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# Classification of linear planar systems with hybrid feedback control

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*Abstract*— **Continuing the authors' studies of hybrid dynamical systems, i.e. differential equations governed by finite automata, an efficient and complete classification of control linear systems in the plane is offered. The set of all such systems is divided into equivalence classes which are explicitly characterized by some quantitative invariants. The canonical representatives in each class are determined. It is shown how to use this classification to find out whether a given system is stabilizable or not.**

## I. INTRODUCTION

Consider a linear control $2 \times 2$ - system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \qquad y = \mathbf{C}\mathbf{x}, \tag{1}$$

on $[0, \infty)$, where $\mathbf{x} \in \mathcal{R}^2$ is the state variable of the system, $y \in \mathcal{R}$ is the output variable, $u \in \mathcal{R}$ is the control variable, and $\mathbf{B}, \mathbf{C}$ are real, nonzero 2-dimensional column vector and row vector, respectively. As $\operatorname{rank} \mathbf{B} = \operatorname{rank} \mathbf{C} = 1$, the stabilization of the system is a nontrivial problem, because static output feedback controls may not work, even if the usual controllability and observability assumptions are fulfilled. A typical example is *the controlled harmonic oscillator*. Among dynamical stabilizers (which always exist), the simplest are those having a finite number of states ("automata"). Coupling such a stabilizer with the given continuous dynamical system ("a plant") generates a hybrid feedback control. A way of attaching the automaton to the plant is given by a switching diagram. In [1] it is shown that there exists a linear hybrid feedback control (in short: $\mathcal{LH}$-control, see Definition 2.2 below), which stabilizes the controlled harmonic oscillator. However, in [2] and [5], it is proved that some planar systems cannot be stabilized in this way. That is why, the problem arise how to recognize systems, which admit stabilization by finite state controllers and which are not.

Remark first that the cases, where either $\operatorname{rank} \mathbf{C}$, or $\operatorname{rank} \mathbf{B}$ are not 1 are not of much interest for the systems in the plane. Assume for a moment that $\mathbf{B}, \mathbf{C}$ are matrices of the size $2 \times \ell$ and $m \times 2$, respectively. If the pair $(\mathbf{A}, \mathbf{B})$ is stabilizable, and $\operatorname{rank} \mathbf{C} = 2$ (the case of complete observability of the solutions), or if $\operatorname{rank} \mathbf{B} = 2$ and the pair $(\mathbf{A}, \mathbf{C})$ is detectable, then there always exists a linear static output feedback control $\mathbf{u} = \mathbf{G}\mathbf{y}$ with an $\ell \times m$-matrix $\mathbf{G}$, which exponentially stabilizes the zero solution of the given control system with an arbitrary matrix $\mathbf{A}$ (see e.g. [8]). If, on the contrary, the pair $(\mathbf{A}, \mathbf{B})$ is not stabilizable, or the pair $(\mathbf{A}, \mathbf{C})$ is not detectable (for example, if $\mathbf{B} = 0$ or $\mathbf{C} = 0$), then system (1) can be stabilized neither by linear static output feedback controls, nor by $\mathcal{LH}$-controls. Thus, the assumption posed on system (1) that $\mathbf{B}, \mathbf{C}$ are nonzero vectors is indeed essential and provides no loss of generality: only in this case $\mathcal{LH}$-controls can do the stabilization job better than linear static output feedback controls (see [5], [4] for the details).

In [5] a classification of control planar systems (the so-called "$GT$-classification") was offered, which in particular implied the following stabilization criterion: system (1) is stabilizable by a $\mathcal{LH}$-control ($\mathcal{LH}$-stabilizable) if and only if there exists $\alpha \in \mathcal{R}$ for which the matrix $\mathbf{A} + \alpha\mathbf{B}\mathbf{C}$ has no nonnegative real eigenvalues. A similar result was independently proved by a different method in the papers [2] and [3]. The criterion demonstrates that, indeed, the stabilization condition is considerably weakened if static output feedbacks are replaced by hybrid feedbacks. However, this condition is not always easy to check. In [4] the "$GT$-classification" was exploited to construct verifiable criteria for $\mathcal{LH}$-stabilization of system (1). An efficient algorithm was designed which tests specific systems (1) using the input data $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. This algorithm also recognizes whether the zero solution to (1) can be stabilized by a static output feedback linear control, or by a $\mathcal{LH}$-control, or cannot be stabilized by any of them.

The present paper continues and in some sense completes the study started in [4], [5]. The following are the main results of the paper:

1) A set of invariants is identified, which characterize the $GT$-equivalence classes, where $GT$ is a certain group of transformations of the triples $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. This leads to an efficient classification of systems (1) based on the suggested invariants. In each equivalence class one canonical representative is found whose parameters can explicitly be calculated via the invariants of the class.

2) Certain properties of the equivalence classes are studied, for example, whether or not a system from the given class admits stabilization and, if it does, whether it is possible to find a static output feedback control, or a non-static $\mathcal{LH}$-control should be applied.

3) The final classification is conveniently put in a table.

