{x}}](k=1,\cdots,r)$ and $t_l\in\mathbf{R}[\bar{\boldsymbol{x}}](l=1,\cdots,u)$, define

$$Q(\bar{\boldsymbol{x}}):=f_0(\bar{\boldsymbol{x}})-\sum_{k=1}^{r} s_k(\bar{\boldsymbol{x}})f_k(\bar{\boldsymbol{x}})+\sum_{l=1}^{u} t_l(\bar{\boldsymbol{x}})f_{r+l}(\bar{\boldsymbol{x}})$$

If the condition of Theorem 1 holds, then $Q(\bar{\boldsymbol{x}})\in\Sigma[\bar{\boldsymbol{x}}]$ is satisfied. Note that $s_0:=Q(\bar{\boldsymbol{x}}), s_{0k}:=s_k\in\Sigma[\bar{\boldsymbol{x}}]$and define

$$\begin{aligned} f &:= -f_0(\bar{\boldsymbol{x}})s_0(\bar{\boldsymbol{x}})+\sum_{k=1}^{r}(-f_0(\bar{\boldsymbol{x}}))(f_k(\bar{\boldsymbol{x}}))s_{0k}(\bar{\boldsymbol{x}}) \\ &\in \mathcal{P}(f_1(\bar{\boldsymbol{x}}),\cdots,f_r(\bar{\boldsymbol{x}}),-f_0(\bar{\boldsymbol{x}})) \\ g &:= f_0(\bar{\boldsymbol{x}})\in\mathcal{M}(f_0(\bar{\boldsymbol{x}})) \\ h &:= f_0(\bar{\boldsymbol{x}})\sum_{l=1}^{u} t_l(\bar{\boldsymbol{x}})f_{r+l}(\bar{\boldsymbol{x}}) \\ &\in \mathcal{I}(f_{r+1}(\bar{\boldsymbol{x}}),\cdots,f_{r+u}(\bar{\boldsymbol{x}})) \end{aligned} \quad (4)$$

Then we get $f+g^2+h=0$. This is a necessary and sufficient condition of (3). So, if (1) holds, then (3) is satisfied. This implies that (2) holds. □

In this paper, we will formulate the local stability condition of saturation systems in the form of (2) and will transform it in the structure of (1), to deal with it as polynomial programming. In this regard, we replace nonnegative condition by SOS (sum of squares) condition and use SOS decomposition to them. In computation, we use SOSTOOLS, the Matlab toolbox.

III. System Description and Problem Setting

We consider the anti-windup control system depicted in Figure 1, where P and K are the plant and the compensator, respectively. The signals $\boldsymbol{z},\boldsymbol{u}$ and $\boldsymbol{y}$ denote the output of the compensator, the input to the plant and the measurement output. The compensator may include anti-windup compensation based on the signal $\boldsymbol{w}$. Let P and K be described by

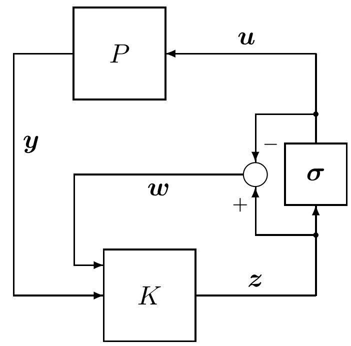

Fig. 1. System with saturation

$$P:\begin{cases} \dot{\boldsymbol{x}}_p(t)=A_p\boldsymbol{x}_p(t)+B_p\boldsymbol{u}(t) \\ \boldsymbol{y}(t)=C_p\boldsymbol{x}_p(t) \end{cases} \quad (5)$$

$$K:\begin{cases} \dot{\boldsymbol{x}}_k(t)=A_k\boldsymbol{x}_k(t)+B_{k1}\boldsymbol{y}(t)+B_{k2}\boldsymbol{w}(t) \\ \boldsymbol{z}(t)=C_k\boldsymbol{x}_k(t)+D_{k1}\boldsymbol{y}(t)+D_{k2}\boldsymbol{w}(t) \end{cases} \quad (6)$$

Note that $\boldsymbol{x}_p(t)\in\mathbf{R}^{\ell}$ and $\boldsymbol{x}_k(t)\in\mathbf{R}^{n-\ell}$ are the states of the plant and the compensator, respectively, and $\boldsymbol{y}(t)\in\mathbf{R}^p$, $\boldsymbol{w}(t)\in\mathbf{R}^m$, $\boldsymbol{z}(t)\in\mathbf{R}^m$.

Moreover, $\boldsymbol{\sigma}$ represents the componentwise saturation nonlinearity consisting of saturation elements as follows.

