Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Polynomial-Time Probabilistic Observability Analysis of Sampled-Data Piecewise Affine Systems

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***Abstract*— This paper proposes a polynomial-time probabilistic approach to solve the observability problem of sampled-data piecewise affine systems. First, an algebraic characterization for the system to be observable is derived. Next, based on the characterization, we propose a randomized algorithm that can determine with a probabilistic accuracy if the system is observable or not. Finally, it is shown with some examples, for which it is hopeless to check the observability in a deterministic way, that the proposed algorithm is very useful.**

I. Introduction

For the last decade, various topics on analysis and control synthesis of hybrid dynamical systems have been studied, while some problems have been negatively solved. In particular, it has been proven in [1], [2] that the controllability and observability problems for a class of hybrid systems are NP-hard or undecidable. This implies that it is hopeless to derive efficient methods for solving in a *deterministic* way these kinds of problems even for the class of relatively small-dimensional systems with a small number of modes. Thus an alternative way to deal with them will be to use a *probabilistic* approach, which allows us to obtain an approximate answer with a low computational complexity even for the NP-hard problems.

Motivated by the above background, this paper proposes a probabilistic approach to approximately solve the observability problem of a class of hybrid systems. More precisely, we focus on the *sampled-data piecewise affine (PWA) systems*, where the switching action is determined at each sampling time fixed in the digital device [3], and propose a randomized algorithm that determines with a probabilistic accuracy by randomly sampling the initial continuous state if this kind of system is observable or not. It is proven under some assumptions that the proposed algorithm is a *polynomial-time* algorithm with respect to *all* parameters of the observability problem such as the observation time period and the dimension of the continuous state. In contrast, in the deterministic approaches based on the mixed-integer programming to the observability problem (e.g., [4], [5]), their computation amounts increase *exponentially* with the observation time period. Thus it is shown by numerical experiments that the proposed algorithm can give an approximate solution within a practically short time for the example for which it is hopeless to check the observability in a deterministic way.

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It is also stressed here that for the observability problem of hybrid systems, *no* probabilistic approach has been obtained so far. In fact, the existing results on this topic, e.g., [6], [7], [8], [9], [10], in addition to [4], [5], are based on a *deterministic* approach. Furthermore, the proposed approach to the observability problem can *not* be directly applied to the case of the *discrete-time* PWA systems, since the proposed method fully exploits the good property of the sampled-data PWA system model, namely, the continuous-time output in the sampled-data PWA system gives more information to estimate the initial state compared with the case of the discrete-time output in the discrete-time PWA system. Note that this does not necessarily imply that the class of the PWA systems we focus on is restrictive. Most controlled plants have the switching decision clock usually implemented in the digital devices or can provide such a switching mechanism in some programming. So we stress here that introducing the sampled-data PWA system model in place of the discrete-time model as a model of such controlled plants enables us to determine the observability in a more efficient way.

Notation: let $\mathcal{R}$ and $\mathcal{N}$ ($\mathcal{N}_+$) denote the real number field and the set of nonnegative (positive) integers, respectively, and let $\mathcal{PC}$ ($\mathcal{PC}_{[0,T]}$) denote the set of all piecewise continuous functions (on the interval $[0, T]$ for the positive scalar T). We denote by $0_{n\times m}$, $\mathtt{I}_n$, and $\mathtt{1}_n$ (or for simplicity of notation, 0, $\mathtt{I}$, and $\mathtt{1}$) the $n \times m$ zero matrix, the $n \times n$ identity matrix, and the $n \times 1$ vector whose all elements are one. The i-th element of the vector x is expressed as $x_{(i)}$, the difference set of the sets $\mathcal{X}_1$ and $\mathcal{X}_2$ is expressed as $\mathcal{X}_1 - \mathcal{X}_2$, and the cardinality of the finite set $\mathcal{I}$ is expressed as $\mathrm{card}(\mathcal{I})$. For the measurable set $\mathcal{X}$ and its Hausdorff dimension d_H, we denote by $\mathrm{vol}(\mathcal{X})$ the d_H-dimensional Hausdorff measure, e.g., for $\mathcal{X}$ given by a 2-dimensional unit ball in $\mathcal{R}^3$, $\mathrm{vol}(\mathcal{X}) = \pi$ holds, although its (3-dimensional Lebesgue) measure is zero. Finally, the set $\mathcal{S}$ given as the form $\mathcal{S} := \{x \in \mathcal{R}^n |\ Ax + b \le 0, Cx + d < 0\}$ is called here the *polyhedron*, where A, C and b, d are some matrices and vectors, respectively.

II. Sampled-Data PWA Systems

The discrete transition, i.e., the transition of the discrete state (mode), of hybrid systems is roughly classified into two groups: the *physical discrete transition* such as the discontinuous phenomena of physical systems (e.g., collision) and the *logical discrete transition* such as the logic actions designed artificially (e.g., emergency measures). Contrary to the physical transition, the logical transition is mostly embedded in the digital device, which means that the switching action of the discrete state is determined depending upon the state at each switching time fixed in the digital device. In fact, the gear switching in an automatic transmission car occurs according to the clock-driven scheme in the digital device. We call here such switching the *sampled-data switching*.

This paper focuses on the class of the hybrid systems with the sampled-data switching in Fig. 1, which involves, in addition to PWA dynamics, the digital device that consists of the sampler, the logic, and the holder. To express the solution behaviors in a rigorous way, the following model Σ, called the *sampled-data PWA system model* [3], is introduced;

$$\Sigma : \begin{cases} \dot{x}(t) = A_{I(t)}x(t) + B_{I(t)}u(t) + a_{I(t)}, \\ y(t) = C_{I(t)}x(t) + D_{I(t)}u(t), \\ I(t) = I(t_k), \qquad \forall t \in [t_k, t_{k+1}), \\ x(t_{k+1}) = \phi(h, I(t_k), x(t_k), u_{[t_k,t_{k+1}]}), \\ I(t_{k+1}) = I_+ \qquad \text{if} \quad x(t_{k+1}) \in \mathcal{S}_{I_+} \end{cases} \tag{1}$$