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## II. $\mathcal{LH}$-CONTROLS AND STABILIZATION

Below are the formal definitions of $\mathcal{LH}$-controls and some relevant objects.

*Definition 2.1:* A discrete automaton is a 6-tuple $\Delta = (Q, I, \mathcal{M}, \mathcal{T}, j, q_0)$, where

(i) $Q$ is a finite set of all automaton states (locations);

(ii) the finite set $I$ contains the input alphabet;

(iii) the transition map $\mathcal{M} : Q \times I \to Q$ indicates the location after a transition time, based on the previous location $q$ and input $i \in I$ at the time of transition;

(iv) $\mathcal{T} : Q \to (0, \infty)$ is a mapping which sets a period $\mathcal{T}(q)$ between transitions times;

(v) $j : \mathcal{R} \to I$ is a function with property $j(\lambda y) = j(y)$, $y \in \mathcal{R}$, $\lambda > 0$;

(vi) $q_0 = q(0)$ is the state of the automaton at the initial time.

For any automaton $\Delta$ satisfying (i)-(vi) a special feedback operator $F_\Delta$ can be iteratively constructed. Given $y : [0, \infty) \to \mathcal{R}$ the function $F_\Delta y : [0, \infty) \to Q$ is defined by:

1. $(F_\Delta y)(0) = q_0$; $t_1 = \mathcal{T}(q_0)$; $F_\Delta y \equiv q_0$ on $[0, t_1)$;

2. If $t_1, \ldots, t_k$ and the values $(F_\Delta y)(t)$ for $t \in [0, t_k)$ are already known, then $t_{k+1}$ and $(F_\Delta y)(t)$ are defined for $t \in [t_k, t_{k+1})$ by the equalities

$(F_\Delta y)(t_k) = \mathcal{M}(q(t_{k-1}), j(y(t_k))) := q(t_k)$;
$t_{k+1} = t_k + \mathcal{T}(q(t_k))$; $\quad F_\Delta y \equiv q(t_k)$ on $[t_k, t_{k+1})$.

Here $\{t_k\}_{k=0}^{\infty}$ ($t_0 = 0$) is a sequence of the automaton's possible switching times.

*Definition 2.2:* Given a discrete automaton $\Delta$ and a set $\{\alpha_q \,|\, q \in Q\} \subset \mathcal{R}$, the pair $(\Delta, \{\alpha_q\})$, will be addressed as a **linear hybrid feedback control**; dependence between the control function $u(\cdot)$ and the output function $y(\cdot)$ is defined by $u(t) = \alpha_{q(t_k)} y(t)$, $t \in [t_k, t_{k+1})$, $k = 0, 1, \ldots,$ where the $\{t_k\}_{k=0}^{\infty}$ is the corresponding sequence of switching times.

The set of all linear hybrid feedback controls has already been denoted by $\mathcal{LH}$. Any specific control $u \in \mathcal{LH}$ has the form $u = (\Delta, \{G_q\})$. According to Definition 2.2, system (1) governed by a control $u = (\Delta, \{G_q\}) \in \mathcal{LH}$ is equivalent to the nonlinear functional differential equation

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{B}\mathbf{G}_{(F_\Delta \mathbf{C}\mathbf{x})(t)}\mathbf{C})\mathbf{x}(t), \quad t \in [0, \infty). \qquad (2)$$

*Lemma 2.1:* [4] For any $\mathcal{LH}$-control $u$ and for any $\mathbf{a} \in \mathcal{R}^2$ there exists the unique trajectory $\mathbf{x} : [0, \infty) \to \mathcal{R}^2$ with the property $\mathbf{x}(0) = \mathbf{a}$ (evidently, $\mathbf{x} \equiv 0$ if $\mathbf{a} = 0$).

*Remark 2.1:* The full dynamics of system (1) governed by a $\mathcal{LH}$-control $u$ is characterized by the triple $H(t) = (x(t), q(t), \mathbf{r}(t))$, where $x(\cdot)$ is a solution to (1), $q(t)$ is the automaton's location at time $t$, and $\mathbf{r}(t)$ is the time remaining till the next transition instance (see [1]). The function $H(\cdot) : [0, \infty) \to \mathcal{R}^2 \times Q \times [0, \infty)$ can be called a **full trajectory** of the (hybrid) system (1). Note that equation (2) explicitly provides the first component of the full trajectory.

Let $\mathcal{LH}_1 \subset \mathcal{LH}$ be the subclass of $\mathcal{LH}$, for which $Q$ contains only one point. Clearly, the subclass $\mathcal{LH}_1$ coincides with the class of linear static output feedback controls of the form $u = \alpha y$.

*Definition 2.3:* System (1) is said to be $u$-stabilizable, where $u \in \mathcal{LH}$, if the trivial solution of the system, subject to the control $u$, is uniformly asymptotically stable. In other words,

(a) for any $\varepsilon > 0$ there is $\delta > 0$ such that every solution $\mathbf{x}(\cdot)$ with the property $|\mathbf{x}(0)| < \delta$ satisfies the estimate $|\mathbf{x}(t)| < \varepsilon$ for $t \geq 0$;

(b) for every solution $\mathbf{x}(\cdot)$ one has $|\mathbf{x}(t)| \to 0$ as $t \to \infty$, the convergence being uniform w.r.t. initial points $\mathbf{x}(0) \in K$ for any bounded $K \subset \mathcal{R}^2$.

