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An Extension of Duality and Hierarchical Decomposition to a Game-Theoretic Framework

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Abstract— **This paper extends some duality results from a standard optimization setup to a noncooperative (Nash) game framework. A coupled constrained Nash game is considered. Solving directly such a coupled Nash game requires coordination among possibly all players. An alternative approach is proposed based on the equivalence to a special constrained optimization problem for an NG-game cost function. By exploiting specific separability properties of the NG-game cost, this duality approach leads to a natural hierarchical decomposition into a lower-level uncoupled Nash game and a higher-level system optimization problem.**

I. INTRODUCTION

A powerful tool for solving standard constrained optimization problems with separable cost function and constraints is the duality approach, [14]. The dual problem can be used to provide lower bounds and in some cases may be easier to solve than the primal problem. For separable problems, the dual problem can be decomposed hierarchically into a set of lower-order optimization problems and a higher-level optimization problem for the Lagrangian multipliers. This decomposition has a computational advantage as the lower-order problems may be analytically tractable. The duality approach and separability has been used successfully in developing congestion control algorithms, where the Lagrangian multipliers pay the role of pricing parameters. In [16], [15] a system problem is defined as a constrained optimization problem with a central separable cost function. Duality is used decompose the problem into a set of decoupled user problems and a network problem.

Recently, as an alternative to the traditional system-wide network optimization, [1]-[4], game theory approaches [20], [13] have been used for optimization and control of networks. In large-scale networks, control decisions are often made by users independently, according to their own performance objectives, [5], [6], and noncooperative game theory is suitable framework.

Game-theoretic models have been employed for flow optimization (congestion control) or for power allocation and control in networks. In flow or congestion control, [15], [17]- [19], each user's utility depends only on its own action and is not coupled to the other users' actions. The coupling appears in the constraints only, specifically the link capacity constraints. This is contrasted with power allocation and control via game theory in wireless networks, [6]-[9], [7], or in optical networks, [10]-[12]. Here each user's utility depends not only on its own action but also on the other users' actions, and hence the utilities are coupled.

In this paper we extend duality results from standard optimization framework to a noncooperative (Nash) game theoretical context.We consider a Nash game with coupled utilities and coupled constraints. Existence conditions for an NE solution are based on an augmented two-argument system cost function, [13]. This cost function is defined in a Nash game (NG)-sense and we call it the NG-game cost function. Due to coupling, solving directly a coupled Nash game requires coordination among possibly all players (Section II). Alternatively, we first show that a coupled Nash game can be converted into an equivalent constrained optimization problem for the NG-game cost function, with respect to the second argument, that admits a fixed-point solution (Section III). We exploit the property that the NG-cost function is separable with respect to the second argument, i.e., in a NG-game sense. Based on this we extend standard duality results, [14], to a NG-game framework. We use duality approach as a natural way to obtain a hierarchical decomposition of the coupled NG game into a lower-level uncoupled Nash game, and a higher-level system optimization problem (Section IV).

We use standard notation. For a two-argument function, $f(\mathbf{u};\mathbf{x})$, $f : R^m \times R^m \to R$, where $\mathbf{u} = (u_1, \ldots, u_m) \in R^m$ and $\mathbf{x} = (x_1, \ldots, x_m) \in R^m$, we denote by $\nabla f(\mathbf{u};\mathbf{x})$ the gradient

$$\nabla f(\mathbf{u};\mathbf{x}) = \left[\begin{array}{cc} \nabla_u f(\mathbf{u};\mathbf{x}) & \nabla_x f(\mathbf{u};\mathbf{x}) \end{array}\right] \quad (1)$$

with $\nabla_u f(\mathbf{u};\mathbf{x}) = \left[\frac{\partial f(\mathbf{u};\mathbf{x})}{\partial u_i}\right]$, $\nabla_x f(\mathbf{u};\mathbf{x}) = \left[\frac{\partial f(\mathbf{u};\mathbf{x})}{\partial x_i}\right]$.

II. NASH NONCOOPERATIVE GAME THEORY

A. *Formulation for Rectangular Action Sets*

In this section we review definitions and results in noncooperative game theory, [13], [20], for rectangular actions sets. A noncooperative (Nash) m-player game is defined where each player minimizes an individual cost function by adjusting its own action, in response to the other players' actions. We give definitions with respect to both individual players' cost functions and an NG-game cost function.

Let $\mathbf{u}$ denote the vector of player actions, and $\mathbf{u}_{-i}$ the vector obtained by deleting the i^{th} element from $\mathbf{u}$,

$$\begin{aligned} \mathbf{u} &= [u_1 \ldots\ldots u_i \ldots\ldots u_m]^T \\ \mathbf{u}_{-i} &= [u_1 \ldots u_{i-1}, u_{i+1} \ldots u_m]^T \end{aligned}$$

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so that $\mathbf{u} = (\mathbf{u}_{-i}, u_i) \in \Omega$, or $\mathbf{u}^* = [u_i^*]$ where Ω is the action space $\Omega = \Omega_1 \times \ldots \times \Omega_m$. Let $\Omega_i = [m_i, M_i]$, $i \in \mathcal{M}$, $\mathcal{M} = \{1, \ldots, m\}$ so that Ω is a rectangular or separable set. The relevant concept is the noncooperative Nash equilibrium (NE), [20], [13].

Definition 1: Consider an m-player game, with each player minimizing the cost function $J_i : \Omega \to R$, over $u_i \in \Omega_i$. A vector $\mathbf{u}^* = [u_i^*] \in \Omega$, or $\mathbf{u}^* = (\mathbf{u}_{-i}^*, u_i^*)$ is called a Nash equilibrium (NE) solution of this game if for every given $\mathbf{u}_{-i}^*$,

$$J_i(\mathbf{u}_{-i}^*, u_i^*) \leq J_i(\mathbf{u}_{-i}^*, x_i) \qquad \forall x_i \in \Omega_i \qquad \forall i \in \mathcal{M}$$

Definition 1 specifies that $\mathbf{u}^*$ is an NE when u_i^* is the solution to the individual optimization problem J_i for player i, given all other players have equilibrium actions, $\mathbf{u}_{-i}^*$. In this sense, each cost function (parameterized by $\mathbf{u}_{-i}^*$) is minimized individually, but the NE solution has to satisfy simultaneously the set of m inequalities in Definition 1.

