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Robust Identification of Periodic Systems with Applications to Texture Inpainting.

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Abstract— **In this paper we address the problem of robust identification of discrete LTI systems that have a periodic impulse response. The main result of the paper shows that this problem can be solved by factoring a suitable defined Hankel operator. These results are illustrated with a practical example arising in the context of image processing: completing a textured image where some pixels are missing.**

I. INTRODUCTION

This paper addresses the problem of robust identification of discrete linear time invariant systems that have a periodic impulse response. This situation arises in the context of many practical problems from widely dissimilar areas, where the use of control–theoretic tools can result in a substantial simplification. For instance, as we will show in the sequel, solving this problem allows for developing efficient algorithms for image restoration. Additional interesting applications include obtaining models of repetitive actions (such as human gait) that can be later used for activity recognition, or low–order approximations, over a finite time horizon, of an input–output system with high–dimensional, linear Hamiltonian dynamics.

The problem of identifying a stable, causal, finite dimensional linear shift invariant system (FDLSI) from samples of its output has been extensively studied, leading to several techniques, roughly divided into subspace identification [8], [13] and operator–theoretic (or worst–case) methods [2], [10]. However, neither of these approaches can directly handle structural constraints of the form $A^N = I$, where A denotes the state matrix, a key requirement in the applications of interest here.

The approach pursued in this paper is related to subspace identification methods, but allows for directly incorporating the structural constraints and handling measurement noise characterized by a set (rather than stochastic) description. Motivated by earlier results in realization theory [12], [14], [6] and N4SID type algorithms [13], our starting point is to form a *circulant* Hankel matrix using the output measurements. The structural properties of this matrix can then be exploited to obtain a state–space realization of the unknown plant that interpolates the experimental data and satisfies the periodicity constraints.

In the second portion of the paper we apply these tools to the non–trivial problem of texture inpainting: seamlessly completing missing portions of a textured image. The main idea is to model images as the (periodic) impulse response of a non-necessarily causal LTI system and use the proposed method to recast the problem into a rank–minimization form. While these problems are known to be generically NP–hard, we show that for this particular problem efficient convex relaxations are available.

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II. NOTATION

Below we summarize the notation used in this paper:

$\mathbf{x}$	column vector.
$\mathbf{x}^H$	Hermitian conjugate of $\mathbf{x}$.
$\Vert\mathbf{x}\Vert_p$	p-norm of a vector: $\Vert\mathbf{x}\Vert_p \doteq (\sum_{k=1}^m \vert x_k\vert^p)^{\frac{1}{p}}$, $p \in [1, \infty)$
$\ell_{2,n}^m$	Banach subspace of length n vector sequences equipped with the norm: $\Vert x\Vert_2 \doteq \left(\sum_{i=0}^{n-1} \Vert\mathbf{x}_i\Vert_2^2\right)^{\frac{1}{2}}$
$\mathcal{L}(\ell_{2,n})$	space of bounded operators in $\ell_{2,n}$.
$\mathbf{I}_p$	$p \times p$ Identity Matrix
$\sigma_i(\mathbf{A})$	singular values of $\mathbf{A}$.
$\mathcal{H}^n$	set of all block circulant Hankel matrix of the form:

$$\mathbf{H}^n = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \dots & \mathbf{h}_n \\ \mathbf{h}_2 & \mathbf{h}_3 & \dots & \mathbf{h}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{h}_n & \mathbf{h}_1 & \dots & \mathbf{h}_{n-1} \end{bmatrix}$$

where $\mathbf{h} \in R^{p \times m}$.

This paper considers finite-dimensional, discrete-time, linear shift invariant (FDLSI) systems. From an input–output viewpoint, such a system G can be represented by its convolution kernel $\{\mathbf{g}_i\}$. Causal LSI systems (i.e $\mathbf{g}_i = 0,\ i < 0$) will also be represented by a minimal state–space realization:

$$\begin{aligned} \mathbf{x}_{k+1} =& \mathbf{A}\mathbf{x}_k + \mathbf{B}u_k \\ y_k =& \mathbf{C}\mathbf{x}_k + \mathbf{D}u_k. \end{aligned} \tag{1}$$

In the sequel, we will associate to any finite sequence $x = \{\mathbf{x}_k\}$, the following circulant Hankel matrix:

$$\mathbf{H}_x^n = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \\ \mathbf{x}_2 & \mathbf{x}_3 & \dots & \mathbf{x}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_n & \mathbf{x}_1 & \dots & \mathbf{x}_{n-1} \end{bmatrix}.$$

Finally, given a system G, we will denote by $\mathbf{H}_{\mathbf{g}}^n$ the circulant Hankel matrix associated with its impulse response.