where $x \in \mathcal{R}^n$ is the continuous state, $I \in \mathcal{I}$ is the discrete state (it is sometimes called the *mode*, and the symbol "J" is also used together with I), $\mathcal{I} := \{0, 1, \dots, M-1\}$ is the set of the discrete state values, $M \in \mathcal{N}_+$ is the number of the discrete state, $u \in \mathcal{R}^m$ is the control input, $y \in \mathcal{R}^p$ is the output, $A_I \in \mathcal{R}^{n\times n}$, $B_I \in \mathcal{R}^{n\times m}$, $a_I \in \mathcal{R}^n$, $C_I \in \mathcal{R}^{p\times n}$, and $D_I \in \mathcal{R}^{p\times m}$ are constant matrices for mode I, $h \in \mathcal{R}$ is the switching decision time period, $t_k \in \mathcal{R}$ is the time defined as $t_k := kh$ for $k \in \mathcal{N}$ (note $t_0 = 0$), which is called here the *switching decision time*, and $I_+ \in \mathcal{I}$ is the value of the discrete state renewed at a switching decision time. Note here that only at every time t_k, it is determined if the mode is switched or not; thus the mode is not be changed within each time interval (t_k, t_{k+1}). We call $(I, x) \in \mathcal{I} \times \mathcal{R}^n$ the hybrid state (or simply the state). In addition, $\phi(h, I(t_k), x(t_k), u_{[t_k,t_{k+1}]})$ denotes the solution $x(t_k + h)$ of $\dot{x}(t) = A_{I(t_k)}x(t) + B_{I(t_k)}u(t) + a_{I(t_k)}$ with the state $(I(t_k), x(t_k))$ and the control input u on the time interval $[t_k, t_{k+1}]$, and $\mathcal{S}_I$ denotes the subregion of the continuous state assigned to $I \in \mathcal{I}$, given by the polyhedron

$$\mathcal{S}_I := \{x \in \mathcal{R}^n |\ S_I x + s_I \le 0,\ \hat{S}_I x + \hat{s}_I < 0\} \tag{2}$$

where $S_I \in \mathcal{R}^{q_I\times n}$, $s_I \in \mathcal{R}^{q_I}$, $\hat{S}_I \in \mathcal{R}^{\hat{q}_I\times n}$, and $\hat{s}_I \in \mathcal{R}^{\hat{q}_I}$. For this subregion, it is assumed that $\bigcup_{I\in\mathcal{I}} \mathcal{S}_I = \mathcal{R}^n$ and $\mathcal{S}_I \cap \mathcal{S}_J = \emptyset$ for every $I, J \in \mathcal{I}$ such that $I \neq J$. This assumption guarantees that I is uniquely determined for each x, in other words, Σ is well-posed.

In this paper, as the first step to develop a probabilistic algorithm for observability analysis, we focus on the autonomous system ($B_I = 0$, $D_I = 0$ for every $I \in \mathcal{I}$). For simplicity of notation, we often use $x(0) = x_0 \in \mathcal{R}^n$ as the initial state instead of the hybrid state $(I(0), x(0)) = (I_0, x_0) \in \{(I, x) \in \mathcal{I}\times\mathcal{R}^n | x \in \mathcal{S}_I\}$, since the initial discrete state $I_0 \in \mathcal{I}$ is uniquely determined by each x_0. Furthermore, let $x(t, x_0)$ and $y(t, x_0)$ denote the continuous state $x(t)$ and the output $y(t)$ under the initial state $x(0) = x_0$, respectively.

III. Characterization of Observability

The following observability notion is considered.

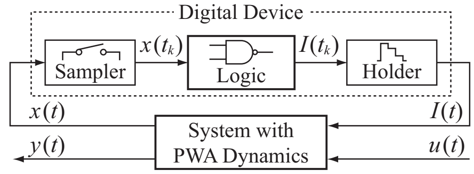

Fig. 1. Hybrid system with PWA dynamics and sampled-data switching.

Definition 1: For Σ, suppose that the observation time period $T \in \{t_1, t_2, \dots\}$ and the set $\mathcal{X}_0 \subseteq \mathcal{R}^n$ of the initial continuous state are given. Then Σ is said to be **$(T, \mathcal{X}_0)$-observable** if for every $x_0 \in \mathcal{X}_0$, there does not exist an $\tilde{x}_0 \in \mathcal{X}_0$ satisfying $x_0 \neq \tilde{x}_0$ and $y(t, x_0) = y(t, \tilde{x}_0)$ for every $t \in [0, T]$ (i.e., the initial state can be uniquely determined by the output y on the time interval $[0, T]$). ■

Note that the observation time period T and the set $\mathcal{X}_0$ of the initial state are explicitly specified. Such a definition will be useful in checking the feasibility of the state estimation with the observation time period fixed. Note also that if Σ is $(T, \mathcal{X}_0)$-observable, then Σ is $(T + \tau, \mathcal{X}_0)$-observable for every $\tau \in [0, \infty)$.

Now, let us derive an observability condition. Using $\bar{x} \in \mathcal{R}^{n+1}$, $\bar{A}_I \in \mathcal{R}^{(n+1)\times(n+1)}$, and $\bar{C}_I \in \mathcal{R}^{p\times(n+1)}$ defined as

$$\bar{x} := \begin{bmatrix} x \\ 1 \end{bmatrix},\ \ \bar{A}_I := \begin{bmatrix} A_I & a_I \\ 0 & 0 \end{bmatrix},\ \ \bar{C}_I := [C_I\ \ 0],$$

for simplicity of notation, we rewrite the first and second equations in (1) as

$$\dot{\bar{x}}(t) = \bar{A}_{I(t)}\bar{x}(t), \qquad y(t) = \bar{C}_{I(t)}\bar{x}(t), \tag{3}$$

respectively. For the number $f \in \mathcal{N}_+$ of the switching decision time on $[0, T)$ (i.e., $T = t_f$), let $\mathbb{I} := [I_0\ \ I_1\ \cdots\ I_{f-1}]^\top \in \mathcal{I}^f$ be a sequence of the mode (i.e., mode sequence) for $I_0, I_1, \dots, I_{f-1} \in \mathcal{I}$ ($\mathbb{J} := [J_0\ \ J_1\ \cdots J_{f-1}]^\top$ is also used for the mode sequence). Let $\mathcal{S}_{\mathbb{I}}$ be the polyhedron defined as $\mathcal{S}_{\mathbb{I}} := \mathcal{S}_{I_0} \times \mathcal{S}_{I_1} \times \cdots \times \mathcal{S}_{I_{f-1}}$, and then the set of $x_0 \in \mathcal{X}_0$ satisfying $[I(t_0)\ I(t_1)\ \cdots\ I(t_{f-1})]^\top = \mathbb{I}$ for the initial state $x(0) = x_0$ is expressed by $\mathcal{X}_0^{\mathbb{I}}(T, \mathcal{X}_0)$ (or simply $\mathcal{X}_0^{\mathbb{I}}$), i.e., $\mathcal{X}_0^{\mathbb{I}}(T, \mathcal{X}_0) := \mathcal{S}_{I_0} \cap \mathcal{X}_0$ for $f = 1$ and

$$\mathcal{X}_0^{\mathbb{I}}(T, \mathcal{X}_0) := \left\{ x_0 \in \mathcal{X}_0 \left| \begin{bmatrix} [\mathtt{I}_n\ 0]\bar{x}_0 \\ [\mathtt{I}_n\ 0]e^{\bar{A}_{I_0}h}\bar{x}_0 \\ \vdots \\ [\mathtt{I}_n\ 0]e^{\bar{A}_{I_{f-2}}h}\cdots e^{\bar{A}_{I_1}h}e^{\bar{A}_{I_0}h}\bar{x}_0 \end{bmatrix} \in \mathcal{S}_{\mathbb{I}} \right. \right\} \tag{4}$$