Existence of an NE solution depends on existence of well-defined reaction curves of all players, that have a common intersection point, [13]. Sufficient conditions for existence are given in Theorem 4.3 in [13]: each cost function is jointly continuous in all arguments and is convex in its "own" argument u_i for every given u_j $j \neq i$, and the action set Ω_i is a compact subset of R^n. Assumption (A.1) below guarantees that an NE solution is inner.

(A.1) $u_i = m_i$ and $u_i = M_i$ are not solutions to the minimization of J_i, i.e., $J_i(\mathbf{u}_{-i}, m_i) > J_i(\mathbf{u}_{-i}, u_i)$ $\forall u_i \neq m_i$ and $J_i(\mathbf{u}_{-i}, M_i) > J_i(\mathbf{u}_{-i}, u_i)$, $\forall u_i \neq M_i$.

Without loss of generality we assume that (A.1) holds and hence an NE solution is inner.

Next we review necessary conditions for an NE solution. Since each cost function J_i is convex in its "own" argument, u_i, there exists a minimizing u_i^*, for any given $\mathbf{u}_{-i}$, such that

$$J_i(\mathbf{u}_{-i}, u_i^*) < J_i(\mathbf{u}_{-i}, u_i), \qquad \forall u_i \neq u_i^*$$

on the compact set Ω_i. Moreover, by (A.1), u_i^* is inner. To find u_i^* we solve the necessary condition

$$\frac{\partial J_i}{\partial u_i}(\mathbf{u}_{-i}, u_i) = 0 \qquad \forall\, i \in \mathcal{M} \tag{2}$$

which defines the reaction curve of the i^{th} player, R_i, [13]. The optimal u_i^* depends on $\mathbf{u}_{-i}$, $u_i^* = R_i(\mathbf{u}_{-i})$, i.e., it is parameterized in $\mathbf{u}_{-i}$. An NE solution to the m-player game, $\mathbf{u}^*$, is a vector solution of the set of m equations, (2), for all $i \in \mathcal{M}$. The procedure for solving the NG-game can be summarized as follows: find a solution to minimization of each individual cost function, J_i, stack the resulting set of m parameterized equations in vector form, and look for a fixed-point solution, $\mathbf{u}^* = [u_i^*]$. We will then denote $J_i^* = J_i(\mathbf{u}_{-i}^*, u_i^*)$ the Nash individual optimal values.

Definition 1 involves a set of m inequalities that have to be satisfied simultaneously. It can be equivalently formulated by using an augmented "system-like" cost function, [13], that we call the NG-game cost function. The NG-game cost function $\widetilde{J}$ is defined as the two-argument function, $\widetilde{J} : \Omega \times \Omega \to R$,

$$\widetilde{J}(\mathbf{u}; \mathbf{x}) := \sum_{i=1}^{m} J_i(\mathbf{u}_{-i}, x_i), \qquad \forall \mathbf{x} \in \Omega \tag{3}$$

with $\mathbf{x} = [x_1 \ldots x_i \ldots x_m]^T$, and $\mathbf{u}$, $\mathbf{u}_{-i}$ defined as before.

The following definition, given with respect to the NG-game cost function, is equivalent to Definition 1.

Definition 2: Consider an m-player game, with each player minimizing the cost function $J_i : \Omega \to R$, over $u_i \in \Omega_i$. Then a vector $\mathbf{u}^* \in \Omega$ is called a NE solution of this game if its NG-game cost function $\widetilde{J}$, (3), satisfies

$$\widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) \leq \widetilde{J}(\mathbf{u}^*; \mathbf{x}) \qquad \forall \mathbf{x} \in \Omega$$

Equivalently, for every given $\mathbf{u}_{-i}^*$,

$$\sum_{i=1}^{m} J_i(\mathbf{u}_{-i}^*, u_i^*) \leq \sum_{i=1}^{m} J_i(\mathbf{u}_{-i}^*, x_i) \qquad \forall \mathbf{x} \in \Omega$$

If $\mathbf{u}^*$ is an NE in the sense of Definition 1, it follows immediately that $\mathbf{u}^*$ satisfies also Definition 2. Conversely, it can be shown by using a contradiction argument that, if $\mathbf{u}^*$ is an NE in the sense of Definition 2, $\mathbf{u}^*$ satisfies all component-wise inequalities in Definition 1, [13].

The NG-cost function $\widetilde{J}$, (3), is separable in the second argument, $\mathbf{x}$, for any given first argument, $\mathbf{u}$, which we call separable in a NG-game sense. This property will be instrumental in the following developments. Theorem 4.3 in [13], gives conditions for existence of an NE solution with respect to the cost functions J_i, and (2) are necessary conditions. Since we will be using the NG-cost function equivalence, we reformulate this result (stated as Theorem 1) with respect to the NG-game cost function $\widetilde{J}$, and prove it for completeness.

Theorem 1: Let $\Omega = \Omega_1 \times \ldots \times \Omega_m$, with Ω_i a closed, bounded and convex subset of R. For each $i \in \mathcal{M}$ the cost functional $J_i : \Omega \to R$ are continuous on Ω and strictly convex in u_i, for every $u_j \in \Omega_j$, $j \in \mathcal{M}, j \neq i$. Then the associated Nash game with NG-game cost function $\widetilde{J}$ (3) admits a Nash equilibrium (NE) solution.

Under (A.1), an NE solution $\mathbf{u}$ satisfies the following necessary conditions with respect to the NG-game cost $\widetilde{J}$,

$$\nabla_x \widetilde{J}(\mathbf{u}; \mathbf{x}) \Big|_{\mathbf{x}=\mathbf{u}} = 0 \tag{4}$$

where the notation used denotes a fixed-point solution.