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{z}) &:= \begin{bmatrix} \sigma_1(z_1) & \ldots & \sigma_m(z_m) \end{bmatrix}^{\mathrm{T}}\in\mathbf{R}^m \\ \sigma_j(z_j) &:= \begin{cases} -1 & , z_j<-1 \\ z_j & , |z_j|\leq 1 \quad (j=1,\cdots,m) \\ 1 & , z_j>1 \end{cases} \end{aligned}$$

Next, we transform the system into the one which consists of the linear system G and the dead-zone nonlinearity ϕ as shown in Fig. 2. Here, G and ϕ are described by

$$\begin{aligned} & G:\begin{cases} \dot{\boldsymbol{x}}(t)=A\boldsymbol{x}(t)+B\boldsymbol{w}(t) \\ \boldsymbol{z}(t)=C\boldsymbol{x}(t)+D\boldsymbol{w}(t) \end{cases} \\ & \phi:\boldsymbol{w}(t)=\boldsymbol{\phi}(\boldsymbol{z}(t))=\boldsymbol{z}-\boldsymbol{\sigma}(\boldsymbol{z}) \end{aligned} \quad (7)$$

and

$$\begin{aligned} \boldsymbol{x} &:= \begin{bmatrix} \boldsymbol{x}_p \\ \boldsymbol{x}_k \end{bmatrix} \quad A:=\begin{bmatrix} A_p+B_pD_{k1}C_p & B_pC_k \\ B_{k1}C_p & A_k \end{bmatrix} \\ B &:= \begin{bmatrix} B_p(D_{k2}-I_m) \\ B_{k2} \end{bmatrix} \\ C &:= \begin{bmatrix} D_{k1}C_p & C_k \end{bmatrix}, D=D_{k2} \end{aligned}$$

The purpose of this paper is to solve the following problem by taking account of saturation nonlinearity in a less conservative way.

Problem 1: Given the system depicted in Figure 2, estimate the domain of initial state vectors $\boldsymbol{x}(0)$ for which the state trajectory satisfies $\boldsymbol{x}(t)\to 0(t\to\infty)$. □

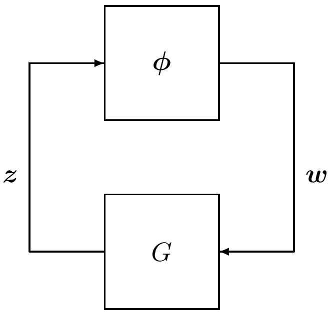

Fig. 2. stable linear system G with dead zone $\boldsymbol{\phi}$

IV. Local Stability Condition

In this section, we derive a local stability condition based on Theorem 1. To this end, let X_i $(i = 1, 2, \cdots 3^m - 1)$ be the division of the space $[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{w}^{\mathrm{T}}]^{\mathrm{T}}$, which corresponds to each saturation case (i.e., $z_i < -1$, $z_i > 1$ or $|z_i| \le 1$ for $i \in [1,\ m]$ except the case $|z_i| \le 1$ for all i). First, we characterize the dead-zone nonlinearity as equality constraints for each X_i, and describe X_i as inequality constraints. Then we derive a stability condition in the form of (1).

A. Characteristics of Dead-Zone Nonlinearity

In the partition X_i, according to the value of $z_j = C_j\boldsymbol{x} + D_j\boldsymbol{w}(j = 1, \cdots, m)$, each element of $\boldsymbol{w}$ is expressed as follows;

$$w_j = \begin{cases} C_j\boldsymbol{x} + D_j\boldsymbol{w} + 1 & , z_j < -1 \\ 0 & , |z_j| \le 1 \\ C_j\boldsymbol{x} + D_j\boldsymbol{w} - 1 & , z_j > 1 \end{cases},$$

where C_j and D_j are j-th rows of C and D, respectively. Here, corresponding to each X_i, we define the following vector

$$\boldsymbol{\eta}_i(\boldsymbol{z}) := \begin{bmatrix} \eta_{i,1}(z_1) \\ \vdots \\ \eta_{i,m}(z_m) \end{bmatrix},\ \eta_{i,j}(z_j) := \begin{cases} -1 & , z_j < -1 \\ 0 & , |z_j| \le 1 \\ 1 & , z_j > 1 \end{cases}$$

to obtain

$$\begin{cases} C_j\boldsymbol{x} + D_j\boldsymbol{w} - w_j - \eta_{i,j}(z_j) = 0 & , |z_j| > 1 \\ w_j = 0 & , |z_j| \le 1 \end{cases}. \tag{8}$$