Let $\mathcal{U} \subset \mathcal{LH}$. System (1) is called $\mathcal{U}$-stabilizable if $\exists u \in \mathcal{U}$ such that (1) is $u$-stabilizable.

Let $\Sigma$ be the set of all triples $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, where $\mathbf{A}$ is a real $2 \times 2$-matrix, $\mathbf{B}$ is a nonzero real column-vector, and $\mathbf{C}$ is a nonzero real row-vector. Clearly, system (1) is uniquely determined by $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and the control $u$. A triple $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \Sigma$ is in the sequel called $u$-stabilizable, resp. $\mathcal{U}$-stabilizable, if the corresponding system (1) is $u$-stabilizable, resp. $\mathcal{U}$-stabilizable.

*Definition 2.4:* Let $\Omega = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \Sigma$, $u \in \mathcal{LH}$. The upper Lyapunov exponent $\kappa(\Omega, u)$ of the $u$-governed system (1) (or of the triple $\Omega$) is the greatest lower bound of $\lambda > 0$ such that for each solution $\mathbf{x}$ of this system the exponential estimate $|\mathbf{x}(t)| \leq M e^{\lambda t} |\mathbf{x}(0)|$ holds for $t \geq 0$, where $M > 0$ may only depend on $\lambda$.

Given a subset $\mathcal{U} \subset \mathcal{LH}$ we put

$$\kappa(\Omega, \mathcal{U}) = \inf_{u \in \mathcal{U}} \kappa(\Omega, u).$$

The number $\kappa(\Omega, \mathcal{U})$ is called the upper Lyapunov exponent of (1) with respect to the subset $\mathcal{U}$.

The number $\kappa(\Omega, \mathcal{U})$ characterizes capability of the controls $u \in \mathcal{U}$ to influence the rate of convergence of solutions to zero (if $\kappa < 0$), or to infinity (if $\kappa > 0$) as $t \to \infty$. In particular, the equality $\kappa(\Omega, \mathcal{U}) = -\infty$ means that the controls can make the solutions tend to zero as fast as necessary.

Let $\sigma(\mathbf{M})$ denote the spectrum of a matrix $\mathbf{M}$, and put $\mathcal{C}_- = \{z \in \mathcal{C} \,|\, \operatorname{Re} z < 0\}$, $\mathcal{C}_-^* = \{z \in \mathcal{C} \,|\, (\operatorname{Re} z < 0) \vee (\operatorname{Im} z \neq 0)\}$. The following known definitions (specified for the particular system (1)) can be found in [8].

*Definition 2.5:* Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \Sigma$.

The pair $(\mathbf{A}, \mathbf{B})$ is *controllable* if $\det(\mathbf{B}\ \mathbf{A}\mathbf{B}) \neq 0$.

The pair $(\mathbf{A}, \mathbf{C})$ is *observable* if the pair $(\mathbf{A}^\top, \mathbf{C}^\top)$ is controllable.

The pair $(\mathbf{A}, \mathbf{B})$ is *stabilizable* if there exists a real, 2-dimensional row-vector $\mathbf{F}$ such that $\sigma(\mathbf{A} + \mathbf{B}\mathbf{F}) \subset \mathcal{C}_-$.

The pair $(\mathbf{A}, \mathbf{C})$ is *detectable* if the pair $(\mathbf{A}^\top, \mathbf{C}^\top)$ is stabilizable.

It is possible to check explicitly [4] whether a triple $\Omega \in \Sigma$, related to system (1), has one or several following properties:

$\mathbf{P_1}$. The pair $(\mathbf{A}, \mathbf{B})$ is controllable,
$\mathbf{P_2}$. The pair $(\mathbf{A}, \mathbf{C})$ is observable,
$\mathbf{P_3}$. The pair $(\mathbf{A}, \mathbf{B})$ is stabilizable,
$\mathbf{P_4}$. The pair $(\mathbf{A}, \mathbf{C})$ is detectable,
$\mathbf{P_5}$. The triple $\Omega$ is $\mathcal{LH}_1$-stabilizable, i.e. $\sigma(\mathbf{A} + \alpha\mathbf{B}\mathbf{C}) \subset \mathcal{C}_-$

for some $\alpha \in \mathcal{R}$,

$\mathbf{P_6}$. The triple $\Omega$ is $\mathcal{LH}$-stabilizable, i.e. $\sigma(\mathbf{A}+\alpha\mathbf{BC}) \subset \mathcal{C}_-^*$ for some $\alpha \in \mathcal{R}$,

$\mathbf{P_7}$. $\kappa(\Omega, \mathcal{LH}) = -\infty$.