Proof:

We use arguments similar to those in Theorem 4.4, [13]. Since $\widetilde{J}$, (3), is separable in NG-game sense, i.e., separable in the second argument, $\mathbf{x}$, using (3) we see that $\nabla_x \widetilde{J}(\mathbf{u}; \mathbf{x})$ is given as

$$\nabla_x \widetilde{J}(\mathbf{u}; \mathbf{x}) = \left[\begin{array}{ccc} \frac{\partial J_1(\mathbf{u}_{-1}, x_1)}{\partial x_1} & \ldots & \frac{\partial J_m(\mathbf{u}_{-m}, x_m)}{\partial x_m} \end{array} \right] \tag{5}$$

for any given $\mathbf{u}$. From (5), it follows that the Hessian of $\widetilde{J}$ with respect to $\mathbf{x}$, $\nabla_{xx}^2 \widetilde{J}(\mathbf{u}; \mathbf{x}) = \left[\frac{\partial^2 \widetilde{J}(\mathbf{u};\mathbf{x})}{\partial x_i\, \partial x_j} \right]$ is a diagonal matrix, with the i^{th} diagonal element given as

$$\left[\nabla^2_{xx}\widetilde{J}(\mathbf{u};\mathbf{x})\right]_{(i,i)} = \frac{\partial^2 J_i(\mathbf{u}_{-i}, x_i)}{\partial x_i^2}$$

By strict convexity of each J_i with respect to its argument it follows that for each given $\mathbf{u}_{-i}$, $\frac{\partial^2 J_i(\mathbf{u}_{-i},x_i)}{\partial x_i^2} > 0$, $\forall x_i \in \Omega_i$. Hence, $\nabla^2_{xx}\widetilde{J}(\mathbf{u};\mathbf{x})$ is positive definite, i.e., $\widetilde{J}$ is strictly convex with respect to its second argument $\mathbf{x}$, for every given $\mathbf{u}$ with $(\mathbf{u},\mathbf{x}) \in \Omega \times \Omega$. Therefore, there exists an $\mathbf{x}^*$ minimizing $\widetilde{J}(\mathbf{u};\mathbf{x})$, over $\mathbf{x}$, for any given $\mathbf{u}$, and we can introduce the reaction set of the game, [13],

$$\Psi(\mathbf{u}) = \{\mathbf{v} \in \Omega | \widetilde{J}(\mathbf{u};\mathbf{v}) < \widetilde{J}(\mathbf{u};\mathbf{x}),\, \forall \mathbf{x} \in \Omega\} \tag{6}$$

so that $\mathbf{x}^* \in \Psi(\mathbf{u})$. The above properties of continuity and convexity of $\widetilde{J}$ on a closed compact set, imply via a fixed-point theorem argument [13], that the reaction set of the game $\Psi(\mathbf{u})$ has a fixed point $\mathbf{u}^*$ such that

$$\mathbf{u}^* \in \Psi(\mathbf{u}^*)$$

Using (6), we see that $\mathbf{u}^*$ satisfies

$$\widetilde{J}(\mathbf{u}^*;\mathbf{u}^*) < \widetilde{J}(\mathbf{u}^*;\mathbf{x}),\, \forall \mathbf{x} \in \Omega \tag{7}$$

i.e., in effect $\mathbf{x}^*$=$\mathbf{u}^*$ minimizes $\widetilde{J}(\mathbf{u};\mathbf{x})$, over $\mathbf{x} \in \Omega$. We write this compactly as

$$\mathbf{u}^* = \arg\left\{\ \min\nolimits_{\mathbf{x}\in\Omega} \widetilde{J}(\mathbf{u};\mathbf{x})\Big|_{\mathbf{x}=\mathbf{u}}\ \right\} \tag{8}$$

and

$$\widetilde{J}(\mathbf{u}^*;\mathbf{u}^*) = \min_{\mathbf{x}\in\Omega} \widetilde{J}(\mathbf{u};\mathbf{x})\Big|_{\mathbf{x}=\mathbf{u}} \tag{9}$$

The above compact notation in (8, 9), denotes minimization of $\widetilde{J}$, with respect to the second argument $\mathbf{x}$ as in (6), followed by solving for a fixed-point solution. This is realized by setting $\mathbf{x} = \mathbf{u}$, and solving $\mathbf{u} = \Psi(\mathbf{u})$ for a solution $\mathbf{u}^*$. From (7), by Definition 2, $\mathbf{u}^*$ is an NE solution to the m-player game. Obviously the individual components of $\mathbf{u}^*$ constitute an NE solution in the sense of Definition 1, and by (A.1), it is inner.

Next we consider the necessary conditions. In order to find an $\mathbf{x}^*$ minimizing $\widetilde{J}(\mathbf{u};\mathbf{x})$, over $\mathbf{x}$ as in (6), we need to solve the necessary condition

$$\nabla_x \widetilde{J}(\mathbf{u};\mathbf{x}) = 0 \tag{10}$$

This will give $\mathbf{x}^*$ parameterized by $\mathbf{u}$, $\mathbf{x}^* \in \Psi(\mathbf{u})$. Since an NE solution $\mathbf{u}$ is a fixed point of $\Psi(\mathbf{u})$, in effect we need to solve (10) for a fixed point solution $\mathbf{x} = \mathbf{u}$, i.e., in a compact notation

$$\nabla_x \widetilde{J}(\mathbf{u};\mathbf{x})\ \Big|_{\mathbf{x}=\mathbf{u}} = 0$$

■

Remark 1: The foregoing procedure involves minimization of the NG-game cost function, $\widetilde{J}$, with respect to the second argument $\mathbf{x}$, which is a standard minimization of a function parameterized by $\mathbf{u}$. To find an NE solution, we just impose that the solution of this minimization is a *fixed point* defined as above. Using (5), we see that (4) is given component-wise as

$$\frac{\partial J_i}{\partial x_i}(\mathbf{u}_{-i}, x_i)\ \Big|_{x_i=u_i} = 0, \qquad \forall\, i \in \mathcal{M}$$

For this set of equations we look for a fixed-point solution, i.e., we set $x_i = u_i$, so that we can write

$$\frac{\partial J_i}{\partial u_i}(\mathbf{u}_{-i}, u_i) = 0, \qquad \forall\, i \in \mathcal{M}$$

which is exactly (2). Hence the two-argument form (4) is equivalent to the component-wise form, (2).