III. Robust Identification of Periodic Systems

A. Problem Statement

Consider the problem of identifying a periodic plant G from measurements of its output $\mathbf{y}_k$, $k = 0, 1, \ldots, n-1$, corrupted by additive bounded noise $\mathbf{v}$ in a given set $\mathcal{N}$, to a known input $\mathbf{u} \in \ell^2$ applied in some past interval $[-T, -1]$.

$$\mathbf{y}_k = \sum_{i=-T}^{-1} \mathbf{g}_{k-i}\mathbf{u}_i + \mathbf{v}_k, \quad k = 0, 1, \ldots, n-1, \quad \mathbf{v} \in \mathcal{N}. \tag{2}$$

Further, the plant is known to be of McMillan degree r or less and satisfy the structural constraint $g_k = g_{k+N}$, where N is given (the case of unknown N will be addressed later in the paper). Assume now that the input signal is applied for an integer number of periods, e.g. $T = \alpha N$[1]. From the periodicity assumption it follows that:

$$\begin{aligned}
\mathbf{y}_k &= \sum_{i=-T}^{-1} \mathbf{g}_{k-i}\mathbf{u}_i + \mathbf{v}_k \\
&= \sum_{i=-N}^{-1} \mathbf{g}_{k-i} \left(\sum_{j=0}^{\alpha-1} \mathbf{u}_{i-jN} \right) + \mathbf{v}_k \\
&= \sum_{i=-N}^{-1} \mathbf{g}_{k-i}\tilde{\mathbf{u}}_i + \mathbf{v}_k, \text{ with } \tilde{\mathbf{u}}_i \doteq \sum_{j=0}^{\alpha-1} \mathbf{u}_{i-jN}
\end{aligned}$$

Thus, without loss of generality, it will be assumed in the sequel that $T = N$. In addition, we will also (temporarily) assume that $n = N$, that is, a full period of output data is available. With these assumptions the identification problem of interest here can be precisely stated as follows.

Problem 1: Given:

(i) *a priori* set descriptions of the measurement noise $\mathcal{N}$ and candidate models $\mathcal{S}$:

$$\mathcal{S} \doteq \{G \in \mathcal{L}(\ell_{2,N}) \colon \text{degree}(G) \le r, g_{k+N} = g_k\}$$

and

(ii) a finite set of N samples of the input $\{\mathbf{u}_k\}_{k=-N}^{-1}$ and the corresponding output $\{\mathbf{y}_k\}_{k=0}^{N-1}$:

Then:

a.- Determine whether the consistency set $\mathcal{T}(\mathbf{y})$ is nonempty, where

$$\mathcal{T}(\mathbf{y}) \doteq \left\{ g \in \mathcal{S} \colon \{y_k - (g * u)_k\}_{k=0}^{N-1} \in \mathcal{N} \right\}$$

b.- If $\mathcal{T}(\mathbf{y}) \neq \emptyset$, find a model $g_{id} \in \mathcal{T}(\mathbf{y})$ and a bound on the worst–case identification error.

IV. An Identification Algorithm for Periodic Systems

Next we present an algorithm that solves Problem 1. The main idea of this algorithm is to form a circulant Hankel matrix from the output data. As we show in the sequel, a suitable model can then be constructed directly from a singular value decomposition of this matrix.

[1] This can be assumed without loss of generality by padding the input sequence with zeros, if necessary.

A. Additional Assumptions and Problem Transformation

In order to simplify the problem, in the sequel we make the following additional assumptions:

A-1 The input $\mathbf{u}$ is selected so that $\mathbf{H}_u^N$ is unitary.

A-2 The (measurement) noise set is of the form $\mathcal{N} = \{\mathbf{v} \colon \|\mathbf{H}_{\mathbf{v}}^N\|_* \le \epsilon\}$ where $\|.\|*$ denotes a suitable unitarily invariant norm.

Remark 1: These assumptions are a deterministic counterpart of those made in stochastic subspace Id methods.

Next, we show that Problem 1 can be reduced to a rank–constrained matrix approximation problem. To this effect, begin by forming the (circulant) Hankel matrices $\mathbf{H}_{\mathbf{y}}^N$, $\mathbf{H}_{\mathbf{u}}^N$, $\mathbf{H}_{\mathbf{v}}$ and $\mathbf{H}_{\mathbf{g}}$ and consider the following optimization problem:

$$\mu = \min_{\mathbf{H}_{\mathbf{g}}, \mathbf{H}_{\mathbf{v}} \in \mathcal{H}^N} \|\mathbf{H}_{\mathbf{v}}\|_* \tag{3}$$

subject to

$$\mathbf{H}_{\mathbf{y}}^N = \mathbf{H}_{\mathbf{g}}\mathbf{H}_{\mathbf{u}}^N + \mathbf{H}_{\mathbf{v}} \tag{4}$$

$$\text{rank}(\mathbf{H}_{\mathbf{g}}) \le r. \tag{5}$$

Consistency of the *a priori* and *a posteriori* information can now be stated in terms of the solution to (3) as follows:

Lemma 1: The *a priori* and *a posteriori* information are consistent if and only if $\mu \le \epsilon$.