This is an equality constraint which corresponds to $f_{r+l}(\bar{\boldsymbol{x}}) = 0$ in Theorem 1. The above implies the following equality, which corresponding to the term $\sum_{l=1}^{u} t_l(\bar{\boldsymbol{x}}) f_{r+l}(\bar{\boldsymbol{x}})$ in Theorem 1.

$$\begin{aligned} &\bar{\boldsymbol{x}}^{\mathrm{T}}(\mathrm{He}\{Q_i(\boldsymbol{x},\boldsymbol{w})[\,C \quad D - I_m \quad -\boldsymbol{\eta_i(z)}\,]\} \\ &+\mathrm{He}\{\bar{Q}_i(\boldsymbol{x},\boldsymbol{w})[\,0 \quad I_m \quad 0\,]\})\bar{\boldsymbol{x}} = 0 \end{aligned} \tag{9}$$

Here, $\bar{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x}^{\mathrm{T}} & \boldsymbol{w}^{\mathrm{T}} & 1 \end{bmatrix}^{\mathrm{T}}$, and $Q_i([\boldsymbol{x}^{\mathrm{T}}, \boldsymbol{w}^{\mathrm{T}}]^{\mathrm{T}})$ are $(n + m + 1) \times m$ matrices such that each element of j-th columns belong to $\mathbf{R}[[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{w}^{\mathrm{T}}]^{\mathrm{T}}]$ when $|z_j| > 1$. They also satisfy that j-th columns are zero vectors when $|z_j| \le 1$. Similarly, $\bar{Q}_i(\boldsymbol{x},\boldsymbol{w})$ are $(n + m + 1) \times m$ matrices such that j-th columns are zero vectors when $|z_j| > 1$, and each element of j-th columns belong to $\mathbf{R}[[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{w}^{\mathrm{T}}]^{\mathrm{T}}]$ if $|z_j| \le 1$ holds. Note that the first term of (9) corresponds to equality constraints under saturation, while the second one considers equality constraints under non-saturation.

Next, we describe the partition X_i in terms of inequality constraints. For each region X_i, there exists an appropriate size matrices $\hat{E}_i(i = 1, \cdots, 3^m - 1)$ and a vector $\boldsymbol{a}_i$ such that $[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{w}^{\mathrm{T}}]^{\mathrm{T}} \in X_i$ is specified by inequality:

$$\hat{E}_i \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{w} \end{bmatrix} \ge \boldsymbol{a}_i \quad (i = 1, \cdots, 3^m - 1), \tag{10}$$

where inequality holds for each element of the vector. The above inequality corresponds to $f_k(\bar{\boldsymbol{x}}) \ge 0$ in Theorem 1. This yields the following inequality which corresponds to the term $\sum_{k=1}^{r} s_k(\bar{\boldsymbol{x}}) f_k(\bar{\boldsymbol{x}})$ in Theorem 1:

$$\bar{\boldsymbol{x}}^{\mathrm{T}} E_i^{\mathrm{T}} S_i(\boldsymbol{x},\boldsymbol{w}) E_i \bar{\boldsymbol{x}} \ge 0, \quad E_i := \begin{bmatrix} \hat{E}_i & -\boldsymbol{a}_i \\ 0 & 1 \end{bmatrix} \tag{11}$$

where $S_i(\boldsymbol{x},\boldsymbol{w})$ are appropriate size symmetric matrices with each entry belonging to $\Sigma[[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{w}^{\mathrm{T}}]^{\mathrm{T}}]$.

B. Quadratic Polynomial Lyapunov Function

Let $\boldsymbol{\eta}_i \ne 0$ be the vector such that each element is the same value as that of $\eta_{ij}(z_j)(j = 1, \cdots, m)$ in the region X_i.

In this section, we consider the Lyapunov function of the form $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} P \boldsymbol{x} + 2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{x}$, and derive a condition for the level set $\Xi(V, 1)$ to be a domain of attraction. Let $Q_i(\boldsymbol{x},\boldsymbol{w})$ and $\bar{Q}_i(\boldsymbol{x},\boldsymbol{w})$ be matrices defined for each region X_i in the previous subsection, then we have the following result.