B. NE Existence for Coupled Action Sets

For the case when the action spaces (or the constraint sets) are coupled and not rectangular, the players' decision variables are coupled. Thus there is a single set $\overline{\Omega} \subset R^m$, to which the m-tuple $\mathbf{u} = (u_1, \ldots, u_i, \ldots, u_m)$ belongs. There are no separate action sets, $\overline{\Omega}_j$ and $\overline{\Omega}_j$, $j \neq i$, from which player i and player j can choose independently their actions. We will review an existence result, (Theorem 4.4 in [13]), for NG games with coupled constraints, or coupled NG games. The NE concept can be defined as in Definition 1, this time using the projection set $\overline{\Omega}_i(\mathbf{u}^*_{-i})$ instead, i.e.,

$$J_i(\mathbf{u}^*_{-i}, u_i^*) \leq J_i(\mathbf{u}^*_{-i}, x_i) \quad \forall x_i \in \overline{\Omega}_i(\mathbf{u}^*_{-i})\, \forall i \in \mathcal{M} \tag{11}$$

where $\overline{\Omega}_i(\mathbf{u}^*_{-i})$ is a subset obtained by the projection

$$\overline{\Omega}_i(\mathbf{u}^*_{-i}) := \{x_i \in R \quad | \quad (\mathbf{u}^*_{-i}, x_i) \in \overline{\Omega}\}$$

for any given $\mathbf{u}^*_{-i}$. Based on a fixed point theorem argument, Theorem 4.4 in [13] gives sufficient conditions for existence of an NE solution for games with coupled constraint sets. We restate this result here as Theorem 2.

Theorem 2: Let $\overline{\Omega}$ be a closed, bounded and convex subset of R^m, and for each $i \in \mathcal{M}$ the cost functional $J_i : \overline{\Omega} \to R$ be continuous on $\overline{\Omega}$ and convex in u_i, for every $u_j \in \overline{\Omega}_j$, $j \in \mathcal{M}, j \neq i$. Then the associated Nash game with NG-cost function $\widetilde{J}$ admits a Nash equilibrium (NE) solution.

■

Due to coupling, solving directly for such a NE solution requires coordination among possibly all players. In the next section, we exploit the fact that NG games can be equivalently defined with respect to the NG-cost function which is separable in an NG-sense. Afterwards we can use duality approach as a natural way to obtain a hierarchical decomposition of the coupled NG game.

III. LAGRANGIAN EXTENSION FOR COUPLED CONSTRAINED NASH GAMES

In this section we formulate a Nash game (NG) game with coupled constraints. Without loss of generality we consider only inequality constraints, since equality constraints can be treated similarly. Consider a Nash game for m players, each player minimizing the individual cost function $J_i : \Omega \to R$, $\Omega = \Omega_1 \times \ldots \times \Omega_m$, subject to R coupled inequality constraints

$$g_r(\mathbf{u}) \leq 0, \qquad\qquad r = 1, \ldots, R$$

where $g_r(\mathbf{u}) = g_r(\mathbf{u}_{-i}, x_i)$, with $\mathbf{u} = (\mathbf{u}_{-i}, x_i) \in \Omega$. Compactly, we write

$$\mathbf{g}(\mathbf{u}) \leq 0 \tag{12}$$

where $\mathbf{g}(\mathbf{u}) = \left[\begin{array}{c} g_1(\mathbf{u}) \dots g_R(\mathbf{u}) \end{array} \right]^T$. So $\mathbf{u} \in \overline{\Omega}$, where the overall action set $\overline{\Omega}$ is coupled, due to coupled constraints,

$$\overline{\Omega} = \{\mathbf{u} \in \Omega |\, g_r(\mathbf{u}) \leq 0, \quad r = 1, \dots, R\} \tag{13}$$

We use the existence results of NE solution for such games (Theorem 2), and the equivalent formulation with respect to the NG-cost function, (3). Similar to (3), we augment the constraints g_r in an equivalent two-argument form, $\widetilde{g}_r$,

$$\widetilde{g}_r(\mathbf{u}; \mathbf{x}) = \sum_{i=1}^{m} g_r(\mathbf{u}_{-i}, x_i), \quad r = 1, \dots, R \tag{14}$$

From (14) we define in a vector notation,

$$\widetilde{\mathbf{g}}(\mathbf{u}; \mathbf{x}) \quad = \sum_{i=1}^{m} \mathbf{g}(\mathbf{u}_{-i}, x_i) \tag{15}$$

with $\mathbf{g}(\mathbf{u}_{-i}, x_i) = \left[\begin{array}{c} g_1(\mathbf{u}_{-i}, x_i) \dots g_R(\mathbf{u}_{-i}, x_i) \end{array} \right]^T$

It can be immediately seen that this form keeps the separability property in the second argument $\mathbf{x}$, i.e., in a NG-game sense. Moreover, due to coupled identical constraints for all players, all components in $\widetilde{\mathbf{g}}$ are equal to $\mathbf{g}$.

We next show that such a coupled NG game can be formulated as a constrained minimization of the NG-cost function, with respect to the second argument, and fixed-point solution.

Lemma 1: Consider a Nash game for m players, each minimizing the individual cost function $J_i(\mathbf{u}_{-i}, u_i)$, $\mathbf{u} \in \Omega \subset R^m$, subject to coupled constraints $g_r(\mathbf{u}) \leq 0$, $r = 1, \dots, R$. Assume that Theorem 2 is satisfied, so that an NE solution $\mathbf{u}^*$ exists, and we denote $J_i^* = J_i(\mathbf{u}_{-i}^*, u_i^*)$ the Nash individual optimal values.

Equivalently, $\mathbf{u}^*$ is a solution for the constrained minimization of the NG-game cost function $\widetilde{J}$ (3),

$$\widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) \leq \widetilde{J}(\mathbf{u}^*; \mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \, \widetilde{\mathbf{g}}(\mathbf{u}^*; \mathbf{x}) \leq 0 \tag{16}$$

with $\widetilde{\mathbf{g}}(\mathbf{u}^*; \mathbf{u}^*) \leq 0$, where the constraint $\widetilde{\mathbf{g}}(\mathbf{u}^*; \mathbf{x})$ is given in (15), and the optimal NG-game cost is

$$\widetilde{J}^* = \widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) = \sum_i J_i^* \tag{17}$$

Proof:

We give a proof for a single coupled inequality constraint,

$$g(\mathbf{u}) \leq 0, \quad g(\mathbf{u}) = g(\mathbf{u}_{-i}, x_i), \quad \mathbf{u} = (\mathbf{u}_{-i}, x_i) \in \Omega$$

since extension to the multiple inequality case can be done immediately by appropriately using vector inequalities. Using (13), we see that the projection set, (11), in Theorem 2 is

$$\overline{\Omega}_i(\mathbf{u}_{-i}^*) = \{x_i \in \Omega_i |\, g(\mathbf{u}_{-i}^*, x_i) \leq 0\} \tag{18}$$