Proof: Begin by noting that satisfaction of (4)–(5) is equivalent to the existence of a system $g \in \mathcal{S}$ and some noise sequence $\mathbf{v}$ that satisfy (2). Equation (3) is just a restatement of the fact that $\mathcal{T}(\mathbf{y}) \neq \emptyset \iff \mathbf{v} \in \mathcal{N}$ ■

The optimization problem above is non–convex, due to the rank constraint (5). Nevertheless, as we show in the sequel, in the case under consideration here, the circulant structure of matrices in $\mathcal{H}^N$ can be exploited to obtain an efficient solution based on SVD decompositions.

Lemma 2: Given $\mathbf{Y} \in \mathcal{H}^N$, consider the following two approximation problems:

$$\mu_{uc} \doteq \min_{\mathbf{H}_r} \|\mathbf{Y} - \mathbf{H}_r\|_* \text{ subject to } \text{rank}(\mathbf{H}_r) \le r \tag{6}$$

$$\mu_c \doteq \min_{\mathbf{H}_r} \|\mathbf{Y} - \mathbf{H}_r\|_* \text{ subject to } \begin{cases} \text{rank}(\mathbf{H}_r) \le r \\ \mathbf{H}_r \in \mathcal{H}^N \end{cases} \tag{7}$$

where $\|.\|_*$ is unitarily invariant. Then $\mu_c = \mu_{uc}$.

Proof: The idea of the proof (constructive) is to find a solution $\mathbf{H}_o$ to (6) that satisfies the additional structural constraints of (7). The actual construction is given in the next result, together with a state–space realization of an LTI system that has $\mathbf{H}_{\mathbf{g}}$ as its associated Hankel matrix. ■

Next we consider the problem of extracting a state space realization with the appropriate structural constraints from a given circulant Hankel matrix $\mathbf{H}^n$.

Algorithm 1:

0.- Data: a matrix $\mathbf{H}^N \in \mathcal{H}^N$, an integer $r \le N$.

1.- Perform a singular value decomposition:

$$\mathbf{H}^N = \begin{bmatrix} \mathbf{U} & \mathbf{U}_\perp \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}^T \\ \mathbf{V}_\perp^T \end{bmatrix}, \tag{8}$$
$$\mathbf{S} = \text{diag}(\sigma_1, \ldots, \sigma_N),\ \sigma_i \ge \sigma_j,\ i \ge j$$

3.- Assume that $\sigma_r > \sigma_{r+1}$ and form the rank r matrix $\mathbf{H}_r = \mathbf{U}_r\mathbf{S}_r\mathbf{V}_r^T$, where $\mathbf{S}_r = \text{diag}(\sigma_1, \ldots, \sigma_r)$ and $\mathbf{U}_r, \mathbf{V}_r$ denote the submatrices formed by the first r columns of $\mathbf{U}$ and rows of $\mathbf{V}^T$, respectively.
4.- Form the following state space realization:

$$\begin{aligned} \mathbf{A}_r &= \mathbf{S}_r^{-\frac{1}{2}}\mathbf{U}_r^T\mathbf{P}_L\mathbf{U}_r\mathbf{S}_r^{\frac{1}{2}},\ \mathbf{B}_r = \mathbf{S}_r^{\frac{1}{2}}\mathbf{V}_r^{(1)} \\ \mathbf{C}_r &= \mathbf{U}_r^{(1)}\mathbf{S}_r^{\frac{1}{2}} \end{aligned} \tag{9}$$

where

$$\mathbf{P}_L = \begin{bmatrix} 0 & \mathbf{I}_p & 0 & \ldots & 0 \\ 0 & 0 & \mathbf{I}_p & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ \mathbf{I}_p & 0 & 0 & \ldots & 0 \end{bmatrix} \tag{10}$$

and where $\mathbf{U}_r^{(1)}$ and $\mathbf{V}_r^{(1)}$ denote the first $p \times r$ block of $\mathbf{U}_r$ and $r \times m$ block of $\mathbf{V}_r^T$, respectively.

Theorem 1: The matrices generated by the algorithm above satisfy the following properties: (i) $\mathbf{A}_r^N = I$, (ii) $\mathbf{H}_r$ is the circulant Hankel matrix associated with the sequence $\mathbf{C}_r\mathbf{A}_r^{i-1}\mathbf{B}_r$, and (iii) The matrix $\mathbf{H}_r$ solves (6) and (7).

Proof: Given in the Appendix ■

Remark 2: It follows that, for the case under consideration here, the optimal rank–r approximation to $\mathbf{Y}$ obtained directly from its SVD also has a Hankel operator structure.