for $f \ge 2$, where $\bar{x}_0 := [x_0^\top\ 1]^\top$ and note the relation $x = [\mathtt{I}_n\ 0]\bar{x}$. It is remarked here that $\mathcal{X}_0^{\mathbb{I}}$ is a polyhedron if $\mathcal{S}_I \cap \mathcal{X}_0$ is a polyhedron for every $I \in \mathcal{I}$, and that $\bigcup_{\mathbb{I}\in\mathcal{I}^f} \mathcal{X}_0^{\mathbb{I}} = \mathcal{X}_0$ and $\mathcal{X}_0^{\mathbb{I}} \cap \mathcal{X}_0^{\mathbb{J}} = \emptyset$ hold for every $\mathbb{I}, \mathbb{J} \in \mathcal{I}^f$ ($\mathbb{I} \neq \mathbb{J}$) because the solution $x(t, x_0)$ is uniquely determined for every $x_0 \in \mathcal{R}^n$. Then we obtain the following necessary and sufficient condition for the two outputs $y(t, x_0)$ and $y(t, \tilde{x}_0)$, which are generated from the initial states $x(0) = x_0$ and $x(0) = \tilde{x}_0$ ($x_0, \tilde{x}_0 \in \mathcal{X}_0$), to be the same on the time interval $[0, T]$.

Lemma 1: For Σ, suppose that $T \in \{t_1, t_2, \dots\}$, $\mathcal{X}_0 \subseteq \mathcal{R}^n$, $\mathbb{I}, \mathbb{J} \in \mathcal{I}^f$ satisfying $\mathcal{X}_0^{\mathbb{I}} \neq \emptyset$ and $\mathcal{X}_0^{\mathbb{J}} \neq \emptyset$, $x_0 \in \mathcal{X}_0^{\mathbb{I}}$,

and $\tilde{x}_0 \in \mathcal{X}_0^{\mathbb{J}}$ are given. Then the following statements are equivalent.
(i) $y(t, x_0) = y(t, \tilde{x}_0)$ for every $t \in [0, T]$.
(ii) $\Delta_y(\mathbb{I}, x_0, \mathbb{J}, \tilde{x}_0) = 0$ for

$$\Delta_y(\mathbb{I}, x_0, \mathbb{J}, \tilde{x}_0) := \begin{bmatrix} x_0 \\ 1 \\ \tilde{x}_0 \\ 1 \end{bmatrix}^\top G_{\mathbb{IJ}} \begin{bmatrix} x_0 \\ 1 \\ \tilde{x}_0 \\ 1 \end{bmatrix}, \tag{5}$$

$$G_{\mathbb{IJ}} := G_{I_0 J_0} + \sum_{k=1}^{f-1} E_{\mathbb{IJ}}(k)^\top G_{I_k J_k} E_{\mathbb{IJ}}(k),$$

$$G_{IJ} := \begin{bmatrix} \int_0^h e^{\bar{A}_I^\top t} \bar{C}_I^\top \bar{C}_I e^{\bar{A}_I t} dt & -\int_0^h e^{\bar{A}_I^\top t} \bar{C}_I^\top \bar{C}_J e^{\bar{A}_J t} dt \\ -\int_0^h e^{\bar{A}_J^\top t} \bar{C}_J^\top \bar{C}_I e^{\bar{A}_I t} dt & \int_0^h e^{\bar{A}_J^\top t} \bar{C}_J^\top \bar{C}_J e^{\bar{A}_J t} dt \end{bmatrix},$$

$$E_{\mathbb{IJ}}(k) := \begin{bmatrix} e^{\bar{A}_{I_{k-1}} h} \cdots e^{\bar{A}_{I_1} h} e^{\bar{A}_{I_0} h} & 0 \\ 0 & e^{\bar{A}_{J_{k-1}} h} \cdots e^{\bar{A}_{J_1} h} e^{\bar{A}_{J_0} h} \end{bmatrix}.$$

Proof: Statement (i) holds if and only if $\int_0^T (y(t, x_0) - y(t, \tilde{x}_0))^\top (y(t, x_0) - y(t, \tilde{x}_0)) dt = 0$. In addition, $y(t, x_0) = \bar{C}_{I_k} e^{\bar{A}_{I_k} t} \bar{x}_0$ for $t \in [t_k, t_{k+1})$, and thus $\int_0^T (y(t, x_0) - y(t, \tilde{x}_0))^\top (y(t, x_0) - y(t, \tilde{x}_0)) dt = \Delta_y(\mathbb{I}, x_0, \mathbb{J}, \tilde{x}_0)$. These facts imply that (i) and (ii) are equivalent. ∎

From Lemma 1, for given $T \in \{t_1, t_2, \dots\}$, $\mathcal{X}_0 \subseteq \mathcal{R}^n$, $\mathbb{I} \in \mathcal{I}^f$ satisfying $\mathcal{X}_0^{\mathbb{I}} \neq \emptyset$, $\mathbb{J} \in \mathcal{I}^f$, and $x_0 \in \mathcal{X}_0^{\mathbb{I}}$, the set of the initial state $\tilde{x}_0 \in \mathcal{X}_0^{\mathbb{J}}$ satisfying $x_0 \neq \tilde{x}_0$ and $y(t, x_0) = y(t, \tilde{x}_0)$ for every $t \in [0, T]$ is expressed as

$$\tilde{\mathcal{X}}_{uob}^{\mathbb{IJ}}(T, \mathcal{X}_0, x_0) := \{\tilde{x}_0 \in \mathcal{X}_0^{\mathbb{J}} \,|\, x_0 \neq \tilde{x}_0, \ \Delta_y(\mathbb{I}, x_0, \mathbb{J}, \tilde{x}_0) = 0\}, \tag{6}$$

and the set of all $\tilde{x}_0 \in \mathcal{X}_0$ satisfying the same condition ($x_0 \neq \tilde{x}_0$ and $y(t, x_0) = y(t, \tilde{x}_0)$ for every $t \in [0, T]$) is denoted as

$$\tilde{\mathcal{X}}_{uob}^{\mathbb{I}}(T, \mathcal{X}_0, x_0) := \bigcup_{\mathbb{J} \in \mathcal{I}^f} \tilde{\mathcal{X}}_{uob}^{\mathbb{IJ}}(T, \mathcal{X}_0, x_0). \tag{7}$$

Thus the set of the initial state $x_0 \in \mathcal{X}_0$ for which there does not exist an $\tilde{x}_0 \in \mathcal{X}_0$ satisfying $x_0 \neq \tilde{x}_0$ and $y(t, x_0) = y(t, \tilde{x}_0)$ for every $t \in [0, T]$ is given by

$$\mathcal{X}_{ob}(T, \mathcal{X}_0) := \bigcup_{\mathbb{I} \in \mathcal{I}^f} \{x_0 \in \mathcal{X}_0^{\mathbb{I}} \,|\, \tilde{\mathcal{X}}_{uob}^{\mathbb{I}}(T, \mathcal{X}_0, x_0) = \emptyset\}. \tag{8}$$

Based on the above notation, the following result is straightforwardly obtained.

