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Interval Analysis in Dioid : Application to Robust Open-Loop Control for Timed Event Graphs

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Abstract— **This paper deals with robust open-loop control synthesis for timed event graphs, where the number of initial tokens and time delays are only known to belong to intervals. We discuss here the existence and the computation of a greatest interval of control included in the robust control set for uncertain systems that can be described by parametric models, the unknown parameters of which are assumed to vary between known bounds. Each control is computed in order to guarantee that the controlled system behavior is greater than the lower bound of a desired output reference set and is lower than the upper bound of this set. The synthesis presented here is mainly based on dioid, interval analysis and residuation theory.**

I. INTRODUCTION

Discrete Event Systems (DES) appear in many applications in manufacturing systems [1], computer and communication systems [4] and are often described by the Petri Net formalism. Timed-Event Graphs (TEG) are Timed Petri Nets in which all places have single upstream and single downstream transition and appropriately model DES characterized by delay and synchronization phenomena. TEG can be described by linear equations in the dioid algebra [2], [6] and this fact has permitted many important achievements on the control of DES modelled by TEG [6], [7], [15], [13], [8]. TEG control problems are usually stated in a just-in-time context. The design goal is to achieve some performance while minimizing internal stocks. In [2], [15] an optimal open-loop control law is given. In [7] linear closed-loop controllers synthesis are given in a model matching objective, *i.e.*, the controller synthesis is done in order that the controlled system will behave as close as possible to a reference model and will delay as much as possible the tokens input in the system. The reference model is *a priori* known and depicts the desired behavior of the controlled system.

This paper aims at designing robust open-loop control when the system includes some parametric uncertainties which can be described by intervals. First, by using interval analysis, we give a model to depict TEG with number of tokens and time delays which are assumed to vary between known bounds[1]. Next, we consider an open-loop control synthesis for these uncertain systems. The open-loop control synthesis is done in order to maintain the output of controlled system in a set of reference outputs. We assume that the upper and lower bounds of this specifications set are *a priori* known, the synthesis yields a set[2] of control laws which guarantees that the closed loop system behavior is both greater than the lower bound of the specifications set and lower than the upper bound of this same set. The open-loop control synthesis is obtained by considering residuation theory which allows the inversion of mapping defined over ordered sets, and interval analysis which is known to be efficient to characterize set of robust controls in a guaranteed way [10]. An example of manufacturing system illustrates the theory developed.

II. DIOIDS AND RESIDUATION

Definition 1: A *dioid* $\mathscr{D}$ is a set endowed with two internal operations denoted by $\oplus$ (addition) and $\otimes$ (multiplication), both associative and both having neutral elements denoted by ε and e respectively, such that $\oplus$ is also commutative and idempotent (i.e. $a \oplus a = a$). The $\otimes$ operation is distributive with respect to $\oplus$, and ε is absorbing for the product (*i.e.* $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon, \quad \forall a$). When $\otimes$ is commutative, the dioid is said to be commutative. The symbol $\otimes$ is often omitted.
Dioids can be endowed with a natural *order* : $a \succeq b$ iff $a = a \oplus b$. Then they become sup-semilattices and $a \oplus b$ is the least upper bound of a and b. A dioid is *complete* if sums of infinite number of terms are always defined, and if multiplication distributes over infinite sums too. In particular, the sum of all elements of the dioid is defined and denoted by $\top$ (for 'top'). A complete dioid (sup-semilattice) becomes a lattice by constructing the greatest lower bound of a and b, denoted by $a \wedge b$, as the least upper bound of the (nonempty) subset of all elements which are less than a and b (see [2, §4]).

Definition 2 (Subdioid): A subset $\mathscr{C}$ of a dioid is called a subdioid of $\mathscr{D}$ if

- $\varepsilon \in \mathscr{C}$ and $e \in \mathscr{C}$;
- $\mathscr{C}$ is closed for $\oplus$ and $\otimes$, i.e.,$\forall a, b \in \mathscr{C}$, $a \oplus b \in \mathscr{C}$ and $a \otimes b \in \mathscr{C}$.

Theorem 1 ([2]): Over a complete dioid $\mathscr{D}$, the implicit equation $x = ax \oplus b$ admits $x = a^*b$ as least solution, where $a^* = \bigoplus_{i \in \mathbb{N}} a^i$ (Kleene star operator) with $a^0 = e$. Next, this operator will be sometimes represented by the following mapping $\mathscr{K} : \mathscr{D} \to \mathscr{D}, x \mapsto x^*$. Furthermore, let $x, y \in \mathscr{D}$, we

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[1]In a manufacturing context, these systems can represent production systems in which the number of resources varies in the time (e.g. due to some maintenance operations, or to some machines breakdowns,...) or in which the processing times are not well known but vary in known intervals.

[2]It is a set of robust control law which ensures that, for all the possible behaviors of the uncertain system, the controlled system is slower than a reference output and is faster than another one.

have

$$x(yx)^* = (xy)^* x, \quad (1)$$
$$(x^*)^* = x^* \quad (2)$$

Definition 3 (Residual and residuated mapping): An isotone mapping $f : \mathscr{D} \to \mathscr{E}$, where $\mathscr{D}$ and $\mathscr{E}$ are ordered sets, is a *residuated mapping* if for all $y \in \mathscr{E}$, the least upper bound of the subset $\{x | f(x) \preceq y\}$ exists and belongs to this subset. It is then denoted by $f^\sharp(y)$. Mapping $f^\sharp$ is called the residual of f. When f is residuated, $f^\sharp$ is the unique isotone mapping such that

$$f \circ f^\sharp \preceq \mathsf{Id}_{\mathscr{E}} \quad \text{and} \quad f^\sharp \circ f \succeq \mathsf{Id}_{\mathscr{D}}, \quad (3)$$

where Id is the identity mapping respectively on $\mathscr{D}$ and $\mathscr{E}$.

Given a mapping $f : \mathscr{D} \to \mathscr{E}$, we define as usual $\mathsf{Im} f = \{f(x) \mid x \in \mathscr{D}\}$.

Property 1 (Projection [5]): Let $f : \mathscr{D} \to \mathscr{E}$ be a residuated mapping and let $y \in \mathscr{E}$, then $f \circ f^\sharp(y)$ is the "*best approximation from below of* y *in* $\mathsf{Im} f$", that is, the greatest $z \in \mathsf{Im} f$ less than y. This operator $f \circ f^\sharp$ will later on be denoted Π_f and called "least projector on $\mathsf{Im} f$".