Theorem 2: Given input/output sequences $\{\mathbf{u}_k\}_{-N}^{-1}$, $\{\mathbf{y}_k\}_0^{N-1}$ and a set membership description of the measurement noise $\mathcal{N}$ satisfying assumptions A1–A2, define the matrix $\mathbf{Y}^N \doteq \mathbf{H}_\mathbf{y}^N(\mathbf{H}_\mathbf{u}^N)^T$. Let $\mathbf{Y}^N = \mathbf{USV}^T$ be a singular value decomposition of $\mathbf{Y}^N$, with $\mathbf{S} = \text{diag}(\sigma_1, \ldots, \sigma_N)$, $\sigma_i \geq \sigma_j$, $i \geq j$ and assume that $\sigma_r > \sigma_{r+1}$. Then

(i) The consistency set $\mathcal{T}(\mathbf{y}) \neq \emptyset$ if and only if $\|\mathbf{S}_{N-r}\|_* \leq \epsilon$, where $\mathbf{S}_{N-r} = \text{diag}(\sigma_{r+1}, \ldots, \sigma_N)$.
(ii) In this case, a suitable state–space realization of the unknown plant can be obtained applying algorithm 1 to the matrix $\mathbf{Y}^N$.

Proof: From Lemma 1 we have that $\mathcal{T}(\mathbf{y}) \neq \emptyset \iff \mu \leq \epsilon$ Since $\|.\|_*$ is unitarily invariant it follows that:

$$\|\mathbf{H}_\mathbf{y}^N - \mathbf{H}_\mathbf{g}^N\mathbf{H}_\mathbf{u}^N\|_* = \|\mathbf{H}_\mathbf{y}^N(\mathbf{H}_\mathbf{u}^N)^T - \mathbf{H}_\mathbf{g}^N\|_* = \|\mathbf{Y}^N - \mathbf{H}_\mathbf{g}^N\|_*$$

Moreover, since $\mathcal{H}$ is closed under multiplication and addition, it follows that if $\mathbf{H}_\mathbf{y}^N, \mathbf{H}_\mathbf{u}^N, \mathbf{H}_\mathbf{g}^N \in \mathcal{H} \Rightarrow \mathbf{H}_\mathbf{v}^N \in \mathcal{H}$. Thus, problems (3)-(5) and (6) are equivalent, which together with Theorem 1 implies that $\mu = \|S_{N-r}\|_*$. The proof follows now from Lemma 1. ■

B. Analysis of the Identification Error

Since the proposed algorithm is interpolatory and the *a priori* sets $\mathcal{S}, \mathcal{N}$ are convex, symmetric with respect to 0, it follows (Lemma 10.4 in [10]) that the worst case identification error satisfies:

$$\|e\| \leq 2 \sup_{g \in \mathcal{T}(0)} \|g\|$$

where

$$\mathcal{T}(0) = \{g \in \mathcal{S} \colon \mathbf{H}_\mathbf{g}\mathbf{H}_\mathbf{u} + \mathbf{H}_\mathbf{v} = \mathbf{0}, \text{for some } \mathbf{v} \in \mathcal{N}\} \tag{11}$$

Since $\mathbf{H}_\mathbf{u}$ is unitary, it follows that $\|e\|_* \leq 2\epsilon$.

Remark 3: As we will see in the sequel, using $\|.\|_*$ to measure the identification error captures the basic features of the applications of interest in this paper. In particular, in the context of images, if the Frobenious norm is used, then the identification error has a natural interpretation as the mean square error between the pixels of two textured images from the same family.

C. Removing some of the Assumptions

In this section we briefly discuss how to remove the assumptions that the period N and the input signal are known:

Estimating the Period: Assume that $n > N$ output measurements are available, together with bound of the period $N_l \leq N \leq N_u$. Consider now the Hankel matrix

$$\mathbf{H}_\mathbf{y}^{N_u} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \ldots & \mathbf{y}_{N_u} \\ \mathbf{y}_2 & \mathbf{y}_3 & \ldots & \mathbf{y}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_{N_u} & \mathbf{y}_1 & \ldots & \mathbf{y}_{N_u-1} \end{bmatrix}$$

It can be easily seen that if the underlying impulse response is periodic, then this matrix contains (within the measurement noise level) a lower rank submatrix of dimension $N \cdot p \times N \cdot m$. Thus, the period N can be estimated by considering submatrices of $\mathbf{H}_\mathbf{y}^{N_u}$ formed by taking the first $Np \times Nm$ elements, $N_L \leq N \leq N_u$ and selecting the value of N corresponding to the lowest rank. This procedure will be illustrated in the next section with by finding the "texton" in a textured image.

Dealing with unknown inputs: Assume now that the only information available about the input is $\mathbf{H}_\mathbf{u}^T\mathbf{H}_\mathbf{u} = \mathbf{I}$, leading to a blind identification problem. Since $\|.\|_*$ is unitarily invariant, any choice of unitary $\mathbf{H}_\mathbf{u}$ leads to a system $G \in \mathcal{S}$, and all of these systems have the same worst–case identification error bound. Thus, in this case, and in the absence of additional information, one can always select $\mathbf{H}_\mathbf{u} = I$, or, equivalently, $\mathbf{u}_k = \delta_{-N}$. Roughly speaking, this amounts to absorbing the phase of the unknown input in the dynamics of the identified plant.