In addition, the following implications hold: 1)⇒3), 2)⇒4), 5)⇒6)⇐7), 6)⇒3)&4). The first three are well-known, the last one is proved in [5]. Observe also that $\forall\Omega \in \Sigma \;\; \kappa(\Omega, \mathcal{L}H_1) > -\infty$.

These properties seem to be sufficiently exhaustive for stability analysis of the control system (1). It is convenient to write $\Psi(\Omega) = \{i_1, i_2, \ldots, i_7\}$ for $\Omega \in \Sigma$, where for each $k = 1, 2, \ldots, 7$ either $i_k = 1$ (if property $\mathbf{P_k}$ above holds), or $i_k = 0$ (if it does not).

## III. THE TRANSFORMATION GROUP $GT$ AND $GT$-INVARIANTS

Let $GL_2$ denote the multiplicative group of nonsingular real $2 \times 2$-matrices. Define the transformations $T_i(\cdot) : \Sigma \to \Sigma$, $i = 1, 2, 3$ by

$$T_1(\mathbf{D}): \; (\mathbf{A},\mathbf{B},\mathbf{C}) \mapsto (\mathbf{DAD}^{-1}, \mathbf{DB}, \mathbf{CD}^{-1}), \quad \mathbf{D} \in GL_2;$$

$$T_2(m_1, m_2, m_3): (\mathbf{A},\mathbf{B},\mathbf{C}) \mapsto (m_1\mathbf{A}, m_2\mathbf{B}, m_3\mathbf{C}), \quad m_1 > 0,\ m_2, m_3 \in \mathcal{R} \setminus \{0\};$$

$$T_3(\alpha): \; (\mathbf{A},\mathbf{B},\mathbf{C}) \mapsto (\mathbf{A}+\alpha\mathbf{BC}, \mathbf{B}, \mathbf{C}), \quad \alpha \in \mathcal{R}.$$

The family of all transformations of the set $\Sigma$, generated by $T_i$, $i = 1, 2, 3$ is called $GT$. This family consists of transformations $T : \Sigma \to \Sigma$, which can be represented as a finite product of $T_i(\cdot)$, where $(\cdot)$ is replaced by any admissible parameter described above.

*Lemma 3.1 ([5]):* Any transformation $T \in GT$ can, in fact, be represented as $T = T_1(\mathbf{D}) \circ T_2(m_1, m_2, m_3) \circ T_3(\alpha)$ for some $\mathbf{D} \in GL_2$, $m_1 > 0$, $m_2, m_3 \in \mathcal{R} \setminus \{0\}$ and $\alpha \in \mathcal{R}$. A similar representation $T = T_i \circ T_j \circ T_k$ is also valid for any of the six permutation $\{i, j, k\}$ of the set $\{1, 2, 3\}$ (however, the parameter values $\mathbf{D}, m_i, \alpha$ can be different for different permutations).

From the definitions of $T_i$ and from Lemma 3.1 it follows that $GT$ indeed is a group of transformations of the set $\Sigma$, where the group multiplication is the composition (multiplication) of set transformations. Observe that $GT$ is not commutative (see [5]).

The group $GT$ gives rise to the natural equivalence relation on $\Sigma$, where $\Omega_1 \sim \Omega_2$ iff there exists $T \in GT$ such that $\Omega_2 = T(\Omega_1)$. The generated factor set is denoted by $\Sigma/GT$. It consists of all $GT$-equivalence classes.

*Definition 3.1:* Let $\mathcal{A}$ be a property (condition), which is assigned to some triples $(\mathbf{A},\mathbf{B},\mathbf{C}) \in \Sigma$ (or to system (1) corresponding to the triple). The property $\mathcal{A}$ is said to be a $GT$-invariant if its validity for some $\Omega_1 \in \Sigma$ implies its validity for any other $\Omega_2 \in \Sigma$ that is $GT$-equivalent to $\Omega_1$.

A $GT$-invariant is called quantitative if there exist finitely many families $\Sigma = \Sigma_1 \supset \ldots \supset \Sigma_k \supset \Sigma_{k+1} = \emptyset$, sets $E_i \in \mathcal{R}$ and functions $f_i : \Sigma_i \to \mathcal{R}$, $i = 1, 2 \ldots, k$ such that $\mathcal{A}$ can be described by the set of conditions $f_i(\Omega) \in E_i$ ($\Omega \in \Sigma$, $i = 1, 2, \ldots, k_0(\Omega)$), where $k_0(\Omega) \in \mathcal{N}$ is chosen in such a way that $\Omega \in \Sigma_{k_0(\Omega)} \setminus \Sigma_{k_0(\Omega)+1}$.

A complete quantitative invariant (CQI) is a quantitative $GT$-invariant $\mathcal{A}$ satisfying the following condition: if $\mathcal{A}$ is valid for $\Omega_1, \Omega_2 \in \Sigma$, then $\Omega_1 \sim \Omega_2$.

A natural question arises of how the factorization $\Sigma/GT$ is connected with the problem of stabilization (in particular, $\mathcal{LH}$-stabilization) of system (1). The answer is given by the following result.