Property 2: Let $f : \mathscr{D} \to \mathscr{E}$ be a residuated mapping, then

$$y \in f(\mathscr{D}) \quad \Leftrightarrow \quad f(f^\sharp(y)) = y.$$

Property 3 ([2, Th. 4.56]): If $h : \mathscr{D} \to \mathscr{C}$ and $f : \mathscr{C} \to \mathscr{B}$ are residuated mappings, then $f \circ h$ is also residuated and

$$(f \circ h)^\sharp = h^\sharp \circ f^\sharp. \quad (4)$$

Theorem 2 ([2, §4.4.2]): Consider the mapping $f : \mathscr{E} \to \mathscr{F}$ where $\mathscr{E}$ and $\mathscr{F}$ are complete dioids. Their bottom elements are, respectively, denoted by $\varepsilon_{\mathscr{E}}$ and $\varepsilon_{\mathscr{F}}$. Then, f is residuated iff $f(\varepsilon_{\mathscr{E}}) = \varepsilon_{\mathscr{F}}$ and $f(\bigoplus_{x \in \mathscr{G}} x) = \bigoplus_{x \in \mathscr{G}} f(x)$ for each $\mathscr{G} \subseteq \mathscr{E}$ (i.e. f is *lower-semicontinuous abbreviated l.s.c.*).

Corollary 1: The mappings $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$ defined over a complete dioid $\mathscr{D}$ are both residuated.[3] Their residuals are usually denoted, respectively, by $L_a^\sharp(x) = a \backslash\!\!\circ x$ and $R_a^\sharp(x) = x \circ\!\!/ a$ in $(\max, +)$ literature.[4]

The problem of mapping restriction and its connection with the residuation theory is now addressed.

Proposition 1 ([3]): Let $\mathsf{Id}_{|\mathscr{D}_{sub}} : \mathscr{D}_{sub} \to \mathscr{D}$, $x \mapsto x$ be the canonical injection from a complete subdioid into a complete dioid. The injection $\mathsf{Id}_{|\mathscr{D}_{sub}}$ is residuated and its residual is a projector which will be denoted by Pr_{sub}, therefore :

$$\mathsf{Pr}_{sub} = \left(\mathsf{Id}_{|\mathscr{D}_{sub}}.\right)^\sharp = \mathsf{Pr}_{sub} \circ \mathsf{Pr}_{sub}$$

Definition 4 (Restricted mapping): Let $f : \mathscr{E} \to \mathscr{F}$ be a mapping and $\mathscr{A} \subseteq \mathscr{E}$. We will denote[5] $f_{|\mathscr{A}} : \mathscr{A} \to \mathscr{F}$ the mapping defined by $f_{|\mathscr{A}} = f \circ \mathsf{Id}_{|\mathscr{A}}$ where $\mathsf{Id}_{|\mathscr{A}} : \mathscr{A} \to \mathscr{E}$. Identically, let $\mathscr{B} \subseteq \mathscr{F}$ with $\mathsf{Im} f \subseteq \mathscr{B}$. Mapping ${}_{\mathscr{B}|}f : \mathscr{E} \to \mathscr{B}$ is defined by $f = \mathsf{Id}_{|\mathscr{B}} \circ {}_{\mathscr{B}|}f$, where $\mathsf{Id}_{|\mathscr{B}} : \mathscr{B} \to \mathscr{F}$.

Proposition 2: Let $f : \mathscr{D} \to \mathscr{E}$ be a residuated mapping and $\mathscr{D}_{sub}$ (resp. $\mathscr{E}_{sub}$) be a complete subdiod of $\mathscr{D}$ (resp. $\mathscr{E}$).

1. Mapping $f_{|\mathscr{D}_{sub}}$ is residuated and its residual is given by :
$$(f_{|\mathscr{D}_{sub}})^\sharp = (f \circ \mathsf{Id}_{|\mathscr{D}_{sub}})^\sharp = \mathsf{Pr}_{sub} \circ f^\sharp$$
2. If $\mathsf{Im} f \subset \mathscr{E}_{sub}$ then mapping ${}_{\mathscr{E}_{sub}|}f$ is residuated and its residual is given by:
$$\left({}_{\mathscr{E}_{sub}|}f\right)^\sharp = f^\sharp \circ \mathsf{Id}_{|\mathscr{E}_{sub}} = \left(f^\sharp\right)_{|\mathscr{E}_{sub}}.$$

Proof: Statement 1 follows directly from Property 3 and Proposition 1. Statement 2 is obvious since f is residuated and $\mathsf{Im} f \subset \mathscr{E}_{sub} \subset \mathscr{E}$. ■

Definition 5 (Closure mapping): An isotone mapping $f : \mathscr{E} \to \mathscr{E}$ defined on an ordered set $\mathscr{E}$ is a *closure mapping* if $f \succeq \mathsf{Id}_{\mathscr{E}}$ and $f \circ f = f$.

Proposition 3 ([7]): Let $f : \mathscr{E} \to \mathscr{E}$ be a closure mapping. A closure mapping restricted to its image ${}_{\mathsf{Im} f|}f$ is a residuated mapping whose residual is the canonical injection $\mathsf{Id}_{|\mathsf{Im} f} : \mathsf{Im} f \to \mathscr{E}$, $x \mapsto x$.

Corollary 2: The mapping ${}_{\mathsf{Im}\mathscr{K}|}\mathscr{K}$ is a residuated mapping whose residual is $\left({}_{\mathsf{Im}\mathscr{K}|}\mathscr{K}\right)^\sharp = \mathsf{Id}_{|\mathsf{Im}\mathscr{K}}$. This means that $x = a^*$ is the greatest solution to inequality $x^* \preceq a^*$. Actually, the greatest solution achieves equality.

III. Dioid of pairs

The set of pairs (x', x'') with $x' \in \mathscr{D}$ and $x'' \in \mathscr{D}$ endowed with two coordinate-wise algebraic operations :

$$(x', x'') \oplus (y', y'') = (x' \oplus y', x'' \oplus y'')$$
$$\text{and} \quad (x', x'') \otimes (y', y'') = (x' \otimes y', x'' \otimes y''),$$

is a dioid denoted by $\mathscr{C}(\mathscr{D})$ with $(\varepsilon, \varepsilon)$ as the zero element and (e, e) as the identity element (see definition 1).