Theorem 1: For Σ, suppose that $T \in \{t_1, t_2, \dots\}$ and $\mathcal{X}_0 \subseteq \mathcal{R}^n$ are given. Then Σ is $(T, \mathcal{X}_0)$-observable if and only if the relation $\mathcal{X}_{ob}(T, \mathcal{X}_0) = \mathcal{X}_0$ holds. ∎

The condition $\mathcal{X}_{ob}(T, \mathcal{X}_0) = \mathcal{X}_0$ can be further checked by solving optimization problems as follows.

Lemma 2: For Σ, suppose that $T \in \{t_1, t_2, \dots\}$ and $\mathcal{X}_0 \subseteq \mathcal{R}^n$ are given, and assume

(A1) $\mathcal{S}_I \cap \mathcal{X}_0$ is a polyhedron for every $I \in \mathcal{I}$.

Then the following statements hold.
(i) The relation $\mathcal{X}_{ob}(T, \mathcal{X}_0) = \mathcal{X}_0$ holds if and only if for every $\mathbb{I}, \mathbb{J} \in \mathcal{I}^f$, the optimization problem $\boldsymbol{OP\,1(\mathbb{I}, \mathbb{J})}$ **:**

$$\min_{(x_0, \tilde{x}_0, \epsilon_1) \in \mathcal{R}^{2n} \times [-1, \infty)} \ \epsilon_1,$$
$$\text{subject to} \begin{cases} W_{\mathbb{I}} x_0 + w_{\mathbb{I}} \leq 0, & \hat{W}_{\mathbb{I}} x_0 + \hat{w}_{\mathbb{I}} \leq \epsilon_1 \mathbf{1}, \\ W_{\mathbb{J}} \tilde{x}_0 + w_{\mathbb{J}} \leq 0, & \hat{W}_{\mathbb{J}} \tilde{x}_0 + \hat{w}_{\mathbb{J}} \leq \epsilon_1 \mathbf{1}, \\ x_0 \neq \tilde{x}_0, & \Delta_y(\mathbb{I}, x_0, \mathbb{J}, \tilde{x}_0) = 0 \end{cases}$$

is infeasible or its optimal value, denoted by ϵ_1^*, is nonnegative, where ϵ_1 is a scalar variable, and $W_{\mathbb{I}}$, $\hat{W}_{\mathbb{I}}$ and $w_{\mathbb{I}}$, $\hat{w}_{\mathbb{I}}$ are matrices and vectors, respectively, satisfying

$$\mathcal{X}_0^{\mathbb{I}} = \{x_0 \in \mathcal{X}_0 \,|\, W_{\mathbb{I}} x_0 + w_{\mathbb{I}} \leq 0, \ \hat{W}_{\mathbb{I}} x_0 + \hat{w}_{\mathbb{I}} < 0\}, \tag{9}$$

and $W_{\mathbb{J}}$, $\hat{W}_{\mathbb{J}}$, $w_{\mathbb{J}}$, $\hat{w}_{\mathbb{J}}$ are defined in a similar way (note that $\mathcal{X}_0^{\mathbb{I}}$ can be expressed as (9) under (A1)).
(ii) The optimization problem $OP\,1(\mathbb{I}, \mathbb{J})$ corresponds to a convex quadratic programming (QP) problem for $\mathbb{I} \neq \mathbb{J}$ and to a mixed-integer quadratic programming (MIQP) problem for $\mathbb{I} = \mathbb{J}$. ∎

Lemma 2 is obtained from (6)–(8) and the fact that in $OP\,1(\mathbb{I}, \mathbb{J})$, the constraint $x_0 \neq \tilde{x}_0$ for $\mathbb{I} \neq \mathbb{J}$ is redundant ($\mathcal{X}_0^{\mathbb{I}} \cap \mathcal{X}_0^{\mathbb{J}} = \emptyset$ if $\mathbb{I} \neq \mathbb{J}$) and that for $\mathbb{I} = \mathbb{J}$ can be expressed by mixed-integer *linear* inequalities with $2n$ 0-1 variables. Statement (i) implies that if (A1) holds, the condition $\mathcal{X}_{ob}(T, \mathcal{X}_0) = \mathcal{X}_0$ can be checked by solving the optimization problem $OP\,1(\mathbb{I}, \mathbb{J})$ for every $\mathbb{I}, \mathbb{J} \in \mathcal{I}^f$, and (ii) implies that $OP\,1(\mathbb{I}, \mathbb{J})$ corresponds to a convex QP problem or an MIQP problem. Thus we can verify the $(T, \mathcal{X}_0)$-observability based on the optimization problem $OP\,1(\mathbb{I}, \mathbb{J})$.

However, such a deterministic approach will not be practical from the viewpoint of computational complexity. More precisely, as T (i.e., f) and/or n are taken larger, the computation amount becomes exponentially large because $\sum_{i=1}^{M^f} M^f - (i-1)$ (very large) optimization problems (including some MIQP problems with $2n$ 0-1 integer variables) have to be solved in the worst case. Note also that $OP\,1(\mathbb{J}, \mathbb{I})$ is a kind of dual problem of $OP\,1(\mathbb{I}, \mathbb{J})$; so we do not have to solve the M^{2f} optimization problems in practice.

IV. Probabilistic Observability Analysis

In this section, we propose a *polynomial-time probabilistic* approach to the $(T, \mathcal{X}_0)$-observability analysis, which can determine if Σ is $(T, \mathcal{X}_0)$-observable with a probabilistic accuracy or Σ is not $(T, \mathcal{X}_0)$-observable. The key idea is to successively check the *local observability* (for given $x_0 \in \mathcal{X}_0$, whether there exists an $\tilde{x}_0 \in \mathcal{X}_0$ satisfying $x_0 \neq \tilde{x}_0$ and $y(t, x_0) = y(t, \tilde{x}_0)$ for every $t \in [0, T]$, i.e., whether $x_0 \notin \mathcal{X}_{ob}(T, \mathcal{X}_0)$ or not) by randomly sampling the initial continuous state $x_0 \in \mathcal{X}_0$.