V. Applications: Finding Textons and Inpaiting

Texture modelling has been a long standing problem in computer vision. Statistical approaches proceed by modelling texture as a stochastic process and attempting to capture the relevant properties [4], [5]. The approach that we pursue in this paper is related to these in the sense that, motivated by the work in [3], [11], we will also model images exhibiting a given texture as realizations of a second order stationary stochastic process. Our starting point is to consider the intensity values $\mathcal{I}(k,:)$ of the k^{th} row of the image as the output, at step k, of a discrete linear shift-invariant, not necessarily causal, system driven by white noise. In this context, texture modelling can be recast into the problem of identifying the relevant system model from the given images. However, a potential difficulty here is that the unknown system is *not necessarily causal:* the intensity value at a pixel is likely to depend on the values of all pixels in its neighborhood, not just on those preceding it in some ordering of the image pixels.

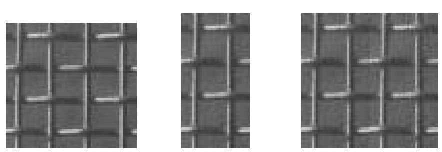
We propose to circumvent this difficulty by considering a given $n \times m$ image as one period of an infinite 2D signal with period (n, m). Thus, at any given location (i, j) in the image, the intensity values $\mathcal{I}(r, s)$ at other pixels are available also at position $(r - qn, s - qm)$, and the integer q can always be chosen so that $r - qn < i, s - qm < j$. From this observation, it follows that the unknown system $\mathcal{S}$ admits a state space representation of form:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k,\ \mathbf{A}^n = \mathbf{I} \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{v}_k. \end{aligned} \tag{12}$$

where for each k, the output vector $\mathbf{y}_k \in R^m$ contains all the intensity values $\mathcal{I}(k, l),\ 1 \le l \le m$ of the pixels in the k^{th} row of the image, and where the (deterministic) input $\mathbf{u}_k$ satisfies $\mathbf{H}_u^T \mathbf{H}_u = \mathbf{I}$. Further, as discussed in section IV-C, we can assume that $\mathbf{H}_u = I$. Thus, the problem reduces to identifying a state–space realization from its impulse response data, with the additional constraint $\mathbf{A}^n = \mathbf{I}$.

A. Application 1: Finding Textons

Consider the problem of finding "textons" in an image, that is, a subimage that, when tiled, reproduces the original image. Assuming that at least one full period is available in the sample image, in the context discussed above, the problem becomes that of jointly identifying a model and its corresponding period, which can be solved proceeding as outlined in section IV-C.



original image texton expanded image

Fig. 1. Finding textons as a rank minimization problem

The potential of this approach is illustrated in Figure 1 where it was used to (i) find a texton, (ii) extract the corresponding model, and (iii) expand the original image. For comparison, an algorithm based on finding the peak of the autocorrelation function [7], fails to identify the correct periodicity. Additional examples, omitted for space reasons, can be found at `http://robustsystems.ee.psu.edu`.

B. Application 2: Texture Inpainting

Consider now the problem of *restoring* a textured image where some pixels are entirely missing. Formally, given an image $\mathcal{I}(x, y)$ and a set of indexes of missing pixels $\mathcal{S} = \{(i_1, j_1), \dots, (i_s, j_s)\}$, the goal is to determine the intensity values $\mathcal{I}(i, j); (i, j) \in \mathcal{S}$ that best fit, in some sense, the rest of the image. As we show next, this problem can be recast into a rank minimization problem.

Restoration as a Rank Minimization Problem: Given an $n \times m$ image $\mathcal{I}(x, y)$, let $\mathbf{H}$ denote the associated Hankel matrix. Finally, denote by $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ the state–space matrices of the corresponding model, and assume that the image $\mathcal{I}$ contains at least one complete period, that is $\mathbf{A} \in R^{r \times r}$, with $\mathbf{A}^r = \mathbf{I}$ and $r < \min\{m, n\}$.

Consider now the situation where a portion of the image is missing. As we show next, this missing portion can be recovered by minimizing the rank of $\mathbf{H}$, provided that enough information is left in the image to recover at least one period. Start by considering an *ideal, noiseless* image, containing an integer number of periods (this assumption will be removed later). Assume now that $\mathbf{R}_1$, the first row of the image, is missing. The corresponding Hankel matrix is given by:

$$\mathbf{H}(\mathbf{x}) \doteq \begin{bmatrix} \mathbf{x} & \mathbf{R}_r & \dots & \mathbf{R}_1 & \dots & \mathbf{R}_2 \\ \mathbf{R}_2 & \mathbf{x} & \dots & \mathbf{R}_2 & \dots & \mathbf{R}_1 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ \mathbf{R}_1 & \mathbf{R}_r & \dots & \mathbf{x} & \dots & \mathbf{R}_r \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots \\ \mathbf{R}_r & \mathbf{R}_{r-1} & \dots & \mathbf{R}_r & \dots & \mathbf{x} \end{bmatrix} \tag{13}$$