Remark 1: The operation $\oplus$ generates the corresponding canonical partial order $\preceq_{\mathscr{C}}$ in $\mathscr{C}(\mathscr{D})$:
$(x', x'') \oplus (y', y'') = (y', y'') \Leftrightarrow (x', x'') \preceq_{\mathscr{C}} (y', y'') \Leftrightarrow x' \preceq_{\mathscr{D}} y'$ and $x'' \preceq_{\mathscr{D}} y''$ where $\preceq_{\mathscr{D}}$ is the order relation in $\mathscr{D}$.

Proposition 4 ([12]): If the dioid $\mathscr{D}$ is complete, then the dioid $\mathscr{C}(\mathscr{D})$ is complete and its top element is given by $(\top, \top)$.

Notation 1: Let us consider the following mappings over $\mathscr{C}(\mathscr{D})$:

$$L_{(a',a'')} : (x', x'') \mapsto (a', a'') \otimes (x', x'');$$
$$R_{(a',a'')} : (x', x'') \mapsto (x', x'') \otimes (a', a'').$$

Proposition 5: The mappings $L_{(a',a'')}$ and $R_{(a',a'')}$ defined over $\mathscr{C}(\mathscr{D})$ are both residuated. Their residuals are equal to $L^\sharp_{(a',a'')}(b', b'') = (a', a'') \backslash\!\!\circ (b', b'') = (a' \backslash\!\!\circ b', a'' \backslash\!\!\circ b'')$ and $R^\sharp_{(a',a'')}(b', b'') = (b', b'') \circ\!\!/ (a', a'') = (b' \circ\!\!/ a', b'' \circ\!\!/ a'')$.

Proof: Observe that $L_{(a',a'')}\left(\bigoplus_{(x',x'') \in X} (x', x'')\right) = \bigoplus_{(x',x'') \in X} L_{(a',a'')}(x', x'')$, (for every subset X of $\mathscr{C}(\mathscr{D})$), moreover $L_{(a',a'')}(\varepsilon, \varepsilon) = (a'\varepsilon, a''\varepsilon) = (\varepsilon, \varepsilon)$. Then $L_{(a',a'')}$ is residuated (follows from Theorem 2). Therefore, we have to find, for given (b', b'') and (a', a''), the greatest solution (x', x'') for inequality $(a', a'') \otimes (x', x'') \preceq_{\mathscr{C}} (b', b'') \Leftrightarrow (a' \otimes x', a'' \otimes x'') \preceq_{\mathscr{C}} (b', b'')$, moreover according to Remark 1 on the order relation induced by $\oplus$ on $\mathscr{C}(\mathscr{D})$ we have,

$$a' \otimes x' \preceq_{\mathscr{D}} b' \quad \text{and} \quad a'' \otimes x'' \preceq_{\mathscr{D}} b''.$$

[3]This property concerns as well a matrix dioid product, for instance $X \mapsto AX$ where $A, X \in \mathscr{D}^{n \times n}$. See [2] for the computation of $A \backslash\!\!\circ B$ and $B \circ\!\!/ A$.

[4]$a \backslash\!\!\circ b$ is the greatest solution of $ax \preceq b$.

[5]These notations are borrowed from classical linear system theory see [17].

Since the mappings $x' \mapsto a' \otimes x'$ and $x'' \mapsto a'' \otimes x''$ are residuated over $\mathscr{D}$ (cf. Corollary 1), we have $x' \preceq_{\mathscr{D}} a' \circ\!\!\!\backslash b'$ and $x'' \preceq_{\mathscr{D}} a'' \circ\!\!\!\backslash b''$. Then, we obtain $L^{\sharp}_{(a',a'')}(b',b'') = (a' \circ\!\!\!\backslash b', a'' \circ\!\!\!\backslash b'')$. ∎

Notation 2: The set of pairs $(\widetilde{x}',\widetilde{x}'')$ s.t. $\widetilde{x}' \preceq \widetilde{x}''$ is denoted by $\mathscr{C}_0(\mathscr{D})$.

Proposition 6: Let $\mathscr{D}$ be a complete dioid. The set $\mathscr{C}_0(\mathscr{D})$ is a complete subdioid of $\mathscr{C}(\mathscr{D})$.

Proof: Clearly $\mathscr{C}_0(\mathscr{D}) \subset \mathscr{C}(\mathscr{D})$ and it is closed for $\oplus$ and $\otimes$ since: $\widetilde{x}' \oplus \widetilde{y}' \preceq \widetilde{x}'' \oplus \widetilde{y}''$ and $\widetilde{x}' \otimes \widetilde{y}' \preceq \widetilde{x}'' \otimes \widetilde{y}''$ whenever $\widetilde{x}' \preceq \widetilde{x}''$ and $\widetilde{y}' \preceq \widetilde{y}''$. Moreover zero element $(\varepsilon,\varepsilon)$, unit element (e,e) and top element $(\top,\top)$ of $\mathscr{C}(\mathscr{D})$ are in $\mathscr{C}_0(\mathscr{D})$. ∎

Proposition 7: The canonical injection $\mathsf{Id}_{|\mathscr{C}_0(\mathscr{D})} : \mathscr{C}_0(\mathscr{D}) \to \mathscr{C}(\mathscr{D})$ is residuated. Its residual $(\mathsf{Id}_{|\mathscr{C}_0(\mathscr{D})})^{\sharp}$ is a projector denoted by $\mathsf{Pr}_{\mathscr{C}_0(\mathscr{D})}$. Its practical computation is given by :

$$\mathsf{Pr}_{\mathscr{C}_0(\mathscr{D})}((x',x'')) \quad = \quad (x' \wedge x'', x'') = (\widetilde{x}', \widetilde{x}''). \qquad (5)$$

Proof: It is a direct application of Proposition 1, since $\mathscr{C}_0(\mathscr{D})$ is a subdioid of $\mathscr{C}(\mathscr{D})$. Moreover, let $(x',x'') \in \mathscr{C}(\mathscr{D})$, we have $\mathsf{Pr}_{\mathscr{C}_0(\mathscr{D})}((x',x'')) = (\widetilde{x}',\widetilde{x}'') = (x' \wedge x'', x'')$, which is the greatest pair such that :

$$\widetilde{x}' \preceq x', \quad \widetilde{x}'' \preceq x'' \quad \text{and} \quad \widetilde{x}' \preceq \widetilde{x}''.$$

∎

Definition 6: An isotone mapping f defined over $\mathscr{D}$ admits a natural extension over $\mathscr{C}_0(\mathscr{D})$, which is defined as $f(\widetilde{x}',\widetilde{x}'') = (f(\widetilde{x}'), f(\widetilde{x}''))$. For example, the Kleene star mapping in $\mathscr{C}_0(\mathscr{D})$ is defined by $\mathscr{K}(\widetilde{x}',\widetilde{x}'') = (\mathscr{K}(\widetilde{x}'), \mathscr{K}(\widetilde{x}'')) = (\widetilde{x}'^{*}, \widetilde{x}''^{*})$.