In the following subsections, for simplicity of notation, the symbols $\tilde{\mathcal{X}}_{uob}^{\mathbb{IJ}}(x_0)$, $\tilde{\mathcal{X}}_{uob}^{\mathbb{I}}(x_0)$, and $\mathcal{X}_{ob}$ are often used instead of $\tilde{\mathcal{X}}_{uob}^{\mathbb{IJ}}(T, \mathcal{X}_0, x_0)$, $\tilde{\mathcal{X}}_{uob}^{\mathbb{I}}(T, \mathcal{X}_0, x_0)$, and $\mathcal{X}_{ob}(T, \mathcal{X}_0)$, respectively.

A. Main Algorithm for Probabilistic Observability Analysis

Let us consider the following algorithm, called **Algorithm POA**, based on the random sampling of the initial continuous state $x_0 \in \mathcal{X}_0$.

```
0: Given T ∈ {t_1, t_2, ...}, X_0 ⊆ R^n, and N ∈ N;
1: i := 1;
2: while i ≤ N
    2.1: Generate an i.i.d. random vector x_0^i ∈ X_0;
    2.2: If x_0^i ∉ X_ob, then Halt: return "N";
    2.3: i := i+1;
3: Halt: return "Y_p";
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Algorithm POA repeatedly checks whether an i.i.d. random vector $x_0^i \in \mathcal{X}_0$ satisfies the condition $x_0^i \notin \mathcal{X}_{ob}$ or not; if $x_0^i \notin \mathcal{X}_{ob}$ holds for some $i \in \{1, 2, \ldots, N\}$, the algorithm outputs the symbol "N" (line 2.2); otherwise, that is, if $x_0^i \in \mathcal{X}_{ob}$ holds for all $i \in \{1, 2, \ldots, N\}$, it outputs the symbol "Y*" (line 3).

Algorithm POA provides a practical method to approximately solve the $(T, \mathcal{X}_0)$-observability problem as follows.

Let x_0 be a random vector with a uniform probability density function on $\mathcal{X}_0$, and under assumption

(A2) The set $\mathcal{X}_0$ is bounded and measurable,

we formally define

$$\mathrm{Prob}\{x_0 \in \mathcal{X}_{ob}\} := \frac{\mathrm{vol}(\mathcal{X}_{ob})}{\mathrm{vol}(\mathcal{X}_0)}. \tag{10}$$

Note that $\mathcal{X}_{ob} \subseteq \mathcal{X}_0$ and $\mathrm{Prob}\{x_0 \in \mathcal{X}_0 - \mathcal{X}_{ob}\} = 1 - \mathrm{Prob}\{x_0 \in \mathcal{X}_{ob}\}$ hold. Then the following result is straightforwardly obtained from the result in [11], [12], [13].

Lemma 3: For Σ, suppose that $T \in \{t_1, t_2, \ldots\}$ and $\mathcal{X}_0 \subseteq \mathcal{R}^n$ satisfying (A2) are given. In addition, for Algorithm POA, suppose that $N \in \mathcal{N}$ is given as the smallest integer satisfying

$$N \geq \frac{\ln \frac{1}{\delta}}{\ln \frac{1}{1-\varepsilon}} \tag{11}$$

for arbitrarily given $\varepsilon \in (0, 1)$ and $\delta \in (0, 1)$. Then the following statements hold.
(i) If Algorithm POA outputs "Y*", then the following relation holds:

$$\mathrm{Prob}\{\mathrm{Prob}\{x_0 \in \mathcal{X}_0 - \mathcal{X}_{ob}\} \leq \varepsilon\} \geq 1 - \delta. \tag{12}$$

(ii) If Algorithm POA outputs "N", then Σ is not $(T, \mathcal{X}_0)$-observable. ■

Lemma 3 (i) implies that if Algorithm POA outputs "Y*", $\mathrm{vol}(\mathcal{X}_0 - \mathcal{X}_{ob}) / \mathrm{vol}(\mathcal{X}_0) \leq \varepsilon$ holds with the probability more than or equal to $1 - \delta$. Thus if ε and δ are sufficiently small, it is guaranteed with sufficiently high probability $1 - \delta$ that for almost all $x_0 \in \mathcal{X}_0$ (i.e., for all $x_0 \in \mathcal{X}_0$ except for elements in a subset of $\mathcal{X}_0$ whose volume is less than $\varepsilon\,\mathrm{vol}(\mathcal{X}_0)$), there does not exist an $\tilde{x}_0 \in \mathcal{X}_0$ satisfying $x_0 \neq \tilde{x}_0$ and $y(t, x_0) = y(t, \tilde{x}_0)$ for every $t \in [0, T]$. On the other hand, (ii) implies that if Algorithm POA outputs "N", then it is verified that Σ is not $(T, \mathcal{X}_0)$-observable. In fact, if there exists an $i \in \{1, 2, \ldots, N\}$ satisfying $x_0^i \notin \mathcal{X}_{ob}$, then $\mathcal{X}_{ob} \neq \mathcal{X}_0$ holds. In this way, for given ε and δ as accuracy parameters, Algorithm POA with N satisfying (11) can determine if Σ is $(T, \mathcal{X}_0)$-observable in the sense of (12) or not $(T, \mathcal{X}_0)$-observable. Note here that the number N of samples is bounded by a polynomial function of ε and δ (see (11)).

However, Algorithm POA may not terminate in a practically short time due to the following drawback. From (6)–(8), it follows that for $\mathbb{I} \in \mathcal{I}^f$ satisfying $x_0^i \in \mathcal{X}_0^{\mathbb{I}}$, $x_0^i \notin \mathcal{X}_{ob}$ holds if and only if there exists a $\mathbb{J} \in \mathcal{I}^f$ such that $\tilde{\mathcal{X}}_{uob}^{\mathbb{IJ}}(x_0^i) \neq \emptyset$, and that the condition $\tilde{\mathcal{X}}_{uob}^{\mathbb{IJ}}(x_0^i) \neq \emptyset$ holds if and only if the optimization problem $\boldsymbol{OP\,2(\mathbb{I}, x_0^i, \mathbb{J})}$ **:**

$$\min_{(\tilde{x}_0, \epsilon_2) \in \mathcal{R}^n \times [-1, \infty)} \epsilon_2,$$
$$\text{subject to} \begin{cases} W_{\mathbb{J}}\tilde{x}_0 + w_{\mathbb{J}} \leq 0, & \hat{W}_{\mathbb{J}}\tilde{x}_0 + \hat{w}_{\mathbb{J}} \leq \epsilon_2 \mathbf{1}, \\ x_0^i \neq \tilde{x}_0, & \Delta_y(\mathbb{I}, x_0^i, \mathbb{J}, \tilde{x}_0) = 0 \end{cases}$$

is feasible and its optimal value, denoted by ϵ_2^*, is negative, where ϵ_2 is a scalar variable, and $W_{\mathbb{J}}$, $\hat{W}_{\mathbb{J}}$, $w_{\mathbb{J}}$, $\hat{w}_{\mathbb{J}}$ are defined in a similar way to the case of $OP\,1(\mathbb{I}, \mathbb{J})$. Note that $OP\,2(\mathbb{I}, x_0^i, \mathbb{J})$ also corresponds to a convex QP problem for $\mathbb{I} \neq \mathbb{J}$ and to an MIQP problem with $2n$ 0-1 integer variables for $\mathbb{I} = \mathbb{J}$ (it is proven in a similar way to Lemma 2 (ii)). Thus the condition $x_0^i \notin \mathcal{X}_{ob}$ is verified by M^f times solving $OP\,2(\mathbb{I}, x_0^i, \mathbb{J})$ in the worst case (note $T = t_f$). This implies that the computation amount for verifying the condition $x_0^i \notin \mathcal{X}_{ob}$ becomes, in general, exponentially larger with T and n. Thus by focusing on a limited class of the systems, we consider here an efficient algorithm for verifying $x_0^i \notin \mathcal{X}_{ob}$.