Then, for an NE solution $\mathbf{u}^*$, we have (cf. Definition 1 and (11)), for all $i = 1, \dots, m$,

$$J_i(\mathbf{u}_{-i}^*, u_i^*) \leq J_i(\mathbf{u}_{-i}^*, x_i), \quad \forall x_i \in \Omega_i, \, g(\mathbf{u}_{-i}^*, x_i) \leq 0 \tag{19}$$

with $g(\mathbf{u}_{-i}^*, u_i^*) \leq 0$. Note that $g(\mathbf{u}_{-i}^*, u_i^*) \leq 0$ is equivalent to $\widetilde{g}(\mathbf{u}^*; \mathbf{u}^*) \leq 0$. Using (3), it can be seen that (19) implies that

$$\widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) \leq \widetilde{J}(\mathbf{u}^*; \mathbf{x}), \qquad \forall \mathbf{x} \in \Omega, \, \widetilde{g}(\mathbf{u}^*; \mathbf{x}) \leq 0 \tag{20}$$

with $\widetilde{g}(\mathbf{u}^*; \mathbf{u}^*) \leq 0$, and $\widetilde{g}(\mathbf{u}^*; \mathbf{x}) : \Omega \times \Omega \to R$ defined as in (15). Conversely, we show by a contradiction argument that if $\mathbf{u}^*$ is an NE in the sense of (20), then $\mathbf{u}^*$ satisfies (19). Therefore, assume that $\mathbf{u}^*$ is an NE in the sense of (20) but not in the sense of (19). Then, it follows that there exists an $i_0 \in \mathcal{M}$ and some $\bar{x}_{i_0} \in \Omega_{i_0}$, with

$$g(\mathbf{u}_{-i_0}^*, \bar{x}_{i_0}) \leq 0 \tag{21}$$

such that

$$J_{i_0}(\mathbf{u}_{-i_0}^*, \bar{x}_{i_0}) < J_{i_0}(\mathbf{u}_{-i_0}^*, u_{i_0}^*) \tag{22}$$

Then adding $\sum_{i \neq i_0} J_i(\mathbf{u}^*)$, on both sides of (22) yields

$$\widetilde{J}(\mathbf{u}^*; \bar{\mathbf{x}}_0) < \widetilde{J}(\mathbf{u}^*; \mathbf{u}^*), \quad for \quad \bar{\mathbf{x}}_0 = (\mathbf{u}_{-i_0}^*, \bar{x}_{i_0}) \in \Omega$$

Now from (21) and $g(\mathbf{u}_{-i}^*, u_i^*) \leq 0$, it follows that

$$\widetilde{g}(\mathbf{u}^*; \bar{\mathbf{x}}_0) = \sum_{i \neq i_0} g(\mathbf{u}_{-i}^*, u_i^*) + g(\mathbf{u}_{-i_0}^*, \bar{x}_{i_0}) \leq 0, \, \forall i \neq i_0$$

so that $\mathbf{x}_0$ is feasible. The two foregoing inequalities imply that $\mathbf{u}^*$ is not an NE solution in the sense of (20), contradicting the hypothesis. Hence, (20) is equivalent to (19), which is (16) for a single constraint and (17) follows. ■

Remark 2: Using the same compact notation as in (9), from (16) we can write

$$\widetilde{J}(\mathbf{u}^*; \mathbf{u}^*) = \left[\begin{array}{c} \min_{\mathbf{x} \in \Omega, \widetilde{\mathbf{g}}(\mathbf{u}; \mathbf{x}) \leq 0} \widetilde{J}(\mathbf{u}; \mathbf{x}) \end{array} \right] |_{\mathbf{x} = \mathbf{u}} \tag{23}$$

In the following we use Lemma 1 and standard optimality conditions for constrained optimization [14], applied to the NG-cost function $\widetilde{J}$. For standard constrained optimization problems, the set of optimality conditions involves a set of auxiliary variables called Lagrange multipliers, [14], which form the basis of duality results. Accordingly, we introduce an augmented Lagrangian function and augmented constraints in a two-argument form,

$$\widetilde{L}(\mathbf{u}; \mathbf{x}; \mu) = \widetilde{J}(\mathbf{u}; \mathbf{x}) + \mu^T \, \widetilde{\mathbf{g}}(\mathbf{u}; \mathbf{x}) \tag{24}$$

where μ is a Lagrange multiplier vector. The next result gives necessary conditions in terms of Lagrange multipliers.

Lemma 2: Let $\mathbf{u}$ be a NE solution of the Nash game with individual cost function $J_i(\mathbf{u}_{-i}, u_i)$, $i = 1, \dots, m$, $\mathbf{u} \in \Omega \subset R^m$, subject to the coupled constraints $g_r(\mathbf{u}) \leq 0$, $r = 1, \dots, R$. J_i and g_r are continuously differentiable functions, and $\mathcal{A}(\mathbf{u}) = \{r | g_r(\mathbf{u}) = 0\}$ denotes the set of active constraints at $\mathbf{u}$.

Consider the two-argument Lagrangian function $\widetilde{L}$, (24). Then there exist unique μ^*, $\mu_r^* \geq 0$, such that

$$\nabla_x \widetilde{L}(\mathbf{u}; \mathbf{x}; \mu^*) \, \Big|_{\mathbf{x} = \mathbf{u}} = 0$$

with $\mu_r^* = 0, \quad \forall r \notin \mathcal{A}(\mathbf{u})$, i.e.,

$$\mu_r^* \, g_r(\mathbf{u}) = 0, \qquad \forall r = 1, \dots, R$$

Proof: From Lemma 1, an NE solution to the game $\mathbf{u}$ can be equivalently found by solving the constrained minimization problem (16), or (23), with respect to the second argument $\mathbf{x}$ and finding a fixed-point solution. Since the first step is a standard minimization for the augmented NG-game cost, $\widetilde{J}(\mathbf{u};\mathbf{x})$ (parameterized in $\mathbf{u}$), it follows that necessary conditions are expressed in terms of the Lagrangian

$$\widetilde{L}(\mathbf{u};\mathbf{x};\mu) := \widetilde{J}(\mathbf{u};\mathbf{x}) + \sum_{r=1}^{R} \mu_r \, \widetilde{g}_r(\mathbf{u};\mathbf{x})$$