Proposition 8: Let $(\widetilde{a}',\widetilde{a}'') \in \mathscr{C}_0(\mathscr{D})$, mapping ${}_{\mathscr{C}_0(\mathscr{D})|}L_{(\widetilde{a}',\widetilde{a}'')|_{\mathscr{C}_0(\mathscr{D})}} : \mathscr{C}_0(\mathscr{D}) \to \mathscr{C}_0(\mathscr{D})$ is residuated. Its residual is given by

$$\left({}_{\mathscr{C}_0(\mathscr{D})|}L_{(\widetilde{a}',\widetilde{a}'')|_{\mathscr{C}_0(\mathscr{D})}}\right)^{\sharp} \quad = \quad \mathsf{Pr}_{\mathscr{C}_0(\mathscr{D})} \circ \left(L_{(\widetilde{a}',\widetilde{a}'')}\right)^{\sharp} \circ I_{|\mathscr{C}_0(\mathscr{D})}.$$

Proof: Since $(\widetilde{a}',\widetilde{a}'') \in \mathscr{C}_0(\mathscr{D}) \subset \mathscr{C}(\mathscr{D})$, it follows directly from Proposition 5 that mapping $L_{(\widetilde{a}',\widetilde{a}'')}$ defined over $\mathscr{C}(\mathscr{D})$ is residuated. Furthermore, $\mathscr{C}_0(\mathscr{D})$ being closed for $\otimes$ we have $\mathsf{Im}L_{(\widetilde{a}',\widetilde{a}'')|_{\mathscr{C}_0(\mathscr{D})}} \subset \mathscr{C}_0(\mathscr{D})$, it follows from Definition 4 and proposition 2 that :

$$\begin{aligned}\left({}_{\mathscr{C}_0(\mathscr{D})|}L_{(\widetilde{a}',\widetilde{a}'')|_{\mathscr{C}_0(\mathscr{D})}}\right)^{\sharp} &= \left(L_{(\widetilde{a}',\widetilde{a}'')} \circ I_{|\mathscr{C}_0(\mathscr{D})}\right)^{\sharp} \circ I_{|\mathscr{C}_0(\mathscr{D})} \\ &= \mathsf{Pr}_{\mathscr{C}_0(\mathscr{D})} \circ \left(L_{(\widetilde{a}',\widetilde{a}'')}\right)^{\sharp} \circ I_{|\mathscr{C}_0(\mathscr{D})}\end{aligned}$$

Then, by considering $(\widetilde{b}',\widetilde{b}'') \in \mathscr{C}_0(\mathscr{D}) \subset \mathscr{C}(\mathscr{D})$, the greatest solution in $\mathscr{C}_0(\mathscr{D})$ of $L_{(\widetilde{a}',\widetilde{a}'')}((\widetilde{x}',\widetilde{x}'')) = (\widetilde{a}',\widetilde{a}'') \otimes (\widetilde{x}',\widetilde{x}') \preceq (\widetilde{b}',\widetilde{b}'')$ is $L^{\sharp}_{(\widetilde{a}',\widetilde{a}'')}((\widetilde{b}',\widetilde{b}'')) = (\widetilde{x}',\widetilde{x}'') = (\widetilde{a}',\widetilde{a}'') \circ\!\!\!\backslash (\widetilde{b}',\widetilde{b}'') = \mathsf{Pr}_{\mathscr{C}_0(\mathscr{D})}((\widetilde{a}' \circ\!\!\!\backslash \widetilde{b}', \widetilde{a}'' \circ\!\!\!\backslash \widetilde{b}'')) = (\widetilde{a}' \circ\!\!\!\backslash \widetilde{b}' \wedge \widetilde{a}'' \circ\!\!\!\backslash \widetilde{b}'', \widetilde{a}'' \circ\!\!\!\backslash \widetilde{b}'')$. ∎

IV. DIOID AND INTERVAL MATHEMATICS

Interval mathematics was pioneered by R.E. Moore as a tool for bounding and rounding errors in computer programs. Since then, interval mathematics had been developed into a general methodology for investigating numerical uncertainty in numerous problems and algorithms, and is a powerful numerical tool for calculating guaranteed bounds on functions using computers.

In [12] the problem of interval mathematics in dioids is addressed. The authors give a weak interval extensions of dioids and show that idempotent interval mathematics appears to be remarkably simpler than its traditional analog. For example, in the traditional interval arithmetic, multiplication of intervals is not distributive with respect to addition of intervals, while idempotent interval arithmetic keeps this distributivity. Below, we state that residuation theory has a natural extension in dioid of intervals.

Definition 7: A (closed) interval in dioid $\mathscr{D}$ is a set of the form $\mathbf{x} = [\underline{x},\overline{x}] = \{t \in \mathscr{D} | \underline{x} \preceq t \preceq \overline{x}\}$, where $(\underline{x},\overline{x}) \in \mathscr{C}_0(\mathscr{D})$, $\underline{x}$ (respectively, $\overline{x}$) is said to be lower (respectively, upper) bound of the interval $\mathbf{x}$.

Proposition 9: The set of intervals, denoted by $\mathrm{I}(\mathscr{D})$, endowed with two coordinate-wise algebraic operations :

$$\mathbf{x} \mathbin{\bar{\oplus}} \mathbf{y} = \left[\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}\right] \quad \text{and} \quad \mathbf{x} \mathbin{\bar{\otimes}} \mathbf{y} = \left[\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}\right] \qquad (6)$$

is a dioid, where the interval $\boldsymbol{\varepsilon} = [\varepsilon,\varepsilon]$ (respectively, $\mathbf{e} = [e,e]$) is zero (respectively, unit) element of $\mathrm{I}(\mathscr{D})$.