*Theorem 3.1:* Any value of the (vector) function $\Psi$ from Section 2 is a $GT$-invariant. In other words, properties $\mathbf{P_k}$, $k = 1, ..., 7$ of the triples $(\mathbf{A},\mathbf{B},\mathbf{C}) \in \Sigma$ are $GT$-invariant. In particular, Theorem 3.1 justifies that the corresponding factor-mapping $\tilde{\Psi} : \Sigma/GT \to \{0; 1\}^{\{1;2;\ldots;7\}}$ is well-defined.

To be able to suggest an efficient $GT$-classification it is, however, necessary to find CQI. The first, yet crucial step in this direction is described in the next theorem. Some notation used in the theorem is introduced below.

Let $\Sigma_1 = \{\Omega \in \Sigma | \mathbf{CB} \neq 0\}$. Define the functions $\omega, \mathbf{m} : \Sigma_1 \to \mathcal{R}$ by

$$\omega(\Omega) = \operatorname{tr}\mathbf{A} - \frac{\mathbf{CAB}}{\mathbf{CB}}, \;\; \mathbf{m}(\Omega) = \frac{\mathbf{CAB}}{\mathbf{CB}} \cdot \omega(\Omega) - \det\mathbf{A}.$$

Let $\Sigma_2 = \{\Omega \in \Sigma_1 | \mathbf{m}(\Omega) \neq 0\}$. Then the function $\tau : \Sigma_2 \to \mathcal{R}$ is given by

$$\tau(\Omega) = \frac{\omega(\Omega)}{\sqrt{|\mathbf{m}(\Omega)|}}.$$

*Theorem 3.2:* Each of the conditions listed below is a quantitative $GT$-invariant:

$\mathbf{Q_1}$. $\mathbf{CAB} = \mathbf{CB} = 0$;

$\mathbf{Q_2}$. $\mathbf{CAB} \neq 0$, $\mathbf{CB} = 0$;

$\mathbf{Q_3}$. $\mathbf{CAB} \cdot \mathbf{CB} \neq 0$, $\operatorname{sign}\omega(\Omega) = \mu$, $\operatorname{sign}\mathbf{m}(\Omega) = \nu$ $(\forall \mu, \nu \in \{-1, 0, 1\})$;

$\mathbf{Q_4}$. $\mathbf{CAB} \cdot \mathbf{CB} \cdot \mathbf{m}(\Omega) \neq 0$, $\tau(\Omega) = \mu$ $(\forall \mu \in \mathcal{R})$.

## IV. SYSTEMS WITH $\mathcal{LH}$-CONTROLS AND THEIR $GT$-CLASSIFICATION

The quantitative invariants described in Theorem 3.2 can be used in the $GT$-classification. A brief overview of what has been done is presented below without proofs. The final $GT$-classification is presented at the end of the section as a table. The proofs are technical and will be published elsewhere.

a) A convenient description of the factor-set $\Sigma/GT$ is offered, where a certain CQI is assigned to each element of this factor-set (i.e. to each $GT$-equivalence class). This provides an efficient way to verify to which $GT$-equivalence class does a given triple $\Omega = (\mathbf{A},\mathbf{B},\mathbf{C}) \in \Sigma$ belong.

b) A canonical triple in any $GT$-equivalence class is found. This is a triple which seems to be most suitable to study various properties of system (1), which are invariant under transformations from $GT$, in particular, to calculate the vector $\Psi(\Omega)$. An algorithm of how a given triple $\Omega \in \Sigma$ can be transformed to its canonical form is constructed as well.

c) Using the canonical triples, Theorem 3.1 and the results from [5], [6], [4], and [7], the function $\tilde{\Psi}$ can explicitly be calculated in all cases. In other words, for any $GT$-equivalence class $\tilde{\Omega} \in \Sigma/GT$ it is found which of the seven properties $\mathbf{P_k}$, $k = 1, ..., 7$ are valid for the representatives $\Omega$ of the class $\tilde{\Omega}$.

The results of the $GT$-classification, in particular those described in a)-c), are summarized in the table below. The algorithm of transforming a triple to its canonical form is, however, omitted because of the limitations on the size of the paper.

The first column in the table contains an explicit (below the dotted lines) and symbolic (above the dotted lines) description of different $GT$-equivalence classes. The symbolic description depends on three parameters $H = H(\rho_1, \rho_2, \rho_3)$, which constitute CQI, thus characterizing the equivalence class in question. The explicit description is added for the sake of convenience. The first invariant $\rho_1$ takes only 4 integer values, from 1 to 4, and corresponds the number $k$ of the condition $\mathbf{Q_k}$ in Theorem 3.2. For example, if $\rho_1 = 1$, then the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ satisfies $\mathbf{CAB} = \mathbf{CB} = 0$, and this is also written below the dotted line. The systems with the same $\rho_1$ have similar properties, while the systems with different $\rho_1$ are less similar. That is why the systems with different $\rho_1$ are placed in the different parts of the table separated by a double line. The second invariant $\rho_2$ takes only 3 values: -1, 0, 1 depending on some special properties of the triple $\Omega$, which are formulated differently for different values of the other two invariants. The third invariant may take infinitely many values and can be calculated in the way shown in the table with the help of the functions $\omega(\Omega)$, $\mathbf{m}(\Omega)$, $\tau(\Omega)$ introduced just before Theorem 3.2, as well as the eigenvalues of the matrix $\mathbf{A}$: $\sigma(\mathbf{A}) = \{\lambda_1, \lambda_2\}$. If the eigenvalues $\lambda_1, \lambda_2$ are real and different, then it is assumed, without loss of generality, that $\lambda_1 < \lambda_2$.