with $\mu = [\mu_1, \ldots, \mu_r, \ldots, \mu_R]^T$ and $\widetilde{g}_r$ defined as in (14). Recalling (15), we can write $\widetilde{L}$ as in (24). The necessary conditions for a $\mathbf{x}$ (parameterized by $\mathbf{u}$) to be a minimizing solution, is that there exist Lagrange multipliers, μ^* (parameterized by $\mathbf{u}$) satisfying (cf. Proposition 3.3.1 in [14]),

$$\nabla_x \widetilde{L}(\mathbf{u};\mathbf{x};\mu^*) \;\; = 0$$

and

$$\mu_r^* \, \widetilde{g}_r(\mathbf{u};\mathbf{x}) = 0, \qquad \forall r = 1, \ldots, R$$

which will give a set $\mathbf{x} = \Psi(\mathbf{u})$. Since for an NE solution we search for a fixed-point solution from this set, $\mathbf{u} = \Psi(\mathbf{u})$, we express this compactly as

$$\nabla_x \widetilde{L}(\mathbf{u};\mathbf{x};\mu^*) \;\Big|_{\mathbf{x}=\mathbf{u}} = 0 \tag{25}$$

and

$$\mu_r^* \, \widetilde{g}_r(\mathbf{u};\mathbf{u}) = 0, \qquad \forall r = 1, \ldots, R$$

As before, the procedure involves minimization of $\widetilde{L}$ with respect to $\mathbf{x}$, parameterized by $\mathbf{u}$, and then looking for a fixed-point solution $\mathbf{x} = \mathbf{u}$. ■

Remark 3: Note that alternatively, the proof can be based on a penalized cost function for each player, and transforming into an unconstrained penalized Nash game. Assuming existence of an NE solution for this game and using the necessary conditions in terms of a set of m equations, it can be shown that the resulting system of m equations has a fixed point solution.

Similarly the following sufficient result can be proved.

Lemma 3: Consider the Nash game with individual cost function $J_i(\mathbf{u}_{-i}, u_i)$, $i = 1, \ldots, m$, $\mathbf{u} \in \Omega \subset R^m$, subject to the inequality constraints $g_r(\mathbf{u}) \leq 0$, $r = 1, \ldots, R$. J_i and g_r are continuously differentiable functions. Let $\mathbf{u}^*$ be a feasible point that together with a vector μ, satisfies $\mu_r \geq 0$, $r = 1, \ldots, R$, with $\mu_r = 0$, $\forall r \notin \mathcal{A}(\mathbf{u})$, $\quad \mathcal{A}(\mathbf{u}) = \{r | g_r(\mathbf{u}) = 0\}$, or, succinctly,

$$\mu_r \, g_r(\mathbf{u}) = 0, \; r = 1, \ldots, R \tag{26}$$

and that minimizes the Lagrangian function $\widetilde{L}(\mathbf{u};\mathbf{x};\mu)$, (24), over $\mathbf{x} \in \Omega$, as a fixed point, i.e., as in (8),

$$\mathbf{u}^* = \arg \Big\{ \; \Big[\; \min_{x \in \Omega} \widetilde{L}(\mathbf{u};\mathbf{x};\mu) \; \Big] \, |_{\mathbf{x}=\mathbf{u}} \; \Big\} \tag{27}$$

Then $(\mathbf{u};\mathbf{x}) = (\mathbf{u}^*;\mathbf{u}^*)$ is a NE solution of the game, in the sense of (16), i.e.,

$$\widetilde{J}(\mathbf{u}^*;\mathbf{u}^*) \leq \widetilde{J}(\mathbf{u}^*;\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \, \widetilde{\mathbf{g}}(\mathbf{u}^*;\mathbf{x}) \leq 0$$

Proof: Since $\mathbf{u}^*$ is a feasible point satisfying (26), using (15) it follows that $\mu^T \, \widetilde{\mathbf{g}}(\mathbf{u}^*;\mathbf{u}^*) = 0$. Therefore, using (24), we have

$$\widetilde{L}(\mathbf{u}^*;\mathbf{u}^*;\mu) = \widetilde{J}(\mathbf{u}^*;\mathbf{u}^*)$$

Also from (27), $\mathbf{u}^*$ minimizes the augmented Lagrangian $\widetilde{L}(\mathbf{u};\mathbf{x};\mu)$, with respect to the second argument and is a fixed point-solution. Then similar to (7) we have

$$\widetilde{L}(\mathbf{u}^*;\mathbf{u}^*;\mu) \leq \widetilde{L}(\mathbf{u}^*;\mathbf{x};\mu), \quad \forall \mathbf{x} \in \Omega$$

SInce $\mu \geq 0$, using (24), on the right-hand side of the foregoing, if follows that, for any $\mathbf{x}$ such that $\widetilde{\mathbf{g}}(\mathbf{u}^*;\mathbf{x}) \leq 0$, we can write

$$\widetilde{J}(\mathbf{u}^*;\mathbf{u}^*) \leq \widetilde{J}(\mathbf{u}^*;\mathbf{x}), \quad \forall x \in \Omega, \quad \widetilde{\mathbf{g}}(\mathbf{u}^*;\mathbf{x}) \leq 0$$

i.e., (16) holds. Then using Lemma 1, the foregoing is equivalent to $\mathbf{u}^*$ being an NE game solution. ■

Remark 4: Note that if J_i and g_r are differentiable convex functions and $\Omega = R^m$, the Lagrangian function $\widetilde{L}(\mathbf{u};\mathbf{x};\mu)$ is convex with respect to $\mathbf{x}$, so the Lagrangian minimization is equivalent to the first order necessary condition. Thus in the presence of convexity the first order optimality conditions are also sufficient.

IV. DUALITY EXTENSION AND HIERARCHICAL DECOMPOSITION

In this section we consider duality extension for coupled NE games. From Lemma 1, solving a Nash game with cost functions J_i, $i = 1, m$, is equivalent to minimizing the NG-game cost function $\widetilde{J}$, with respect to second argument for a fixed-point solution. We introduce a dual cost function, related to the minimization of the associated Lagrangian function, similar to standard optimization, [14]. For convex inequality constraints, duality enables hierarchical decomposition into a lower-level modified Nash game and a higher-level coordination problem, with a Stackelberg game [13] (leader-follower) interpretation.