Theorem 2: Suppose that there exist $\epsilon > 0$, scalar polynomials $a_i \in \Sigma[[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{w}^{\mathrm{T}}]](i = 1, \cdots, 3^m - 1)$, a constant matrix $P = P^{\mathrm{T}} \in \mathbf{R}^{n \times n}$, matrices $Q_i(\boldsymbol{x},\boldsymbol{w})$, $\bar{Q}_i(\boldsymbol{x},\boldsymbol{w})$ and $S_i(\boldsymbol{x},\boldsymbol{w})$ whose all entries belong to $\Sigma[[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{w}^{\mathrm{T}}]^{\mathrm{T}}]$ such that the following relations hold:

$$V(\boldsymbol{x}) - \epsilon \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \in \Sigma[\boldsymbol{x}] \tag{12}$$

$$-\begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} PA + A^{\mathrm{T}}P & A^{\mathrm{T}}\boldsymbol{q} \\ \boldsymbol{q}^{\mathrm{T}}A & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} - \epsilon \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \in \Sigma[\boldsymbol{x}] \tag{13}$$

$$\begin{aligned} &\bar{\boldsymbol{x}}^{\mathrm{T}} \left(-\begin{bmatrix} PA + A^{\mathrm{T}}P & PB & A^{\mathrm{T}}\boldsymbol{q} \\ B^{\mathrm{T}}P & 0 & B^{\mathrm{T}}\boldsymbol{q} \\ \boldsymbol{q}^{\mathrm{T}}A & \boldsymbol{q}^{\mathrm{T}}B & 0 \end{bmatrix} - \epsilon I_{\boldsymbol{x}} \right. \\ &-a_i(\boldsymbol{x},\boldsymbol{w}) \begin{bmatrix} -P & 0 & -\boldsymbol{q} \\ 0 & 0 & 0 \\ -\boldsymbol{q}^{\mathrm{T}} & 0 & 1 \end{bmatrix} - E_i^{\mathrm{T}} S_i(\boldsymbol{x},\boldsymbol{w}) E_i \\ &+\mathrm{He}\{Q_i(\boldsymbol{x},\boldsymbol{w})[\,C \quad D - I_m \quad -\boldsymbol{\eta}_i(\boldsymbol{z})\,] \\ &\left. +\bar{Q}_i(\boldsymbol{x},\boldsymbol{w})[\,0 \quad I_m \quad 0\,]\} \right) \bar{\boldsymbol{x}} \in \Sigma[[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{w}^{\mathrm{T}}]^{\mathrm{T}}] \end{aligned} \tag{14}$$

Then every $\boldsymbol{x}(t)$ satisfying $\boldsymbol{x}(0) \in \Xi(V, 1)$ converges to the origin as $t \to \infty$.

Proof: First, (12) implies that $V(\boldsymbol{x})$ is positive definite.

Second, in the case where input saturation does not occur (i.e., $\boldsymbol{w}=0$), we have

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= \dot{\boldsymbol{x}}^{\mathrm{T}}P\boldsymbol{x}+\boldsymbol{x}^{\mathrm{T}}P\dot{\boldsymbol{x}}+\boldsymbol{q}^{\mathrm{T}}\dot{\boldsymbol{x}}+\dot{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{q}\\ &= \boldsymbol{x}^{\mathrm{T}}AP\boldsymbol{x}+\boldsymbol{x}^{\mathrm{T}}PA\boldsymbol{x}+\boldsymbol{q}^{\mathrm{T}}A\boldsymbol{x}+\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{q}\\ &= \begin{bmatrix}\boldsymbol{x}\\1\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}PA+A^{\mathrm{T}}P & A^{\mathrm{T}}\boldsymbol{q}\\ \boldsymbol{q}^{\mathrm{T}}A & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\1\end{bmatrix}\end{aligned} \quad (15)$$

Therefore, (13) implies that $\dot{V}(\boldsymbol{x})<0$.

Finally, we consider the case where saturation occurs (i.e., at least one element of $\boldsymbol{z}$ is saturated). For region X_i specified by (10), let $\mathcal{J}_i$ be the set of indexes j's such that $|z_j|>1$, and let $\bar{\mathcal{J}}_i$ be those for $|z_j|\le 1$, and $\eta_{ij}=\eta_{ij}(z_j)$. Then a local stability condition based on Lyapunov stability in each $X_i(i=1,\cdots,3^m-1)$ is described as follows:

$$\begin{aligned}&\Xi(V,1)\cap\left\{\boldsymbol{x}\in X_i \;\middle|\; \hat{E}_i\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{w}\end{bmatrix}\ge \boldsymbol{a}_i,\right.\\ &\qquad\left. \begin{array}{l} C_j\boldsymbol{x}+D_j\boldsymbol{w}-w_j-\eta_{i,j}=0\ (j\in\mathcal{J}_i)\\ w_j=0\ (j\in\bar{\mathcal{J}}_i)\end{array}\right\}\\ &\subseteq\{\boldsymbol{x}\in\mathbf{R}^n|\dot{V}(\boldsymbol{x})\le-\epsilon\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}\}\ (i=1,\cdots,3^m-1)\end{aligned} \quad (16)$$

So it is enough to show that the above condition is satisfied if (14) holds. Note that, as for $\dot{V}(\boldsymbol{x})<0$, the following relation holds.