Proof: First, $\underline{x} \oplus \underline{y} \preceq \overline{x} \oplus \overline{y}$ and $\underline{x} \otimes \underline{y} \preceq \overline{x} \otimes \overline{y}$ whenever $\underline{x} \preceq \overline{x}$ and $\underline{y} \preceq \overline{y}$, then $\mathrm{I}(\mathscr{D})$ is closed with respect to the operations $\bar{\oplus}, \bar{\otimes}$. From definition 1, it follows directly that it is a dioid. ∎

Remark 2: According to remark 1, the order relation on $\mathrm{I}(\mathscr{D})$ is not an inclusion relation, but must be understood as follows :

$$\mathbf{x} \mathbin{\bar{\oplus}} \mathbf{y} = \mathbf{y} \iff \mathbf{x} \preceq_{\mathrm{I}(\mathscr{D})} \mathbf{y} \iff \underline{x} \preceq_{\mathscr{D}} \underline{y} \text{ and } \overline{x} \preceq_{\mathscr{D}} \overline{y}. \qquad (7)$$

Definition 8: Let $\mathscr{D}$ be a complete dioid and $\{\mathbf{x}_\alpha\}$ be an infinite subset of $\mathrm{I}(\mathscr{D})$, the infinite sum of elements of this subset is :

$$\overline{\bigoplus_{\alpha}} \mathbf{x}_\alpha = \left[\bigoplus_{\alpha} \underline{x}_\alpha, \bigoplus_{\alpha} \overline{x}_\alpha\right].$$

Remark 3: If $\mathscr{D}$ is a complete dioid then $\mathrm{I}(\mathscr{D})$ is a complete dioid by considering definition 8. Its top element is given by $\mathsf{T} = [\top,\top]$.

Note that if $\mathbf{x}$ and $\mathbf{y}$ are intervals in $\mathrm{I}(\mathscr{D})$, then $\mathbf{x} \subset \mathbf{y}$ iff $\underline{y} \preceq \underline{x} \preceq \overline{x} \preceq \overline{y}$. In particular, $\mathbf{x} = \mathbf{y}$ iff $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$.

An interval for which $\underline{x} = \overline{x}$ is called *degenerate*. Degenerate intervals allow to represent numbers without uncertainty. In this case we identify $\mathbf{x}$ with its element by writing $\mathbf{x} \equiv x$.

Proposition 10: Mapping $L_{\mathbf{a}} : \mathrm{I}(\mathscr{D}) \to \mathrm{I}(\mathscr{D}), \mathbf{x} \mapsto \mathbf{a} \mathbin{\bar{\otimes}} \mathbf{x}$ is residuated. Its residual is equal to $L^{\sharp}_{\mathbf{a}}(\mathbf{b}) = \mathbf{a} \circ\!\!\!\backslash \mathbf{b} = [\underline{a} \circ\!\!\!\backslash \underline{b} \wedge \overline{a} \circ\!\!\!\backslash \overline{b}, \overline{a} \circ\!\!\!\backslash \overline{b}]$.

Proof: Let $\Psi : \mathscr{C}_0(\mathscr{D}) \to \mathrm{I}(\mathscr{D}), (\widetilde{x}',\widetilde{x}'') \mapsto [\underline{x},\overline{x}] = [\widetilde{x}',\widetilde{x}'']$, the mapping which maps an interval to an ordered pair. This mapping defines an isomorphism of dioid, since it is sufficient to handle the bounds to handle an interval. Then the result follows directly from proposition 8. ∎

Remark 4: We would show in the same manner that mapping $R_{\mathbf{a}} : \mathrm{I}(\mathscr{D}) \to \mathrm{I}(\mathscr{D}), \mathbf{x} \mapsto \mathbf{x} \mathbin{\bar{\otimes}} \mathbf{a}$ is residuated.

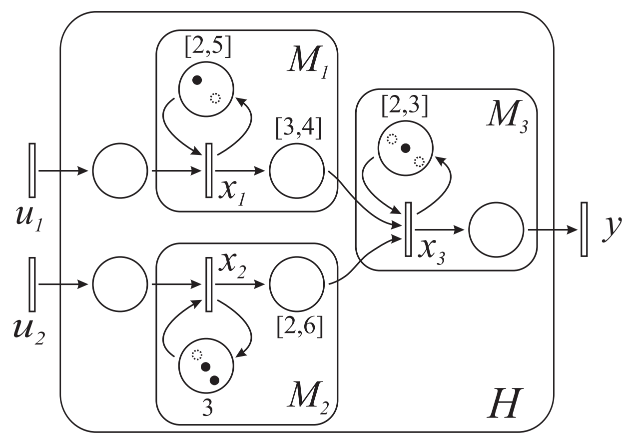

Fig. 1. A uncertain TEG with a controller (bold dotted lines)

V. INTERVAL ARITHMETIC AND TIMED EVENT GRAPHS

It is well known that the behavior of a TEG can be expressed by linear state equations over some dioids, *e.g.*, over dioid of formal power series with coefficients in $\overline{\mathbb{Z}}_{\max}$ and exponents in $\mathbb{Z}$ namely $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$.

$$X = AX \oplus BU \tag{8}$$
$$Y = CX \tag{9}$$

Where $X \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^n$ represents the internal transitions behavior, $U \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^p$ represents the input transitions behavior, $Y \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^q$ represents the output transitions behavior, and $A \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{n\times n}$, $B \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{n\times p}$ and $C \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{q\times n}$ represent the link between transitions.

Remark 5: A,B,C entries are periodic and causal series (i.e., rational and realizable series see [2]). We refer the reader to [6] and [7] for a complete presentation.