where $\mathbf{x}$ denotes the missing pixels. Let $(r_o, \mathbf{x}_o)$ denote the solution to the rank minimization problem $r_o = \min_{\mathbf{x}} \text{rank}\{\mathbf{H}(\mathbf{x})\}$. Since by assumption the image contains at least one full period, and the minimal realization of this period requires r states, it follows that $\mathbf{H}(\mathbf{x})$ contains at least one rank r submatrix $\mathbf{M}_r = [\mathbf{R}_{j_i}]$. Hence $r_o \ge r$ and the minimum can be achieved for instance when $\mathbf{x}_o$ is set to the correct value. Thus, for any minimizing solution $\tilde{\mathbf{x}}$, there exist r columns $\mathbf{H}(:, i)$ and scalars α_i such that $\mathbf{H}(:, 1) = \sum_{i=1}^r \alpha_i \mathbf{H}(:, i)$. By contradiction, assume now that $\tilde{\mathbf{x}} \ne \mathbf{R}_1$. Since all indexes i appear in $\mathbf{H}(:, 1)$, this implies, (by selecting an appropriate subset of rows of $\mathbf{H}$), that $\mathbf{R}_1 = \sum_{i=1}^{r-1} \beta_i \mathbf{R}_i$, for some β_i not all zero, which contradicts the hypothesis that rank$(\mathbf{M}_r) = r$.

In the case of real images, corrupted by noise, let $\mathbf{Y}(\mathbf{x})$ and $\mathbf{H}_g$ denote the corresponding image and the underlying low rank Hankel operator. From the reasoning above, it follows that the missing pixels $\mathbf{x}$ can be found by solving the following problem:

$$\min_{\mathbf{x}} \{r\} \text{ subject to: } \begin{cases} \|\mathbf{Y} - \mathbf{H}\|_* \le \epsilon \\ \mathbf{H} \in \mathcal{H},\ \text{rank}(\mathbf{H}) = r \end{cases} \tag{14}$$

If the noise is characterized in terms of the $\|\mathbf{H}_v\|_2$, this leads to the following (non–convex) optimization problem:

$$\min_{\mathbf{x}} \{r\} \text{ subject to:} \sigma_i(\mathbf{x}) \le \epsilon, i \ge r$$

where $\sigma_i(.)$ denote the singular values of $\mathbf{Y}$.

Finally, the case where the image does not contain an integer number of periods can be solved by combining the idea above with the technique proposed for finding textons: A sequence of rank minimization problems can be solved, for submatrices of increasing dimensions. The texton and missing pixels can be jointly determined by finding the region, along with the corresponding minimizer $\mathbf{x}_o(\mathcal{T})$ that minimizes rank$[\mathbf{H}(\mathbf{x}, \mathcal{T})]$. Note also that the proofs above

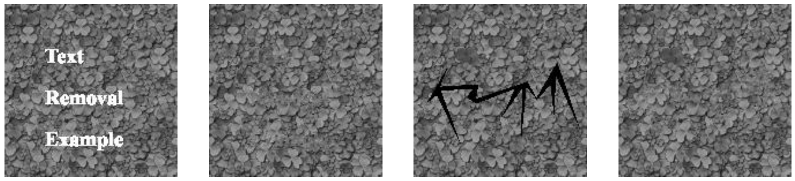


Fig. 2. Corrupted and Restored Images

generalize to other regions as long as the hypothesis that $\mathbf{H}$ contains a rank r submatrix holds.

Reducing the Computational Complexity: A potential difficulty with the approach outlined above stems from the fact that rank minimization problems are known to be generically NP–hard [1]. However, as we briefly show in the sequel, in this case the specific structure of the problem can be exploited to obtain computationally tractable convex relaxations. Begin by noting that if the Hankel matrix (13) has rank r, so does the Toeplitz matrix:

$$\mathbf{T}(\mathbf{x}) \doteq \begin{bmatrix} \mathbf{R}_1 & \mathbf{x} & \dots & \mathbf{R}_{n-1} \\ \mathbf{R}_2 & \mathbf{R}_1 & \dots & \mathbf{x} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x} & \mathbf{R}_{n-1} & \dots & \mathbf{R}_1 \end{bmatrix} \tag{15}$$

(this can be easily shown by noting that $\mathbf{H}^T\mathbf{H} = \mathbf{T}^T\mathbf{T}$. Moreover, it is not hard to show that the singular values of $\mathbf{T}$ are given by the magnitude of the Fourier Transform of its first column, evaluated at the frequencies $\omega_i = \frac{2\pi i}{n}$, $i = 0, 1, \dots, n-1$, that is:

$$\sigma(i) = \left[F^H(\omega_i)F(\omega_i)\right]^{\frac{1}{2}}, \; F(\omega_i) \doteq \sum_{k=1,n} \mathbf{R}_k e^{j(k-1)\omega_i}$$

Since $\mathbf{T}$ is an affine function of the missing pixels $\mathbf{x}$, it follows that $\sigma(i)$ is a convex function of $\mathbf{x}$. One can then attempt to solve Problem (14) by solving the following optimization problem:

$$\min_{\mathbf{x}} \sum_i log(\sigma(i)^2 + \epsilon) \tag{16}$$

The idea behind this function is to drive as many singular values as possible below the noise threshold ϵ. Consistent numerical experience shows that this relaxation achieves a value of the rank within 1 to 2% of the actual minimum.