The second column in the table describes the canonical triple $\Omega = (\mathbf{A}, \mathbf{B}, \mathbf{C})$ in each $GT$-equivalence class. Each canonical triple depends on a parameter which coincides with the third invariant of the equivalence class. The 7-dimensional vector $\Psi(\Omega)$ below the dotted line, which takes the values 0 or 1, is described in Theorem 3.1. It indicates whether the properties $\mathbf{P_k}$ are valid or not for the triple $\Omega$. Remark that according to Section 3 this vector is also equal to $\tilde{\Psi}(H(\rho_1, \rho_2, \rho_3))$.

## V. DISCUSSION

The described $GT$-classification is complete. This means that the union of all $H(\cdot)$ in the table coincides with $\Sigma/GT$.

It is interesting to remark that the sets $H(\rho_1, \cdot, \cdot)$, $\rho_1 = 1, 2, 3$, are far smaller than the set $H(4, \cdot, \cdot)$. To be more precise, consider the set of all triples $\Sigma = \{(\mathbf{A}, \mathbf{B}, \mathbf{C})\}$ as a subset of the space $\mathcal{R}^8$. Then the set consisting of all $\Omega$ from the equivalence classes $H(\rho_1, \cdot, \cdot)$, where $\rho_1 = 1, 2, 3$, is a nowhere dense set of measure 0 in $\mathcal{R}^8$. The sets $\Sigma_2^- = \{\Omega \mid \exists \mu \in \mathcal{R},\ \Omega \in H(4, -1, \mu)\}$ and $\Sigma_2^+ = \{\Omega \mid \exists \mu \in \mathcal{R},\ \Omega \in H(4, 1, \mu)\}$ are, on the contrary, open in $\mathcal{R}^8$, so that they are invariant under small perturbations of the matrices from system (1). In addition, $\Sigma_2 = \Sigma_2^- \cup \Sigma_2^+$ is dense in $\mathcal{R}^8$. Notice also that the invariant $\mu = \tau(\Omega)$, which also is the diagonal entry in $\mathbf{A}$ in the canonical form for $H(4, \pm 1, \mu)$ depends continuously on $\Omega$.