Recall the Lagrangian function $\widetilde{L}$, (24) associated with the coupled NE game, and its minimization as in (27). For each μ the resulting fixed-point solution $\mathbf{x} = \mathbf{u}$, will be a function of μ, $\mathbf{u}^* = \mathbf{u}^*(\mu)$. Consider the *dual cost function* $D(\mu)$ defined as

$$D(\mu) := \Big[\; \min_{\mathbf{x} \in \Omega} \widetilde{L}(\mathbf{u};\mathbf{x};\mu) \; \Big]\Big|_{\mathbf{x}=\mathbf{u}} \tag{28}$$

where $\widetilde{\mathbf{g}}(\mathbf{u};\mathbf{u}) \leq 0$, i.e.,

$$D(\mu) = \widetilde{L}(\mathbf{u}^*;\mathbf{u}^*;\mu)$$

The *dual NG problem* can be defined as *maximizing* $D(\mu)$ subject to $\mu \geq 0$, with the dual optimal value defined as

$$D^* = \max_{\mu \geq 0} D(\mu) \tag{29}$$

Finally the following results characterizes the primal and dual optimal solution pairs.

Theorem 3: $(\mathbf{u}^*;\mu^*)$ is an optimal NE solution-Lagrange multiplier pair in the sense of (16) and (29), if and only if

(NG feasibility)

$$\mathbf{u}^* \in \Omega \qquad \widetilde{g}_r(\mathbf{u}^*;\mathbf{u}^*) \leq 0, \quad r = 1, \ldots, R$$

(Dual feasibility)

$$\mu^* \geq 0$$

(Lagrangian optimality)

$$\mathbf{u}^* = \arg\left\{ \left[\min_{x\in\Omega} \widetilde{L}(\mathbf{u};\mathbf{x};\mu) \right]|_{\mathbf{x}=\mathbf{u}} \right\}$$

(Complementary slackness)

$$\mu_r^* \widetilde{g}_r(\mathbf{u}^*;\mathbf{u}^*) = 0, \quad r = 1,\ldots,R$$

Proof: If $(\mathbf{u}^*;\mu^*)$ is an optimal NE solution-Lagrange multiplier pair, then $\mathbf{u}^*$ is primal feasible and μ^* is dual feasible and the first two relations follow directly. The last two relations follow from Lemma 2.

Conversely, for sufficiency, using Lagrangian optimality, we obtain

$$\widetilde{L}(\mathbf{u}^*;\mathbf{u}^*,\mu^*) = \left[\min_{\mathbf{x}\in\Omega} \widetilde{L}(\mathbf{u};\mathbf{x};\mu^*) \right]\Big|_{\mathbf{x}=\mathbf{u}}$$

so that

$$\widetilde{L}(\mathbf{u}^*;\mathbf{u}^*,\mu^*) \le \widetilde{L}(\mathbf{u}^*;\mathbf{x};\mu^*), \quad \forall \mathbf{x}\in\Omega$$

Recall that $\widetilde{L}$ is defined as in (24), so that, using the complementary slackness condition, we can write

$$\widetilde{L}(\mathbf{u}^*;\mathbf{u}^*,\mu^*) = \widetilde{J}(\mathbf{u}^*;\mathbf{u}^*)$$

Using this together with (24) into the foregoing inequality we can write

$$\widetilde{J}(\mathbf{u}^*;\mathbf{u}^*) \le \widetilde{J}(\mathbf{u}^*;\mathbf{x}), \quad \forall x\in\Omega, with \widetilde{\mathbf{g}}(\mathbf{u}^*;\mathbf{x}) \le 0$$

Therefore (16) holds and using Lemma 1, it follows that $\mathbf{u}^*$ is an optimal NE game solution and

$$\widetilde{J}^* = \widetilde{J}(\mathbf{u}^*;\mathbf{u}^*)$$

On the other hand using (28) evaluated at μ^* and the foregoing relations, yields

$$D(\mu^*) = \left[\min_{\mathbf{x}\in\Omega} \widetilde{L}(\mathbf{u};\mathbf{x};\mu^*) \right]\Big|_{\mathbf{x}=\mathbf{u}}$$

$$= \widetilde{L}(\mathbf{u}^*;\mathbf{u}^*,\mu^*) = \widetilde{J}(\mathbf{u}^*;\mathbf{u}^*)$$

Then, using the definition of the optimal dual cost D^*, (29),

$$D^* \ge D(\mu^*) = \widetilde{J}(\mathbf{u}^*;\mathbf{u}^*) = \widetilde{J}^*$$

■

Remark 5: If a Lagrange multiplier μ is known then all optimal NE solutions $(\mathbf{u}^*,\mathbf{u}^*)$ can be found by minimizing the Lagrangian over $\mathbf{x}\in\Omega$, in the fixed-point sense as (27). However among those solutions $\mathbf{u}^*$, there may be vectors that do not satisfy the NG-feasibility condition $\mathbf{g}(\mathbf{u}^*) \le 0$, so this has to be checked.

Recall that both the coupled Nash cost and the constraints are separable in the second argument, which we will exploit in the following. We show that the dual NG cost function $D(\mu)$ can be decomposed and equivalently found by solving a modified uncoupled Nash game.

Corollary 1: Consider the coupled Nash game with individual cost function $J_i(\mathbf{u}_{-i},u_i)$, $i=1,\ldots,m$, $\mathbf{u}\in\Omega\subset R^m$, $\Omega=\Omega_1\times\ldots\times\Omega_m$, subject to the inequality constraint $g(\mathbf{u}) \le 0$. J_i and g are continuously differentiable and convex functions.

The dual cost function $D(\mu)$ can be decomposed as

$$D(\mu) = \sum_{i=1}^{m} \left[\min_{x_i\in\Omega_i} L_i(\mathbf{u}_{-i},x_i,\mu) \right]|_{x_i=u_i} \quad (30)$$

$$= \sum_{i=1}^{m} L_i(\mathbf{u}_{-i}^*(\mu), u_i^*(\mu), \mu)$$

where $\mathbf{u}^*(\mu) = [u_i^*(\mu)]$ is a a fixed-point solution to the set of minimizations and

$$L_i(\mathbf{u}_{-i},x_i,\mu) = J_i(\mathbf{u}_{-i},x_i) + \mu\, g(\mathbf{u}_{-i},x_i) \quad (31)$$

Alternatively, $D(\mu)$ can be obtained by solving the uncoupled Nash game with cost functions L_i, (31).