$$\begin{aligned}&-\dot{V}(\boldsymbol{x})-\epsilon\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}\\ &=-\boldsymbol{x}P\dot{\boldsymbol{x}}^{\mathrm{T}}-\dot{\boldsymbol{x}}^{\mathrm{T}}P\boldsymbol{x}-\boldsymbol{q}^{\mathrm{T}}\dot{\boldsymbol{x}}-\dot{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{q}-\epsilon\bar{\boldsymbol{x}}^{\mathrm{T}}I_{\boldsymbol{x}}\bar{\boldsymbol{x}}\\ &=-\bar{\boldsymbol{x}}^{\mathrm{T}}A_0\bar{\boldsymbol{x}}\end{aligned}$$

where

$$A_0 := -\begin{bmatrix}PA+A^{\mathrm{T}}P & PB & A^{\mathrm{T}}\boldsymbol{q}\\ B^{\mathrm{T}}P & 0 & B^{\mathrm{T}}\boldsymbol{q}\\ \boldsymbol{q}^{\mathrm{T}}A & \boldsymbol{q}^{\mathrm{T}}B & 0\end{bmatrix}-\epsilon I_{\boldsymbol{x}}.$$

So if $\bar{\boldsymbol{x}}^{\mathrm{T}}A_0\bar{\boldsymbol{x}}>0$ holds, we have $\dot{V}(\boldsymbol{x})<0$. While, $\boldsymbol{x}\in\Xi(V,1)$ holds, if we have

$$1-V(\boldsymbol{x})=\bar{\boldsymbol{x}}^{\mathrm{T}}A_1\bar{\boldsymbol{x}}\ge 0$$

where

$$A_1 := \begin{bmatrix}-P & 0 & -\boldsymbol{q}\\ 0 & 0 & 0\\ -\boldsymbol{q}^{\mathrm{T}} & 0 & 1\end{bmatrix}.$$

Also, (10) holds when $\boldsymbol{x}^{\mathrm{T}}E_i^{\mathrm{T}}S_i(\boldsymbol{x},\boldsymbol{w})E_i\boldsymbol{x}\ge 0$ is satisfied. In addition, (9) expresses the dead-zone nonlinearity. Therefore, if (14) holds then we obtain (16) by Theorem 1. This proves the theorem. ■

C. Algorithm

In the local stability analysis, it is desired to find $V(\boldsymbol{x})$ whose level set $\Xi(V,1)$ is larger than the exiting ones. To this end, we try to minimize the $\mathrm{trace}(P)$. The optimization problem to solve is as follows:

Problem 2: Solve $\min_{P,\boldsymbol{q},a_i,Q_i,\bar{Q}_i,S_i}\mathrm{trace}(P)$ subject to (12), (13) and (14) where $i=1,\cdots,3^m-1$. □

This problem is not convex. Therefore we will fix some variables and seek others by applying SOS decomposition. To reduce each SOS constraint to positive semidefinite programming problem, we may use SOSTOOLS[9], which solves semidefinite programming problem by SeDuMi[15]. The algorithm is as follows:

Algorithm 1:

step 0: Find an initial Lyapunov function $V_o(\boldsymbol{x})=\boldsymbol{x}^{\mathrm{T}}P_0\boldsymbol{x}+2\boldsymbol{q}_0^{\mathrm{T}}\boldsymbol{x}$ by an existing method (e.g., circle criterion and so on). Set $P^{(0)}=P_0,\boldsymbol{q}^{(0)}=\boldsymbol{q}_0$. Set the index $\ell=1$.

step 1 : For $P^{(\ell-1)}$ and $\boldsymbol{q}^{(\ell-1)1}y$, find $a_i(\boldsymbol{x},\boldsymbol{w})\in\Sigma[[\boldsymbol{x}^{\mathrm{T}}\ \ \boldsymbol{w}^{\mathrm{T}}]]$ and parameter dependent multipliers $Q_i(\boldsymbol{x},\boldsymbol{w}),\bar{Q}_i(\boldsymbol{x},\boldsymbol{w}),S_i(\boldsymbol{x},\boldsymbol{w})$ satisfying (14).

step 2 : For $a_i(\boldsymbol{x},\boldsymbol{w})$ obtained in **step 1**, find $P^{(l)},\boldsymbol{q}^{(l)}$, $Q_i(\boldsymbol{x},\boldsymbol{w})$, $\bar{Q}_i(\boldsymbol{x},\boldsymbol{w})$, $S_i(\boldsymbol{x},\boldsymbol{w})$ minimizing $\mathrm{trace}(P^{(l)})$ subject to (12), (13) and (14).

step 3 : If $\mathrm{trace}(P^{(l-1)})-\mathrm{trace}(P^{(l)})(>0)$ is less than a specified tolerance $\epsilon_d>0$, then stop the algorithm. Otherwise set $\ell=\ell+1$ and return to **step 1**.