B. Polynomial-Time Algorithm for Verifying $x_0 \notin \mathcal{X}_{ob}$

We first give the following result.

Lemma 4: For Σ, suppose that $\mathcal{X}_0 \subseteq \mathcal{R}^n$, $I_0, J_0 \in \mathcal{I}$, and $x_0 \in \mathcal{X}_0^{I_0}(t_1, \mathcal{X}_0)$ are given. Then if

(A3) (C_I, A_I) is observable for every $I \in \mathcal{I}$,

the following statements hold (note that $\mathcal{X}_0^{I_0}(t_1, \mathcal{X}_0) = \mathcal{S}_{I_0} \cap \mathcal{X}_0$ holds, and $\Delta_y(I_0, x_0, J_0, \tilde{x}_0)$ and $\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0)$ are defined in (5) and (6) for $T := t_1$, $\mathbb{I} := I_0$, $\mathbb{J} := J_0$).
(i) $\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0)$ is a finite set satisfying $\mathrm{card}(\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0)) \leq 1$.
(ii) Let $\tilde{x}_0^*(I_0, x_0, J_0) := -\frac{1}{2} G_1^{-1} G_2^\top [\bar{x}_0^\top \;\; 1]^\top$ for the nonsingular matrix $G_1 \in \mathcal{R}^{n \times n}$ and the matrices $G_2 \in \mathcal{R}^{(n+2) \times n}$, $G_3 \in \mathcal{R}^{(n+2) \times (n+2)}$ satisfying

$$\Delta_y(I_0, x_0, J_0, \tilde{x}_0) = \tilde{x}_0^\top G_1 \tilde{x}_0 + \begin{bmatrix} \bar{x}_0 \\ 1 \end{bmatrix}^\top G_2 \tilde{x}_0 + \begin{bmatrix} \bar{x}_0 \\ 1 \end{bmatrix}^\top G_3 \begin{bmatrix} \bar{x}_0 \\ 1 \end{bmatrix}$$

for every $\tilde{x}_0 \in \mathcal{R}^n$. Then if $\tilde{x}_0^*(I_0, x_0, J_0) \in \mathcal{X}_0^{J_0}(t_1, \mathcal{X}_0)$, $x_0 \neq \tilde{x}_0^*(I_0, x_0, J_0)$, and $\Delta_y(I_0, x_0, J_0, \tilde{x}_0^*(I_0, x_0, J_0)) = 0$, the relation $\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0) = \{\tilde{x}_0^*(I_0, x_0, J_0)\}$ holds; otherwise $\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0) = \emptyset$ holds.

Proof: (i) The behavior of Σ on the time interval $[0, t_1)$ is the same as the behavior of the so-called affine systems. Thus if there exists an $\tilde{x}_0 \in \mathcal{X}_0^{J_0}(t_1, \mathcal{X}_0)$ satisfying $x_0 \neq \tilde{x}_0$ and $y(t, x_0) = y(t, \tilde{x}_0)$ for every $t \in [0, t_1]$, then it is unique under (A3). This implies that $\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0)$ is a finite set satisfying $\mathrm{card}(\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0)) \leq 1$.
(ii) This is proven from (i), (6), and the fact that $\tilde{x}_0^*(I_0, x_0, J_0)$ is an optimal solution to the problem $\min_{\tilde{x}_0 \in \mathcal{R}^n} \Delta_y(I_0, x_0, J_0, \tilde{x}_0)$ (for given I_0, x_0, and J_0). ■

Lemma 4 provides useful properties for checking the condition $x_0 \notin \mathcal{X}_{ob}$ under (A3) as follows.

For fixed $x_0 \in \mathcal{X}_0$, the discrete state $I_0 \in \mathcal{I}$ satisfying $x_0 \in \mathcal{X}_0^{I_0}(t_1, \mathcal{X}_0)$ and the mode sequence $\mathbb{I} \in \mathcal{I}^f$ satisfying $x_0 \in \mathcal{X}_0^{\mathbb{I}}(T, \mathcal{X}_0)$ are uniquely determined. Then for these x_0, I_0, and $\mathbb{I}$, the condition $x_0 \notin \mathcal{X}_{ob}$ holds if and only if

(C2) There exists an $\tilde{x}_0 \in \tilde{\mathcal{X}}_{uob}^{I_0}(t_1, \mathcal{X}_0, x_0)$ satisfying

$$\tilde{x}_0 \in \tilde{\mathcal{X}}_{uob}^{\mathbb{I}}(T, \mathcal{X}_0, x_0). \tag{13}$$

This is obtained from the facts that $x_0 \notin \mathcal{X}_{ob}$ holds if and only if $\tilde{\mathcal{X}}_{uob}^{\mathbb{I}}(T, \mathcal{X}_0, x_0) \neq \emptyset$ (see (8)), and that $\tilde{\mathcal{X}}_{uob}^{\mathbb{I}}(T, \mathcal{X}_0, x_0) \subseteq \tilde{\mathcal{X}}_{uob}^{I_0}(t_1, \mathcal{X}_0, x_0)$ (see Definition 1). Thus by checking (C2), the condition $x_0 \notin \mathcal{X}_{ob}$ is verified. Then Lemma 4 (i) implies that under (A3), $\tilde{\mathcal{X}}_{uob}^{I_0}(t_1, \mathcal{X}_0, x_0)$ is a finite set satisfying $\mathrm{card}(\tilde{\mathcal{X}}_{uob}^{I_0}(t_1, \mathcal{X}_0, x_0)) \leq M - 1$ (note (7) and $\tilde{\mathcal{X}}_{uob}^{I_0 I_0}(t_1, \mathcal{X}_0, x_0) = \emptyset$); thus the condition (C2) is verified by checking whether (13) holds or not for at most $M-1$ initial states $\tilde{x}_0 \in \tilde{\mathcal{X}}_{uob}^{I_0}(t_1, \mathcal{X}_0, x_0)$. Note that if (A3) does not hold, $\tilde{\mathcal{X}}_{uob}^{I_0}(t_1, \mathcal{X}_0, x_0)$ is not in general a finite set; so in such a case, it is hard to check (C2) in a polynomial time with respect to T and n. On the other hand, Lemma 4 (ii) implies that, under (A3), the set $\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0)$ is easily calculated, and thus $\tilde{\mathcal{X}}_{uob}^{I_0}(t_1, \mathcal{X}_0, x_0)$ can be obtained.

Therefore, by noting from (6) and (7) that for $x_0 \in \mathcal{X}_0$, $I_0 \in \mathcal{I}$ satisfying $x_0 \in \mathcal{X}_0^{I_0}(t_1, \mathcal{X}_0)$, and $\tilde{x}_0 \in \tilde{\mathcal{X}}_{uob}^{I_0}(t_1, \mathcal{X}_0, x_0)$ (note that $x_0 \neq \tilde{x}_0$ holds), (13) holds if and only if

(C3) $\Delta_y(\mathbb{I}, x_0, \mathbb{J}, \tilde{x}_0) = 0$ holds for $\mathbb{J} \in \mathcal{I}^f$ satisfying $\tilde{x}_0 \in \mathcal{X}_0^{\mathbb{J}}(T, \mathcal{X}_0)$,