TABLE I
THE GT-CLASSIFICATION

| Elements of $\Sigma/GT$ and their CQI | The canonical form and $\Psi(\cdot)$ |
|---|---|
| $H(1,0,k)$, $k=-1,0,1$<br>....................<br>$\mathbf{CB}=\mathbf{CAB}=0$,<br>$\lambda_1=\lambda_2$, $\operatorname{sign}\lambda=k$ | $\left(\begin{pmatrix}k&0\\0&k\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},(1\ \ 0)\right)$<br>....................<br>$\{0011110\}$ if $k=-1$<br>$\{0000000\}$ if $k=0,1$ |
| $H(1,-1,\mu)$, $\mu\in\mathcal{R}$<br>....................<br>$\mathbf{CB}=\mathbf{CAB}=0$,<br>$\lambda_1<\lambda_2$, $\mathbf{AB}=\lambda_1\mathbf{B}$<br>$\frac{\lambda_2+\lambda_1}{\lambda_2-\lambda_1}=\mu$ | $\left(\begin{pmatrix}\mu&1\\1&\mu\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},(1\ \ 1)\right)$<br>....................<br>$\{0011110\}$ if $\mu<-1$<br>$\{0001000\}$ if $\mu\in[-1;1)$<br>$\{0000000\}$ if $\mu\geq 1$ |
| $H(1,1,\mu)$, $\mu\in\mathcal{R}$<br>....................<br>$\mathbf{CB}=\mathbf{CAB}=0$,<br>$\lambda_1<\lambda_2$, $\mathbf{AB}=\lambda_2\mathbf{B}$<br>$\frac{\lambda_2+\lambda_1}{\lambda_2-\lambda_1}=\mu$ | $\left(\begin{pmatrix}\mu&1\\1&\mu\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},(1\ \ -1)\right)$<br>....................<br>$\{0011110\}$ if $\mu<-1$<br>$\{0010000\}$ if $\mu\in[-1;1)$<br>$\{0000000\}$ if $\mu\geq 1$ |
| $H(2,0,k)$, $k=-1,0,1$<br>....................<br>$\mathbf{CAB}\neq 0$, $\mathbf{CB}=0$,<br>$\operatorname{sign}\operatorname{tr}\mathbf{A}=k$ | $\left(\begin{pmatrix}k&1\\-1&k\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},(1\ \ 0)\right)$<br>....................<br>$\{1111111\}$ if $k=-1$<br>$\{1111011\}$ if $k=0,1$ |
| $H(3,0,k)$, $k=-1,0,1$<br>....................<br>$\mathbf{CB}\neq 0$, $\mathbf{m}(\Omega)=0$,<br>$\det(\mathbf{B}\ \mathbf{AB})=0$,<br>$\det\begin{pmatrix}\mathbf{C}\\\mathbf{CA}\end{pmatrix}=0$,<br>$\operatorname{sign}\omega(\Omega)=k$ | $\left(\begin{pmatrix}k&0\\0&k\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},(1\ \ 0)\right)$<br>....................<br>$\{0011110\}$ if $k=-1$<br>$\{0000000\}$ if $k=0,1$ |
| $H(3,-1,k)$, $k=-1,0,1$<br>....................<br>$\mathbf{CB}\neq 0$, $\mathbf{m}(\Omega)=0$,<br>$\det(\mathbf{B}\ \mathbf{AB})=0$,<br>$\det\begin{pmatrix}\mathbf{C}\\\mathbf{CA}\end{pmatrix}\neq 0$,<br>$\operatorname{sign}\omega(\Omega)=k$ | $\left(\begin{pmatrix}k&1\\0&k\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},(1\ \ 0)\right)$<br>....................<br>$\{0111110\}$ if $k=-1$<br>$\{0101000\}$ if $k=0,1$ |
| $H(3,1,k)$, $k=-1,0,1$<br>....................<br>$\mathbf{CB}\neq 0$, $\mathbf{m}(\Omega)=0$,<br>$\det(\mathbf{B}\ \mathbf{AB})\neq 0$,<br>$\det\begin{pmatrix}\mathbf{C}\\\mathbf{CA}\end{pmatrix}=0$,<br>$\operatorname{sign}\omega(\Omega)=k$ | $\left(\begin{pmatrix}k&0\\1&k\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},(1\ \ 0)\right)$<br>....................<br>$\{1011110\}$ if $k=-1$<br>$\{1010000\}$ if $k=0,1$ |
| $H(4,-1,\mu)$, $\mu\in\mathcal{R}$<br>....................<br>$\mathbf{CB}\neq 0$, $\mathbf{m}(\Omega)<0$,<br>$\tau(\Omega)=\mu$ | $\left(\begin{pmatrix}\mu&1\\-1&\mu\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},(1\ \ 0)\right)$<br>....................<br>$\{1111111\}$ if $\mu<1$<br>$\{1111011\}$ if $\mu\geq 1$ |
| $H(4,1,\mu)$, $\mu\in\mathcal{R}$<br>....................<br>$\mathbf{CB}\neq 0$, $\mathbf{m}(\Omega)>0$,<br>$\tau(\Omega)=\mu$ | $\left(\begin{pmatrix}\mu&1\\1&\mu\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},(1\ \ 0)\right)$<br>....................<br>$\{1111110\}$ if $\mu<0$<br>$\{1111000\}$ if $\mu\geq 0$ |

For arbitrary $\Omega \in \Sigma$, $T \in GT$, put $T(\Omega) = (T(\mathbf{A}), T(\mathbf{B}), T(\mathbf{C}))$. According to the result, mentioned in Section 1, system (1) is $\mathcal{LH}$-stabilizable if and only if there exists a real $\alpha$ such that the matrix $T_3(\alpha)(\mathbf{A})$ has no nonnegative real eigenvalues. However, it is not enough to achieve an arbitrary rate of convergence of solutions to zero (see e. g. the case $H(1,0,-1)$). The table solves this problem in an efficient way. The corresponding criterion says that $\kappa(\Omega, \mathcal{LH}) = -\infty$ (i.e. the upper Lyapunov exponent of the system can be arbitrarily small) if and only if $\exists T \in GT$, for which $T(\mathbf{A})$ has no real eigenvalues. Indeed, the latter is equivalent to the relation $\Omega \in \Sigma^*$, where

$$\begin{aligned}\Sigma^* =& \{\Omega |\, \exists k \in \{-1,0,1\},\, \Omega \in H(2,0,k)\} \\ &\cup \{\Omega |\, \exists \mu \in \mathcal{R},\, \Omega \in H(4,-1,\mu)\}\end{aligned}$$

Then it is readily seen from the table that this, in turn, is equivalent to $\kappa(\Omega, \mathcal{LH}) = -\infty$.

Of particular interest are the following subsets of $\Sigma^*$: 1) $H(2,0,0)$ and $H(2,0,1)$, the union of which was in [5] denoted by **GT II**, and 2) $H(4,-1,\mu)$ with $\mu \geq 1$, the union of which was in [5] denoted by **GT III**. Only the systems (1) with the matrix triples from these classes are $\mathcal{L}H$-stabilizable (with an arbitrarily small Lyapunov exponent), yet *not* $\mathcal{LH}_1$-stabilizable (see also Theorem 5.1 below). Remark that $\mathcal{L}H_1$-stabilization was proved in [5], the property $\kappa(\Omega, \mathcal{LH}) = -\infty$ for **GT II** and **GT III** was justified in [6] and [7], respectively.