Proof:

Using (3) and (15), we can write the Lagrangian $\widetilde{L}$, (24), as

$$\widetilde{L}(\mathbf{u};\mathbf{x};\mu) = \sum_{i=1}^{m} L_i(\mathbf{u}_{-i},x_i,\mu) \quad (32)$$

where L_i are defined as in (31). Recall the necessary conditions of Lemma 2, with respect to the Lagrangian $\widetilde{L}$,

$$\nabla_x \widetilde{L}(\mathbf{u};\mathbf{x};\mu)|_{\mathbf{x}=\mathbf{u}} = 0, \qquad \mathbf{u}\in\Omega$$

i..e, solving for a fixed point solution in the equation

$$\nabla_x \widetilde{L}(\mathbf{u};\mathbf{x};\mu) = 0 \quad (33)$$

Component-wise, (33) is equivalent to a set of m equations

$$\frac{\partial \widetilde{L}(\mathbf{u};\mathbf{x};\mu)}{\partial x_i} = 0, \qquad i = 1,m$$

to be solved for fixed-point solution, or using (32)

$$\sum_{j=1}^{m} \frac{\partial}{\partial x_i} L_j(\mathbf{u}_{-j},x_j;\mu) = 0, \qquad i=1,m$$

Due to separability with respect to $\mathbf{x}$, we see that the foregoing are equivalent to

$$\frac{\partial}{\partial x_i} L_i(\mathbf{u}_{-i},x_i;\mu) = 0, \qquad i=1,m \quad (34)$$

which are the necessary conditions for minimizing L_i with respect to x_i. Since J_i and g_r are convex, this is also sufficient for minimizing L_i, (31). Therefore, component-wise, the same minimizing solution is found by using (33) or (34). Now for the value functional we have firstly

$$\min_{\mathbf{x}\in\Omega} \widetilde{L}(\mathbf{u};\mathbf{x};\mu) = \min_{\mathbf{x}\in\Omega} \sum_{i=1}^{m} L_i(\mathbf{u}_{-i},x_i,\mu), \qquad \mathbf{x}\in\Omega$$

with L_i given in (31), and $\Omega=\Omega_1\times\ldots\times\Omega_m$. Using again the NG-separability property with respect to $\mathbf{x}=[x_i]$ on the right-hand side of the above it follows that for any given $\mathbf{u}$;

$$\min_{\mathbf{x}\in\Omega} \widetilde{L}(\mathbf{u};\mathbf{x};\mu) = \sum_{i=1}^{m} \min_{x_i\in\Omega_i} L_i(\mathbf{u}_{-i},x_i,\mu) \quad (35)$$

Note that after minimization in each of the m subproblems on the right-hand side we obtain x_i^* as function of $\mathbf{u}_{-i}$, that we denote $x_i^* = x_i(\mathbf{u}_{-i})$, so that

$$x_i(\mathbf{u}_{-i}) = \arg\min_{x_i\in\Omega_i} [L_i(\mathbf{u}_{-i},x_i,\mu)] \quad (36)$$

Moreover, because we look for a fixed-point solution we set $\mathbf{x} = \mathbf{u}$ or simultaneously

$$x_i(\mathbf{u}_{-i}) = u_i, \qquad \forall i = 1, \dots, m$$

and we solve for u_i^*. From (35) we can write for these u_i^*, or in a fixed-point notation

$$\left[\ \min_{\mathbf{x}\in\Omega} \widetilde{L}(\mathbf{u};\mathbf{x};\mu)\ \right]|_{\mathbf{x}=\mathbf{u}} \tag{37}$$

$$= \sum_{i=1}^{m} \left[\ \min_{x_i\in\Omega_i} L_i(\mathbf{u}_{-i}, x_i, \mu)\ \right]|_{x_i=u_i}$$

which using (28) gives the first relation in (30).

A fixed-point solution is a vector solution $\mathbf{u} = [u_i]$ of the set of m equations (34) . We denote such a solution as $\mathbf{u}^* = [u_i^*]$, and note that it will depend on μ. Substituting for this $\mathbf{u}^*(\mu)$ on the right-hand side of (37) yields the second part of (30). Recalling that a fixed-point solution $\mathbf{u}^*$ to the m parallel optimization problems (36) is in fact a NE solution to a modified Nash game with cost functions L_i completes the proof. ∎

Remark 6: Note that the result has the useful form of a lower-level modified Nash game with cost functions L_i and a higher-level optimization problem for coordination. In general, $u_i^*(\mu), i \in \mathcal{M}$ may not be NE optimal (for the given μ) in the sense of attaining the minimum NG cost, i.e., such that $L_i^* = J_i^*$, but by Theorem 3 there exists a dual optimal price $\mu^* \geq 0$ such that $\mathbf{u}(\mu^*) = [u_i(\mu^*)]$ is NE optimal. Hence μ^* can be found as the maximizer in (29), where $D(\mu)$ is given as in (30). A sufficient condition is that the dual cost $D(\mu)$ is strictly concave in μ for $\mathbf{u}^*(\mu)$ as obtained from the lower-level game, (31). Alternatively, the price μ can be adjusted until the slackness conditions in Theorem 3 are satisfied indicating that the dual optimal price μ^* is found.

Remark 7: In effect, the formulation in Corollary 1 has the interpretation of a two-level hierarchical game [13]. The upper level game is a Stackelberg game with the the system being leader that sets the "prices" (Lagrange multipliers) and the players being the m followers. In this game the leader determines prices such that the players will respond with certain actions to maximize the dual cost function, and the optimal cost of the leader is D^*, (29). Given prices as set by the leader, a Nash game is played at the lower level, such that player i chooses the action u_i to minimize its own cost function L_i, (31). Then conditions for existence of a Stackelberg solution to the overall game can be based on Corollary 4.4 in [13]. Each player reacts to given "prices" (Lagrange multipliers) and the price acts as a coordination signal.

V. CONCLUSIONS

In this paper we extended duality results from standard optimization to a noncooperative (Nash) game framework. We started by showing that a coupled Nash game is equivalent to a constrained optimization of the NG-game cost function with respect to the second argument, that admits a fixed-point solution. We exploited the separability of the NG-cost function with respect to the second argument, and we extended the duality approach in a Nash game sense. This approach leads to a natural hierarchical decomposition into a lower-level uncoupled Nash game, and a higher-level system optimization problem, or in effect into a two-level hierarchical game.

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