the following algorithm, called **Algorithm L**, for checking $x_0 \notin \mathcal{X}_{ob}$ is obtained under (A3).

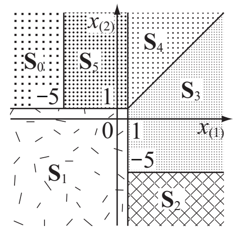
0: Given $T \in \{t_1, t_2, \dots\}$, $\mathcal{X}_0 \subseteq \mathcal{R}^n$, and $x_0 \in \mathcal{X}_0$;
1: Let I_0 and $\mathbb{I}$ be the elements of $\mathcal{I}$ and $\mathcal{I}^f$ satisfying $x_0 \in \mathcal{X}_0^{I_0}(t_1, \mathcal{X}_0)$ and $x_0 \in \mathcal{X}_0^{\mathbb{I}}(T, \mathcal{X}_0)$;
2: Calculate $\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0)$ for every $J_0 \in \mathcal{I}$;
3: $\tilde{\mathcal{X}}_{uob}^{I_0}(t_1, \mathcal{X}_0, x_0) := \bigcup_{J_0 \in \mathcal{I}} \tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0)$;
4: If there exists an $\tilde{x}_0 \in \tilde{\mathcal{X}}_{uob}^{I_0}(t_1, \mathcal{X}_0, x_0)$ satisfying (C3) (i.e., (C2) holds)
  then Halt: return "$x_0 \notin \mathcal{X}_{ob}$";
  else Halt: return "$x_0 \in \mathcal{X}_{ob}$";

In Algorithm L, the condition $x_0 \notin \mathcal{X}_{ob}$ is verified based on (C2), where $\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0)$ is obtained by Lemma 4 (ii) (line 2), and (C2) is verified by checking if there exists an $\tilde{x}_0 \in \tilde{\mathcal{X}}_{uob}^{I_0}(t_1, \mathcal{X}_0, x_0)$ satisfying (C3) or not (line 4). Note that $\mathbb{I} \in \mathcal{I}^f$ satisfying $x_0 \in \mathcal{X}_0^{\mathbb{I}}(T, \mathcal{X}_0)$ can be uniquely obtained by calculating the solution of Σ for the initial state $x(0) = x_0$ (lines 1 and 4).

Finally, it is shown that under (A2) and (A3), the proposed probabilistic observability analysis is efficiently executed by Algorithm POA with Algorithm L.

Theorem 2: For Σ, suppose that $T \in \{t_1, t_2, \dots\}$ and $\mathcal{X}_0 \subseteq \mathcal{R}^n$ are given, and assume (A2) and (A3). Then Algorithm POA with Algorithm L is a polynomial-time algorithm with respect to T, n, ε, and δ.

Proof: From Lemma 4, it follows that $M-1$ sets $(\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0))$ are derived in line 2 and (C3) is checked at most $M-1$ times in line 4. In addition, the computation amounts for obtaining $\tilde{\mathcal{X}}_{uob}^{I_0 J_0}(t_1, \mathcal{X}_0, x_0)$ and for checking (C3) are bounded by a polynomial function of T and n. Hence, by Algorithm L, the condition $x_0 \notin \mathcal{X}_{ob}$ is checked in a polynomial time with respect to T and n. On the other hand, as shown in Section IV-A, the number N of samples in Algorithm POA is bounded by a polynomial function of ε and δ. These implies that Algorithm POA with Algorithm L under (A2) and (A3) is a polynomial-time algorithm with respect to T, n, ε, and δ. ∎



$$\mathcal{S}_0 := \left\{ x \in \mathcal{R}^2 \,\middle|\, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 5 \\ 1 \end{bmatrix} \leq 0 \right\},$$

$$\mathcal{S}_1 := \left\{ x \in \mathcal{R}^2 \,\middle|\, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ -1 \end{bmatrix} < 0 \right\},$$

$$\mathcal{S}_2 := \left\{ x \in \mathcal{R}^2 \,\middle|\, \begin{matrix} [-1 \ \ 0]\, x + 1 \leq 0 \\ [0 \ \ 1]\, x + 5 < 0 \end{matrix} \right\},$$

$$\mathcal{S}_3 := \left\{ x \in \mathcal{R}^2 \,\middle|\, \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix} \leq 0 \right\},$$

$$\mathcal{S}_4 := \left\{ x \in \mathcal{R}^2 \,\middle|\, \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} < 0 \right\},$$

$$\mathcal{S}_5 := \left\{ x \in \mathcal{R}^2 \,\middle|\, \begin{matrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \leq 0 \\ [-1 \ \ 0]\, x - 5 < 0 \end{matrix} \right\}.$$

Fig. 2. Subregions $\mathcal{S}_I$ in the example.

TABLE I

RESULTS OF DETERMINISTIC OBSERVABILITY ANALYSIS.

T	0.2	0.4	0.6	0.8	1.0
$(T, \mathcal{X}_0)$-observability	N	N	N	N	?
Computation time [sec]	0.3	5.9	245.1	11823.2	—

TABLE II

RESULTS OF PROBABILISTIC OBSERVABILITY ANALYSIS.

T		0.2	0.4	0.6	0.8
$(T, \mathcal{X}_0)$-observability		N	N	N	N
Computation time [sec]	max.	0.08	0.08	0.13	0.24
	mean	0.05	0.06	0.09	0.11
	min.	0.03	0.05	0.06	0.06
T		1.0	1.5	2.0	2.5
$(T, \mathcal{X}_0)$-observability		N	N	N	Y*
Computation time [sec]	max.	0.23	0.22	0.83	441.2
	mean	0.16	0.18	0.36	435.7
	min.	0.09	0.09	0.14	432.8

Remark 1: It can be also proven that if $\mathcal{X}_0$ is a polyhedron, Algorithm POA with Algorithm L (under (A2) and (A3)) is a polynomial-time algorithm with respect to the other parameters of the observability problem: the number of the inequalities characterizing $\mathcal{X}_0$, and M, p, q_I, $\hat{q}_I$. ∎

C. Example

Consider Σ with $n = 2$, $M = 6$, and $h = 0.1$, given by

$$(A_0, a_0) := \left(\begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right), \ (A_1, a_1) := \left(\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right),$$

$$(A_2, a_2) := \left(\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right), \ (A_3, a_3) := \left(\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right),$$

$$(A_4, a_4) := \left(\begin{bmatrix} -2 & 0 \\ -1 & -0.5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right), \ (A_5, a_5) := \left(\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right),$$

$C_I := [1 \ \ 1]$ for every $I \in \mathcal{I}$. The subregions of the continuous state assigned to each value of the discrete state are shown in Fig. 2. Then for several $T \in \{0.2, 0.4, \dots\}$ and $\mathcal{X}_0 = [-10, 10]^2$, let us check the $(T, \mathcal{X}_0)$-observability by the deterministic method based on Theorem 1 and Lemma 2, and the proposed probabilistic method. Note that for this $\mathcal{X}_0$