In [5, Th. 6.1] it was proved that system (1) with the triple

$$\Omega_\gamma = \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, (\gamma \;\; 1) \right)$$

is not $\mathcal{LH}$-stabilizable if $-1 \leq \gamma \leq 0$. The set of all $\Omega \in \Sigma$, which are $GT$-equivalent to $\Omega_\gamma$ for some $-1 \leq \gamma \leq 0$, was denoted by **GT IV**. In fact, $\Omega_\gamma$ is not a canonical form in the terminology used in the present paper. However, it is easy to check that $\Omega_\gamma \in H(4,1,\mu)$ if $\gamma \in (-1;0]$, where $\mu = -\frac{\gamma}{\sqrt{1-\gamma^2}} \geq 0$, and $\Omega_{-1} \in H(3,1,1)$.

Given $\mu \in \mathcal{R}$ and $u \in \mathcal{LH}$ it is also readily seen from the table that $x(t)$ is a solution of (1) with the canonical triple from $H(4,1,\mu)$ if and only if $\tilde{\mathbf{x}}(t) := e^{\beta t}\mathbf{x}(t)$, is a solution of (1) with the canonical triple from $H(4,1,\mu+\beta)$. Thus, the property $\kappa(\Omega, \mathcal{LH}) > -\infty$ (i.e. not any rate of convergence is possible) for a triple $\Omega$ from $H(4,1,\mu)$, $\mu \in \mathcal{R}$ follows from the fact that the triple $\Omega_0$ is not $\mathcal{LH}$-stabilizable.

The last result of the paper states that the obtained $GT$-classification easily implies an efficient criteria of $\mathcal{LH}_1$-stabilization and $\mathcal{LH}$-stabilization which were proved in [4]. Below one of the results is presented in somewhat different, yet equivalent form. It again follows directly from the table.

*Theorem 5.1:* A triple $\Omega = (A,B,C)$ is not $\mathcal{LH}_1$-stabilizable, but $\mathcal{LH}$-stabilizable if and only if one of the following statements is true:

1) $\mathbf{CB} = 0$, $\mathbf{CAB} \neq 0$, $\operatorname{tr} A \geq 0$;
2) $\mathbf{m}(\Omega) < 0$, $\tau(\Omega) \geq 1$.

Remark that the second condition of Theorem 5.1 is equivalent to the inequalities $\omega(\Omega) > 0$, $\frac{\mathbf{CAB}}{\mathbf{CB}} < \frac{\det \mathbf{A}}{\omega(\Omega)} \leq \operatorname{tr} \mathbf{A}$. Thus, we obtain Theorem 7.4 from [4].

*Remark 5.1:* The offered classification provides an immediate construction of the stabilizing algorithms for the canonical triples in each equivalence class. A stabilizing algorithm for a general triple $\Omega$ can be found explicitly as well if the $GT$-transformation of $\Omega$ to the corresponding canonical triple is known. This transformation can indeed be calculated for any $\Omega$, although the resulting formulas can be cumbersome.

## REFERENCES


[1] Z. Artstein, "Example of stabilization with hybrid feedback", *in Hybrid Systems III: Verification and Control, vol. 1066,* Lecture Notes in Computer Scient., Springer-Verlag, Berlin, 1996, pp. 173–185.

[2] C. Benassi and A. Gavioli, Hybrid stabilization of planar linear systems with one-dimensional outputs, *Systems Control Letters*, vol. 46, 2002, pp. 303-309.

[3] B. Hu, G. Zhai and A. N. Michel, Hybrid static output feedback stabilization of second-order linear time-invariant systems, *Linear Algebra and its Applications*, vol. 351-352, 2002, pp. 475-485.

[4] E. Litsyn, M. Myasnikova, Yu. Nepomnyashchikh and A. Ponosov, Efficient criteria for stabilization of planar linear systems by hybrid feedback controls, *Abstract and Applied Analysis*, vol. 6, 2004, pp. 487-499.

[5] E. Litsyn, Yu. Nepomnyashchikh, and A. Ponosov, Classification of linear dynamical systems in the plane in admitting a stabilizing hybrid feedback control, *Journal on Dynamical and Control Systems*, vol. 6, 2000, pp. 477-501.

[6] E. Litsyn, Yu. Nepomnyashchikh, and A. Ponosov, Control of Lyapunov exponents via Artstein's hybrid feedback algorithm, *Communications Appl. Nonlinear Analysis*, vol. 8, 2001, pp. 1-15.

[7] E. Litsyn, Yu. Nepomnyashchikh, and A. Ponosov, "Control of Lyapunov exponents via hybrid feedback algorithms", *in Proceedings of the 2nd IFAC Symposium on System, Structure and Control*, Oaxaca, Mexico, 2004, pp. 192-197.

[8] E. Sontag, *Mathematical control theory: deterministic finite dimensional systems*, 2nd ed., Springer-Verlag, NY; 1998.