V. Piecewise Quadratic Lyapunov function

In this section, we adopt piecewise quadratic Lyapunov functions [7] instead of usual quadratic ones. It is expected that the approach may give less conservative estimation of domain of attraction[6], [7]. For simplicity, we will focus on the single input ($m=1$) case.

A. Piecewise Quadratic Lyapunov Function

In case of single input systems, the state-space is divided into three regions;

$$\begin{aligned}X_N &:= \{\boldsymbol{x}\,|\,z\le -1\}\\ X_L &:= \{\boldsymbol{x}\,|\,|z|\le 1\}\\ X_P &:= \{\boldsymbol{x}\,|\,z\ge 1\}\end{aligned}$$

corresponding to negative saturation, linear operation and positive saturation, respectively. We can define a piecewise quadratic Lyapunov function in each region such as V_N,V_L,V_P, respectively. Note that they are continuous on the region boundaries. Although the boundary between X_N and X_L, and the one between X_P and X_L are $z=-1,z=1(z=C\boldsymbol{x}+D\boldsymbol{w})$, respectively. For simplicity we set $C\boldsymbol{x}=-1,C\boldsymbol{x}=1$. Then V_N-V_L and V_P-V_L includes $C\boldsymbol{x}+1$ and $C\boldsymbol{x}-1$ as a factor, respectively. So there exist $Y_i\boldsymbol{x}+r_i$ depending on $\boldsymbol{x}$ satisfying

$$V_i(\boldsymbol{x})=V_L(\boldsymbol{x})+(C\boldsymbol{x}-\eta_i)(Y_i\boldsymbol{x}+r_i),$$

where $Y_i\in\mathbf{R}^n,r_i\in\mathbf{R},i=N,P,\eta_N=-1$ and $\eta_P=1$. Therefore we obtain

$$\begin{aligned}V_i(\boldsymbol{x}) &= \begin{bmatrix}\bar{\boldsymbol{x}}\\1\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}P_i & \boldsymbol{q}_i\\ \boldsymbol{q}_i^{\mathrm{T}} & r_i\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{x}}\\1\end{bmatrix}\\ P_i &:= P+C^{\mathrm{T}}Y_i,\ \ \boldsymbol{q}_i:=\boldsymbol{q}+\frac{1}{2}(\eta_iY_i+r_iC^{\mathrm{T}}).\end{aligned}$$

Note that V_N,V_L,V_P may be positive definite in each region.

B. Local Stability Condition

Before obtaining a local stability condition, we set the domain of attraction as $\Xi(V,\gamma)$ for some $\gamma > 0$ (i.e., γ is not necessary 1 in this section). If we get the largest $\Xi(V_L, 1)$ satisfying following theorem and maximize γ, we may have one of the solutions of Problem 1.

Theorem 3: For $i = N, L, P, \eta_N = -1, \eta_P = 1$, suppose that there exist $a_i \in \Sigma[[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{w}^{\mathrm{T}}]^{\mathrm{T}}]$, $\gamma > 0$, a constant matrix $P = P^{\mathrm{T}} \in \mathbf{R}^{n\times n}$, matrices $Q_i(\boldsymbol{x},\boldsymbol{w}) \in \mathbf{R}^{(n+m+1)\times m}$, symmetric matrices $S_i(\boldsymbol{x},\boldsymbol{w}) \in \mathbf{R}^{(n+m+1)\times(n+m+1)}$, $\bar{S}_L \in \mathbf{R}^{(2m+1)\times(2m+1)}$ and $\bar{S}_i \in \mathbf{R}^{(m+1)\times(m+1)}$ whose each element belongs to $\Sigma[[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{w}^{\mathrm{T}}]^{\mathrm{T}}]$ such that the following relations hold.