TABLE III

COMPARISON BETWEEN PROBABILISTIC OBSERVABILITY AND CONTROLLABILITY ANALYSES.

		Probabilistic Observability Analysis		**Probabilistic Controllability Analysis**
		without (A3)	with (A3)	in [14]
Checked condition for each sample x_0^i		$x_0^i \notin \mathcal{X}_{ob}$		$x_0^i \notin \mathcal{X}_c$
Computation amount for N i.i.d. samples (worst case)		$N(M^f-1+2^{2n})$ times solving QP problem (with $n+1$ variables)	$N(M-1)$ times computation of the set $\tilde{\mathcal{X}}_{uob}^{I_0J_0}(t_1,\mathcal{X}_0,x_0^i)$ and checking (C3) (simple algebraic manipulation)	NM^{f-1} times solving LP problem (with $n(f-1)+1$ variables)
Increase of computation amount	w.r.t. T (f)	Exponential	Polynomial	Exponential
	w.r.t. n	Exponential	Polynomial	Polynomial

and the system, assumptions (A1)–(A3) hold; so Lemma 2 and Algorithm POA with Algorithm L can be applied.

The numerical results based on the deterministic method are given in Table I, where we used MATLAB on the computer with the Pentium M 1.2GHz processor and the 1GB memory, and the deterministic method based on Theorem 1 and Lemma 2 is used for roughly estimating the worst case computation amount of the $(T,\mathcal{X}_0)$-observability analysis (although there may be some techniques for decreasing the computation amount). The symbol "N" expresses that Σ is not $(T,\mathcal{X}_0)$-observable, and the symbol "?" implies that no answer can be obtained within 24 hours. Table I shows that only for sufficiently small T $(T \le 0.8)$, the above $(T,\mathcal{X}_0)$-observability problem can be solved in a practically short time. Similar results were obtained for other examples.

Next, let us apply the proposed probabilistic method. We set $\varepsilon := 10^{-3}$ and $\delta := 10^{-3}$, so $N := 6905$ by (11). Under such a situation, Algorithm POA with Algorithm L can determine if Σ is $(T,\mathcal{X}_0)$-observable in the sense that $\mathrm{Prob}\{x_0 \in \mathcal{X}_0 - \mathcal{X}_{ob}\} \le 0.1[\%]$ holds with a probability more than or equal to 99.9 [%], or Σ is not $(T,\mathcal{X}_0)$-observable.

The numerical results are shown in Table II, where the symbols "Y*" and "N" express the output of Algorithm POA. Each result is based on ten trials, where the algorithm answered the same result in every trial, and the computation time expresses the maximum value, the mean, and the minimum value of the trials. It turns from Table II that Σ is $(2.5,\mathcal{X}_0)$-observable in the sense of (12), and Σ is not $(T,\mathcal{X}_0)$-observable for $T \le 2.0$. Then, in order to know the minimum observation time period for estimating the initial state, we further checked the $(T,\mathcal{X}_0)$-observability for $T = 2.4$, and then it is verified that Σ is not $(2.4,\mathcal{X}_0)$-observable. So it follows that the observation time interval T has to be at least 2.5 for estimating the initial state. In this way, the $(T,\mathcal{X}_0)$-observability problem for which it is hopeless to check in a deterministic way can be solved within a practically short time by the proposed methods.

V. DISCUSSIONS

This section gives some discussions on the gaps between the probabilistic observability analysis (POA) and the probabilistic controllability analysis (PCA) proposed in [14].

The controllability problem, which has been addressed in [14], is to determine if for each $x_0 \in \mathcal{X}_0$ (the initial continuous state set), there exists a $u \in \mathcal{PC}^m_{[0,T]}$ satisfying $x(T) = 0$ under the initial state $x(0) = x_0$. Then for a probabilistic approach to this problem, it is essential to determine if $x_0^i \notin \mathcal{X}_c$ for randomly sampled $x_0^i \in \mathcal{X}_0$, where $\mathcal{X}_c$ denotes the set of the initial state in $\mathcal{X}_0$ such that there exists a $u \in \mathcal{PC}^m_{[0,T]}$ satisfying $x(T) = 0$. We can show that the condition $x_0^i \notin \mathcal{X}_c$ is checked by solving at most M^{f-1} linear programming (LP) problems for each x_0^i, where the dimension of variables is $n(f-1)+1$ (see [14]) and the number of constraints is bounded by a polynomial function of T and n. This fact leads to a polynomial-time algorithm with n, but exponential-time algorithm with T (i.e., f).

In the POA, in a similar way, it is essential to determine if $x_0^i \notin \mathcal{X}_{ob}$ for randomly sampled $x_0^i \in \mathcal{X}_0$. However, as shown in Section IV-A, the condition $x_0^i \notin \mathcal{X}_{ob}$ may not be checked by a polynomial-time algorithm with T and n. This is mainly because we have to solve the MIQP problem with $2n$ 0-1 variables for proving if a sampled $x_0^i \in \mathcal{X}_0$ is a *unique* initial state satisfying some conditions in the POA, while we only have to check if for a sampled $x_0^i \in \mathcal{X}_0$, there exists a control input satisfying some conditions in the PCA.

In this way, these computational complexities are essentially different, in other words, the computational complexity of the POA is in general higher than that of the PCA. However, we have shown in Section IV-B that Algorithm POA is a polynomial-time algorithm with respect to T and n if Algorithm L is used (with (A3)). Table III condenses the above discussions.

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