The uncertain systems, which will be considered, are TEG where the number of tokens and time delays are only known to belong to intervals. Therefore, uncertainties can be described by intervals with known lower and upper bounds and the matrices of equations (8) and (9) are such that $A \in \mathbf{A} \in \mathrm{I}\left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{n\times n}$, $B \in \mathbf{B} \in \mathrm{I}\left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{n\times p}$ and $C \in \mathbf{C} \in \mathrm{I}\left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{q\times n}$, each entry of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are intervals with bounds in dioid $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ with only non-negative exponents and coefficients integer values. By Theorem 1, equation (8) has the minimum solution $X = A^*BU$. Therefore, $Y = CA^*BU$ and the transfer function of the system is

$$H = CA^*B \in \mathbf{H} = \mathbf{CA}^*\mathbf{B} \in \mathrm{I}\left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{q\times p}, \tag{10}$$

where $\mathbf{H}$ represents the interval in which the transfer function will lie for all the variations of the parameters. Figure 1 shows a TEG with 2 inputs and 1 output, which may represent a manufacturing system with 3 machines. Machines M_1 and M_2 produce parts assembled on machine M_3. The token in dotted line means that the resource may or may not to be available to manufacture parts (e.g. a machine may be disabled for maintenance operations ...). Durations in brackets give the interval in which the temporization of the place may evolve. This temporization represents the minimal sojourn time that the token must leave in the place before to contribute to the firing of the downstream transition. This may represent an operation with a processing time which is not well known (e.g. a task executed by an human, ...). For instance, machine M_1 can manufacture 1 or 2 parts and each processing time will last between 2 and 5 time units, this leads to a parameter which evolves in interval which is given by $\mathbf{A}_{1,1} = [2\gamma^2, 5\gamma]$ according to order relation on $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$. The exponent in γ denotes resource number, and the coefficient depicts the processing time. Therefore, we obtain the following interval matrices,

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} [2\gamma^2,5\gamma] & [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] \\ [\varepsilon,\varepsilon] & [3\gamma^3,3\gamma^2] & [\varepsilon,\varepsilon] \\ [3,4] & [2,6] & [2\gamma^3,3\gamma] \end{pmatrix}, \\
\mathbf{B} &= \begin{pmatrix} [e,e] & [\varepsilon,\varepsilon] \\ [\varepsilon,\varepsilon] & [e,e] \\ [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] \end{pmatrix} \\
\mathbf{C} &= \begin{pmatrix} [\varepsilon,\varepsilon] & [\varepsilon,\varepsilon] & [e,e] \end{pmatrix}.
\end{aligned} \tag{11}$$

it follows from Theorem 1 that the transfer function H belongs to the interval matrix $\mathbf{H}$ given below. It characterizes the whole transfer functions arising from (11):

$$\mathbf{H} = \mathbf{CA}^*\mathbf{B} = \begin{pmatrix} [3(2\gamma^2)^*, 4(5\gamma)^*] & [2(3\gamma^3)^*, 6(3\gamma)^*] \end{pmatrix}. \tag{12}$$

VI. ROBUST OPEN-LOOP CONTROL SYNTHESIS

We focus here on the robust open-loop control of a p-input q-output TEG. This problem can be expressed as follows : given the desired output $\mathbf{Z} = [\mathbf{z}_1 \ldots \mathbf{z}_q]^t$, *i.e.* the admissible firing dates for each output transition, find the robust input control $\mathbf{U} = [\mathbf{u}_1 \ldots \mathbf{u}_p]^t$, *i.e.* the firing dates for each input transition, such that the system output $\mathbf{Y} = [\mathbf{y}_1 \ldots \mathbf{y}_q]^t$, *i.e.* the firing dates for each output, be included in the set $\mathbf{Z}$ whatever be the parameters variations of the system $H \subset \mathbf{H}$. Finally, proposition 11 will give assumption in order to compute the greatest interval $\hat{\mathbf{U}}$ included in the set of robust control laws, defined as follows :

$$\mathscr{U} = \left\{ U \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{p\times 1} \mid \mathbf{H}U \subset \mathbf{Z} \right\}. \tag{13}$$

Lemma 1: The greatest interval control $\mathbf{U}$ such that $\mathbf{HU} = \mathbf{CA}^*\mathbf{BU} \preceq \mathbf{Z}$ is given by

$$\hat{\mathbf{U}} = \left[\underline{\hat{U}}, \overline{\hat{U}}\right] = L^\sharp_{\mathbf{H}}(\mathbf{Z}) = \mathbf{H} \lozenge\!\!\!\!\backslash\, \mathbf{Z} = \left[\underline{H} \lozenge\!\!\!\!\backslash\, \underline{Z} \wedge \overline{H} \lozenge\!\!\!\!\backslash\, \overline{Z}, \overline{H} \lozenge\!\!\!\!\backslash\, \overline{Z}\right]. \tag{14}$$

Proof: Since the mapping $L_{\mathbf{H}}(\mathbf{U}) : \mathbf{U} \mapsto \mathbf{HU}$ is residuated, proposition 10 gives directly the result. ■

Remark 6: It is important to note that the order relation considered here is the one of $\mathrm{I}(\mathscr{D})$.

Definition 9 (Reachability): An output interval $\mathbf{Z}$ of system (10) is called a reachable output interval if there exists a control $\mathbf{U}$ such that

$$\mathbf{Z} = \mathbf{HU} = \mathbf{CA}^*\mathbf{BU}.$$

Lemma 2: Let $\mathbf{Z}$ be an arbitrary output interval. The greatest reachability output interval from below of $\mathbf{Z}$ is given by

$$\begin{aligned}
\tilde{\mathbf{Z}} &= \Pi_{L_{\mathbf{H}}}(\mathbf{Z}) = L_{\mathbf{H}} \circ L^\sharp_{\mathbf{H}}(\mathbf{Z}) \\
&= \mathbf{H}\left(\mathbf{H} \lozenge\!\!\!\!\backslash\, \mathbf{Z}\right).
\end{aligned}$$

Proof: It is a direct application of property 1, $\tilde{\mathbf{Z}}$ is the best approximation of $\mathbf{Z}$ in $\mathsf{Im}L_{\mathbf{H}}$, that is, the greatest $\tilde{\mathbf{Z}} \in \mathsf{Im}L_{\mathbf{H}}$ less than $\mathbf{Z}$. ■

Proposition 11: If the desired output interval $\mathbf{Z}$ is reachable, then

$$\hat{\mathbf{U}} \subset \mathscr{U}, \tag{15}$$

where $\hat{\mathbf{U}} = \mathbf{H} \bar{\circ\!\!\backslash} \mathbf{Z} = [\underline{H} \circ\!\!\backslash \underline{Z} \wedge \overline{H} \circ\!\!\backslash \overline{Z}, \overline{H} \circ\!\!\backslash \overline{Z}]$ (see lemma 1).