$$\bar{\boldsymbol{x}}^{\mathrm{T}}\left(\begin{bmatrix} P_i & 0 & \boldsymbol{q}_i \\ 0 & 0 & 0 \\ \boldsymbol{q}_i T & 0 & r_i \end{bmatrix} - \epsilon I_{\boldsymbol{x}} - E_i^{\mathrm{T}}\bar{S}_i E_i\right)\bar{\boldsymbol{x}} \in \Sigma[\boldsymbol{x}] \quad (17)$$

$$-\begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} P_L A + A^{\mathrm{T}}P_L & A\boldsymbol{q}_L \\ \boldsymbol{q}_L^{\mathrm{T}}A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} - \epsilon\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \in \Sigma[\boldsymbol{x}] \ (18)$$

$$\bar{\boldsymbol{x}}^{\mathrm{T}}\Bigg(-\begin{bmatrix} P_iA + A^{\mathrm{T}}P_i & P_iB & A^{\mathrm{T}}\boldsymbol{q}_i \\ B^{\mathrm{T}}P_i & 0 & B^{\mathrm{T}}\boldsymbol{q}_i \\ \boldsymbol{q}_i^{\mathrm{T}}A & \boldsymbol{q}_i^{\mathrm{T}}B & 0 \end{bmatrix} - \epsilon I_{\boldsymbol{x}}$$
$$-a_i(\boldsymbol{w})\begin{bmatrix} -P_i & 0 & -\boldsymbol{q}_i \\ 0 & 0 & 0 \\ -\boldsymbol{q}_i^{\mathrm{T}} & 0 & -\eta_i r_i - \gamma \end{bmatrix} - E_i^{\mathrm{T}}S_i(\boldsymbol{x},\boldsymbol{w})E_i$$
$$+\mathrm{He}\left\{Q_i(\boldsymbol{x},\boldsymbol{w})[\, C \ \ D-1 \ \ -\eta_i \,]\right\}\Bigg)\bar{\boldsymbol{x}} \in \Sigma[[\boldsymbol{x}^{\mathrm{T}}\ \boldsymbol{w}^{\mathrm{T}}]^{\mathrm{T}}] (19)$$

where for $i = N, P$

$$r_L = 0,\ E_L := \begin{bmatrix} C & D & 1 \\ -C & -D & 1 \\ 0 & 0 & 1 \end{bmatrix},\ E_i := \begin{bmatrix} \eta_i C & \eta_i D & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then every $\boldsymbol{x}(t)$ satisfying $\boldsymbol{x}(0) \in \Xi(V, 1)$ converges to the origin as $t \to \infty$. □

An algorithm to compute the piecewise quadratic Lyapunov functions is given as follows:

Algorithm 2:

0) Initialization

Set $\gamma = 1$, $\gamma^{(0)} = 1$, $\ell = 1$ and $k = 1$. Find an initial Lyapunov function $V_0(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}P_0\boldsymbol{x} + \boldsymbol{q}_0\boldsymbol{x}$ through the circle criterion (or any other existing methods). Set $P_i^{(0)} = P_0,\ \boldsymbol{q}_i^{(0)} = \boldsymbol{q}_0$ for $i = L,\ N,\ P$.

1) Solving Lyapunov function

step 1 : For $P_i^{(\ell-1)}, \boldsymbol{q}_i^{(\ell-1)}$, find $a_i(\boldsymbol{w}) \in \Sigma[\boldsymbol{w}]$ and parameter dependent multipliers $Q_i(\boldsymbol{x},\boldsymbol{w}), S_i(\boldsymbol{x},\boldsymbol{w})$ subject to (19).

step 2 : For $a_i(\boldsymbol{w})$ solved in **step 1**, find $P_i^{(\ell)}, \boldsymbol{q}_i^{(\ell)}$, $Q_i(\boldsymbol{x},\boldsymbol{w})$, $\boldsymbol{w}$), $S_i(\boldsymbol{x},\boldsymbol{w})$ that minimize $\mathrm{trace}(P_L^{(\ell)})$ subject to (17),(18),(19). **step 3 :** If $\mathrm{trace}(P_L^{(\ell-1)}) - \mathrm{trace}(P_L^{(\ell)}) (> 0)$ is less than the specified tolerance $\epsilon_d > 0$, stop the iteration. Otherwise set $\ell = \ell + 1$ and return to **step 1**.

2) Finding the Largest Level Set

step 4 : For Lyapunov function solved in **step 3**, find mutilpliers that maximize $\gamma > 0$ subject to (18) and (19). Set $\gamma^{(k)}$ be the obtained maximum γ.

step 5 : Set $\gamma = \gamma^{(k)}$ and find $a_i(\boldsymbol{w}) \in \Sigma[\boldsymbol{w}]$ satisfying (19).

step 6 : If $\gamma^{(k)} - \gamma^{(k-1)} (> 0)$ is less than the specified tolerance, stop the algorithm. Otherwise, go to **step 4**. □

VI. Examples

In this section we will compare our approach to some of existing ones by utilizing two numerical examples from the literature [6], [8]. When we adopt the method based on Theorem 2, we set $\deg(a_i(\boldsymbol{x},\boldsymbol{w})) = 2$ and $N_d = 2$, where N_d is the maximal degree of each element of matrices $S_i(\boldsymbol{x},\boldsymbol{w}), Q_i(\boldsymbol{x},\boldsymbol{w}), \bar{Q}_i(\boldsymbol{x},\boldsymbol{w})$. While, when we use the method based on Theorem 3, we set $N_d = 2$, $\deg(a_i) = 2$.
Example 1 [6]