The potential of this approach is illustrated in Figure 2, where it was used to remove unwanted text and to restore missing pixel values.

VI. Conclusions

Many problems of practical interest require identifying reduction systems having a periodic impulse response from noisy experimental data . Currently available techniques are not well suited for solving these problems, since they cannot guarantee that key structural properties, such as periodicity of the impulse response, will be preserved.

Motivated by existing subspace identification methods and their relationship with well known results in realization theory, in this paper we address these problems by working directly with a *circulant* Hankel $\mathbf{H}_y$ matrix constructed from the measured data $\mathbf{y}$. The main result of the paper shows that a state–space realization of the (unknown) plant can be obtained directly from a SVD–decomposition of $\mathbf{H}_y$. This result was established by noting that rank–constraint approximations obtained by truncating the SVD decomposition of a circulant Hankel matrix automatically inherit the Hankel structure, and that the periodicity constraint induces a circulant structure on the Hankel operator of the plant.

These results are illustrated with two non–trivial practical example arising in the context of image processing of textured images: (i) finding textons and (ii) texture inpaiting, that is, to seamlessly complete a textured image with missing pixels. As we show in the paper, both problems are related to the rank of $\mathbf{H}_y$. The first problem entails finding low rank submatrices, while the second leads to a rank–minimization problem. While these problems are known to be generically NP–hard, in this case the properties of circulant Hankel matrices can be exploited to obtain tight convex relaxations.

Efforts are currently under way to extend the identification approach proposes in this paper to 2-D systems.

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APPENDIX

A. Proof of Theorem 1

In order to prove Theorem 1 we need the following preliminary result.

Lemma 3: Consider the singular value decomposition of a matrix $\mathbf{H} \in \mathcal{H}$:

$$\mathbf{H} = \begin{bmatrix} \mathbf{U}_r & \mathbf{U}_{N-r} & \mathbf{U}_\perp \end{bmatrix} \begin{bmatrix} \mathbf{S}_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{N-r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_r^T \\ \mathbf{V}_{N-r}^T \\ \mathbf{V}_\perp^T \end{bmatrix}.$$

If $\sigma_r > \sigma_{r+1}$ then $\mathbf{P}_L \mathbf{U}_r \in$ span colums$(\mathbf{U}_r)$.

Proof: Let

$$\mathbf{P}_R = \begin{bmatrix} 0 & \mathbf{I}_m & 0 & \dots & 0 \\ 0 & 0 & \mathbf{I}_m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ \mathbf{I}_m & 0 & 0 & \dots & 0 \end{bmatrix} \tag{17}$$

It can be easily verified that $\mathbf{P}_L \mathbf{H} \mathbf{P}_R = \mathbf{H}$. Thus, for any left eigenvector $\mathbf{u}^T$ of $\mathbf{H}\mathbf{H}^T$ we have:

$$\begin{aligned} &\mathbf{u}^T \mathbf{H}\mathbf{H}^T = \sigma \mathbf{u}^T \Rightarrow \mathbf{u}^T \mathbf{H}\mathbf{H}^T \mathbf{P}_L^T = \sigma \mathbf{u}^T \mathbf{P}_L^T \Rightarrow \\ &\mathbf{u}^T \mathbf{P}_L^T \mathbf{P}_L \mathbf{H} \mathbf{P}_R \mathbf{P}_R^T \mathbf{H}^T \mathbf{P}_L^T = \sigma \mathbf{u}^T \mathbf{P}_L^T \Rightarrow \\ &\mathbf{u}^T \mathbf{P}_L^T \mathbf{H}\mathbf{H}^T = \sigma \mathbf{u}^T \mathbf{P}_L^T \end{aligned} \tag{18}$$

where we used the facts that $\mathbf{P}_L^T \mathbf{P} = \mathbf{I}$ and $\mathbf{P}_R \mathbf{P}_R^T = \mathbf{I}$. From the last equation it follows that $\mathbf{u}^T \mathbf{P}_L^T$ is also an eigenvector of $\mathbf{H}\mathbf{H}^T$, with eigenvalue σ. The proof follows now from the facts that subspaces corresponding to different eigenvalues of $\mathbf{H}\mathbf{H}^T$ are orthogonal and that the condition $\sigma_r > \sigma_{r+1}$ guarantees that the subspaces spanned by the columns of $\mathbf{U}_r$ and $\mathbf{U}_{N-r}$ are orthogonal. ■