Proof: Since $\mathbf{Z}$ is reachable, $\mathbf{Z} \in \mathrm{Im} L_{\mathbf{H}}$ then $L_{\mathbf{H}}(\hat{\mathbf{U}}) = L_{\mathbf{H}} \circ L^{\sharp}_{\mathbf{H}}(\mathbf{Z}) = \mathbf{Z}$ due to Property 2, thus $\mathbf{H}\hat{\mathbf{U}} \subset \mathbf{Z}$. Obviously, this is equivalent to $\forall U \in \hat{\mathbf{U}}, \mathbf{H}U \subset \mathbf{Z}$, which leads to the result. ■

Remark 7: This result shows that if the specification is reachable, the optimal solution of inequality problem given in lemma 1 solves an inclusion problem. More precisely all the control law $U \in \hat{\mathbf{U}}$ ensures that the output of the controlled system satisfies $\underline{Z} \preceq Y = HU \preceq \overline{Z}$ for all $H \in \mathbf{H}$.

Corollary 3: If $\mathbf{Z}$ is reachable, then the upper bound of the interval $\hat{\mathbf{U}}$, denoted by $\overline{\hat{U}}$, is the upper bound of the set of robust control $\mathscr{U}$.

Proof: Since $\mathbf{Z}$ is reachable, $\mathbf{Z} \in \mathrm{Im} L_{\mathbf{H}}$ and there exists a control $\hat{\mathbf{U}}$ such that $\mathbf{H}\hat{\mathbf{U}} \subset \mathbf{Z}$ (see proposition 11), where the upper bound of $\hat{\mathbf{U}}$ is given by $\overline{\hat{U}} = \overline{H} \circ\!\!\backslash \overline{Z}$. Then, the upper bound of interval $\mathbf{H}\hat{\mathbf{U}} = [\underline{H\hat{U}}, \overline{H\hat{U}}]$ is equal to $\overline{H}(\overline{H} \circ\!\!\backslash \overline{Z})$ and since $\mathbf{Z} \in \mathrm{Im} L_{\mathbf{H}}$, we have $\overline{H}(\overline{H} \circ\!\!\backslash \overline{Z}) = \overline{Z}$. This means that $\overline{\hat{U}}$ is the upper bound of $\mathscr{U}$, indeed $\overline{\hat{U}}$ is the greatest control such that $H\overline{\hat{U}} = \overline{Z}$. ■

VII. ILLUSTRATION

We consider synthesis of a robust open-loop control for the uncertain TEG depicted in Fig. 1. We consider the following interval

$$\begin{aligned}\mathbf{Z} = [\underline{Z}, \overline{Z}] = (&[18 \oplus 20\gamma^2 \oplus 25\gamma^3 \oplus 32\gamma^4 \oplus 35\gamma^5 \oplus 37\gamma^6 \oplus +\infty\gamma^7,\\ &23 \oplus 28\gamma^2 \oplus 35\gamma^3 \oplus 38\gamma^4 \oplus 41\gamma^5 \oplus 45\gamma^6 \oplus +\infty\gamma^7]).\end{aligned}$$

It is depicted in figure 2, the shaded area is the desired target for the system output.

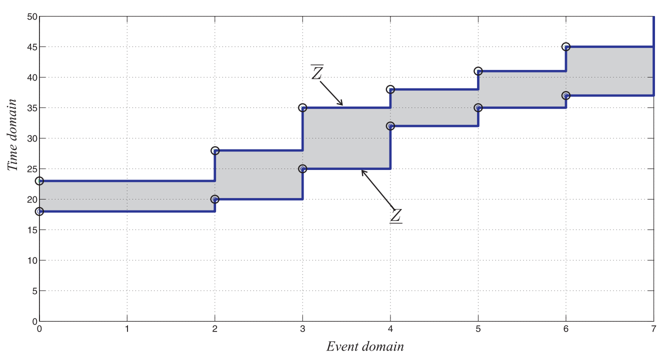

Fig. 2. Desired output interval $\mathbf{Z}$ (the customer demand)

This desired output must be interpreted as follows : for the output we want that parts 0 (the first event is numbered 0) and 1 are available in time interval $[18, 23]$, parts 2 in time interval $[20, 28]$ and so on *mutatis mutandis*, coefficient $+\infty$ means that part 7 never occurs. Thanks to lemma 2 the greatest reachable target interval is given by

$$\begin{aligned}\tilde{\mathbf{Z}} &= \mathbf{H}(\mathbf{H} \circ\!\!\backslash \mathbf{Z})\\ [\underline{\tilde{Z}}, \overline{\tilde{Z}}] &= ([16 \oplus 18\gamma \oplus 20\gamma \oplus 25\gamma^3 \oplus 32\gamma^4 \oplus 35\gamma^5 \oplus 37\gamma^6 \oplus +\infty\gamma^7,\\ &\quad 20 \oplus 23\gamma \oplus 28\gamma^2 \oplus 35\gamma^3 \oplus 38\gamma^4 \oplus 41\gamma^5 \oplus 45\gamma^6 \oplus +\infty\gamma^7]).\end{aligned}$$

It is depicted in figure 3. $\tilde{\mathbf{Z}}$ is the greatest reachable target for the uncertain system $H \subset \mathbf{H}$. Clearly $\mathbf{Z} \neq \tilde{\mathbf{Z}}$, that is $\mathbf{Z}$ is not a reachable target and $\tilde{\mathbf{Z}}$ is the best approximation from below.