Consider the unity feedback system that consists of the linear system P and the saturation element, where the state-space matrices of P are given by:

$$A_p = \begin{bmatrix} -0.333 & -0.86 \\ 1.42 & 0.53 \end{bmatrix}, B_p = \begin{bmatrix} 0.41 \\ -2.27 \end{bmatrix}, C_p^{\mathrm{T}} = \begin{bmatrix} 0.3564 \\ 0.284 \end{bmatrix}$$

Transforming the closed loop system into that in Fig. 2, we have

$$A := A_p + B_pC_p,\ B := -B_p,\ C := C_p,\ D = 0.$$

as the state space matrices of G. The domains of attraction estimated by various methods are shown in Fig. 3.

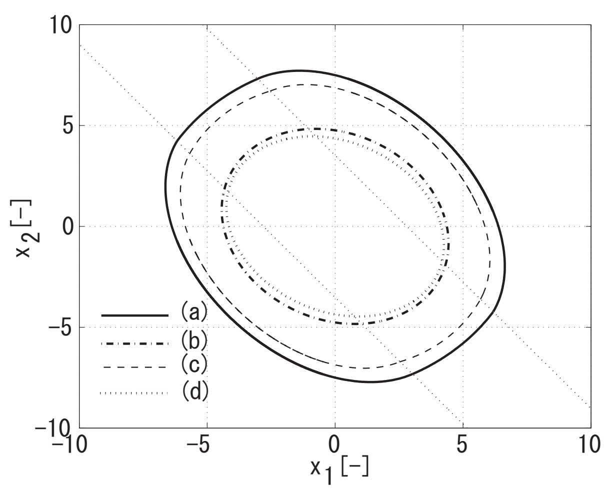

Fig. 3. Domain of Attraction

The lines (a) and (b) show the boundary of domain of attractions through Theorems 3 and 2, respectively. While the line (c) shows the one estimated by Johansson[6], and the line (d) corresponds to the circle criterion [8]. This figure shows that the proposed method gives less conservative domains of attraction, and that the piecewise quadratic Lyapunov functions work better in this case. As for the quadratic Lyapunov functions case, the optimized value of $\mathrm{trace}(P)$ for each method is as follows: (b)0.0965, (d)0.1110.

Example 2[8]

Consider the MIMO system with

$$A = \begin{bmatrix} -0.5 & 5 \\ 0 & -2 \end{bmatrix},\ B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix}$$

For the closed loop system consisting of the linear system and the dead-zone element, the domains of attraction estimated by various method are shown in Fig. 4. In this case, we adopt the quadratic Lyapunov functions only. Note that

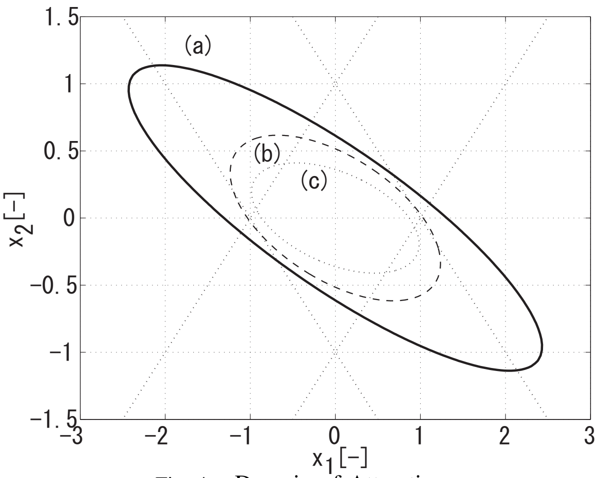

Fig. 4. Domain of Attraction

(a) shows the domain estimated by Theorem 2. While (b) is the domain obtained by [8], and (c) is the one estimated by circle criterion [8]. The value of $\mathrm{trace}(P)$ for each method is as follows: (a)3.23826, (b)4.70, (c)8.969. This figure also demonstrates that the less conservative domain is obtained by the proposed method.

VII. Conclusion

In this paper, we have obtained a local stability condition of saturating systems with direct transmission term, where a generalized *S*-Procedure with parameter dependent multipliers plays a key role to obtain less conservative results. We have estimated the domain of attraction by computing the level sets of two types of Lyapunov functions based on polynomial programming problem, where sum of squares decomposition is utilized for computation. Numerical examples shows the effectiveness of the proposed method.

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