Proof of Theorem 1:

Property (i): Start by partitioning $\mathbf{U} = [\mathbf{U}_r \ \mathbf{U}_{N-r} \ \mathbf{U}_\perp]$. Since $\mathbf{P}_L \mathbf{U}_r$ is orthogonal to $[\mathbf{U}_{N-r} \ \mathbf{U}_\perp]$, it follows that $\mathbf{U}_r \mathbf{U}_r^T \mathbf{P}_L \mathbf{U}_r = \left(\mathbf{I} - \mathbf{U}_{N-r} \mathbf{U}_{N-r}^T - \mathbf{U}_\perp^T \mathbf{U}_\perp\right) \mathbf{P}_L \mathbf{U}_r = \mathbf{P}_L \mathbf{U}_r$. Thus $\mathbf{A}_r^k = \mathbf{S}_r^{\frac{-1}{2}} \mathbf{U}_r^T \mathbf{P}_L^k \mathbf{U}_r \mathbf{S}_r^{\frac{1}{2}}$. The fact that $\mathbf{A}^n = I$ follows directly from $\mathbf{P}_L^N = I$.

Property (ii): Start by defining:

$$\mathbf{E}_L^{(k)} \doteq [\underbrace{\mathbf{0} \dots \mathbf{0}}_{k-1} \mathbf{I}_p \dots \mathbf{0}], \ \mathbf{E}_R^{(k)} \doteq [\underbrace{\mathbf{0} \dots \mathbf{0}}_{k-1} \mathbf{I}_m \dots \mathbf{0}]^T$$

$$\mathbf{h}_{ij} \in R^{p \times m} = \mathbf{E}_L^{(i)} \mathbf{H} \mathbf{E}_R^{(j)}$$

and use the expressions for $\mathbf{C}_r, \mathbf{B}_r$ and $\mathbf{A}_r^k$ to compute $\mathbf{C}_r \mathbf{A}_r^{k-1} \mathbf{B}_r$ leading to:

$$\begin{aligned} &\mathbf{C}_r \mathbf{A}_r^{k-1} \mathbf{B}_r = \mathbf{U}_r^{(1)} \mathbf{U}_r^T \mathbf{P}_L^{k-1} \mathbf{U}_r \mathbf{S}_r \mathbf{V}_r^{(1)} \\ &= \mathbf{E}_L^{(1)} \mathbf{U}_r \mathbf{U}_r^T \mathbf{P}_L^{k-1} \mathbf{U}_r \mathbf{S}_r \mathbf{V}_r^{(1)} = \mathbf{E}_L^{(1)} \mathbf{P}_L^{k-1} \mathbf{U}_r \mathbf{S}_r \mathbf{V}_r \mathbf{E}_R^{(1)} \\ &= \mathbf{E}_L^{(k-1)} \mathbf{H}_r \mathbf{E}_R^{(1)} = \mathbf{h}_{i,1} \end{aligned} \tag{19}$$

Next, compute

$$\begin{aligned} \mathbf{h}_{i,j} &= \mathbf{E}_L^{(i)} \mathbf{H}_r \mathbf{E}_R^{j} = \mathbf{E}_L^{(1)} \mathbf{P}_L^{(i-1)} \mathbf{H}_r \mathbf{P}_R^{-(j-1)} \mathbf{E}_R^{(1)} \\ &= \mathbf{E}_L^{(1)} \mathbf{P}_L^{i+j-2} \mathbf{P}_L^{-(j-1)} \mathbf{H}_r \mathbf{P}_R^{-(j-1)} \mathbf{E}_R^{(1)} \\ &= \mathbf{E}_L^{(i+j-2)} \mathbf{H}_r \mathbf{E}_R^{(1)} = \mathbf{h}_{i+j-2,1} \end{aligned}$$

Thus, $\mathbf{H}_r$ has the required block circulant Hankel structure. Moreover, from (19) and the fact that $\mathbf{A}_r^N = \mathbf{I}$, it follows that $\mathbf{H}_r = \mathcal{O}_N \mathcal{C}_N$, where:

$$\begin{aligned} \mathcal{O}_N &= \begin{bmatrix} \mathbf{C}_r^T & \mathbf{A}_r^T \mathbf{C}_r & \dots & (\mathbf{A}_r^{N-1})^T \mathbf{C}_r^T \end{bmatrix} \\ \mathcal{C}_N &= \begin{bmatrix} \mathbf{B}_r & \mathbf{A}_r \mathbf{B}_r & \dots & \mathbf{A}_r^{N-1} \mathbf{B}_r \end{bmatrix} \end{aligned} \tag{20}$$

Property (iii) The fact that $\mathbf{H}_r$ solves (6) follows directly from Mirsky's Theorem [9], since $\|.\|_*$ is unitarily invariant. The fact that $\mathbf{H}_r$ solves (7) follows from the fact that in general $\mu_{uc} \leq \mu_c$ with $\mathbf{H}_r$ achieving equality in this case.