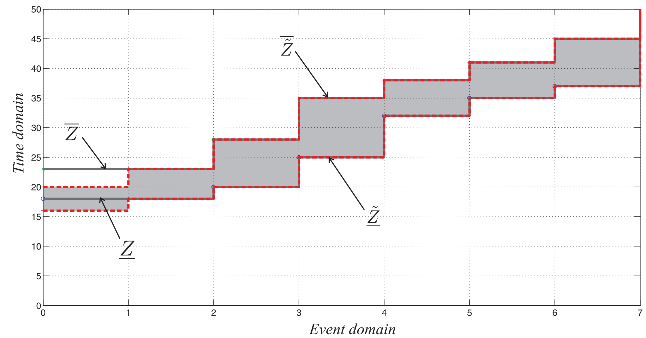

Fig. 3. Greatest reachable output interval $\tilde{\mathbf{Z}}$

Thanks to lemma 1 the greatest input $\hat{\mathbf{U}}$ is given by

$$\hat{\mathbf{U}} = \left(\left[\underline{\hat{U}}_1, \overline{\hat{U}}_1\right], \left[\underline{\hat{U}}_2, \overline{\hat{U}}_2\right]\right) = \mathbf{H} \bar{\circ\!\!\backslash} \mathbf{Z},$$

which is expressed as follows

$$\begin{aligned}\left[\underline{\hat{U}}_1, \overline{\hat{U}}_1\right] = [&11 \oplus 15\gamma \oplus 17\gamma^2 \oplus 22\gamma^3 \oplus 29\gamma^4 \oplus 32\gamma^5 \oplus 34\gamma^6 \oplus \infty\gamma^6,\\ &11 \oplus 16\gamma \oplus 21\gamma^2 \oplus 26\gamma^3 \oplus 31\gamma^4 \oplus 36\gamma^5 \oplus 41\gamma^6 \oplus \infty\gamma^6]\end{aligned}$$

and

$$\begin{aligned}\left[\underline{\hat{U}}_2, \overline{\hat{U}}_2\right] = [&14 \oplus 16\gamma \oplus 18\gamma^2 \oplus 23\gamma^3 \oplus 30\gamma^4 \oplus 33\gamma^5 \oplus 35\gamma^6 \oplus \infty\gamma^7,\\ &14 \oplus 17\gamma \oplus 22\gamma^2 \oplus 29\gamma^3 \oplus 32\gamma^4 \oplus 35\gamma^5 \oplus 39\gamma^6 \oplus \infty\gamma^7].\end{aligned}$$

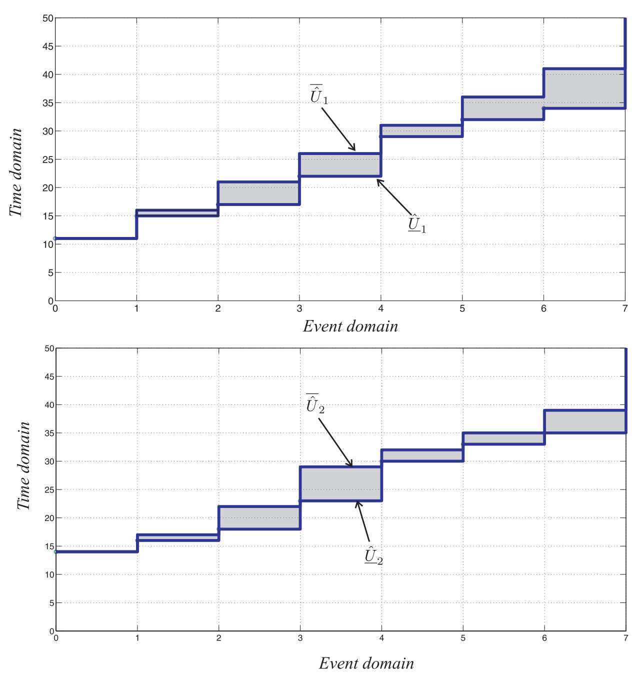

Fig. 4. Margins on each entry of control vector $\mathbf{U}$

Figure 4 represents the entries of the control vector $\hat{\mathbf{U}}$, each control $U \in \hat{\mathbf{U}}$ will guarantee that the output be in the target $\tilde{\mathbf{Z}}$ given figure 3.

Remark 8: The reader can find software tools in order to handle periodic series and solve the illustration (see [16]).

VIII. CONCLUSION

In this paper we assumed that the TEG includes some parametric uncertainties (for example on delays and ressources availability) in a bounded context. We have given a set of robust open-loop control which ensures that the output of the controlled system is in a given interval for all feasible values for the parameters. The next step is to extend this work to other control structure such as the one given in [14] as initiated in [11]. The traditional interval theory is very effective for parameter estimation, it would be interesting to apply the results of this paper to the TEG parameter estimation such as intended in [9], [8].

REFERENCES

[1] H. Ayhan and M.A. Wortman. Job flow control in assembly operations. *IEEE Transactions on Automatic Control*, 44(4):864–868, 1999.

[2] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity : An Algebra for Discrete Event Systems*. Wiley and Sons, 1992.

[3] T.S. Blyth and M.F. Janowitz. *Residuation Theory*. Pergamon press, 1972.

[4] J.-Y Le Boudec and P. Thiran. *Network Calculus*. Springer Verlag, 2002.

[5] G. Cohen, S. Gaubert, and J.P. Quadrat. Kernels, images and projections in dioids. In *Proceedings of WODES'96*, Edinburgh, August. 1996.

[6] G. Cohen, P. Moller, J.P. Quadrat, and M. Viot. Algebraic Tools for the Performance Evaluation of Discrete Event Systems. *IEEE Proceedings: Special issue on Discrete Event Systems*, 77(1):39–58, January 1989.

[7] B. Cottenceau, L. Hardouin, J.-L. Boimond, and J.-L. Ferrier. Model Reference Control for Timed Event Graphs in Dioid. *Automatica*, 37:1451–1458, August 2001.

[8] L. Hardouin. *Sur la commande linéaire de systèmes à événements discrets dans l'algèbre (*max,+*).* habilitation à diriger des recherches, Université d'Angers, Juin 2004.

[9] L. Jaulin, J.-L. Boimond, and L. Hardouin. Estimation of discrete event systems using interval computation. *Reliable Computing*, 5:165–173, 1999.

[10] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter. *Applied Interval Analysis with Examples in Parameter and State Estimation, Robust Control and Robotics*. Springer-Verlag, London, 2001.

[11] M. Lhommeau, L. Hardouin, B. Cottenceau, and L. Jaulin. Interval Analysis and Dioid : Application to Robut Controller Design for Timed Event Graphs. *Automatica*, 40:1923–1930, July 2004.

[12] G.-L. Litvinov and A.-N. Sobolevskii. Idempotent Interval Analysis and Optimization Problems. *Reliable Computing*, 7(5), 2001.

[13] R. Lüders and R. Santos-Mendes. Generalized Multivariable Control of Discrete Event Systems in Dioid. *Worshop on Discrete Event Systems, WODES'02, Zaragoza*, 2002.

[14] C.-A. Maia, L. Hardouin, R. Santos-Mendes, and B. Cottenceau. Optimal Closed-Loop of Timed Event Graphs in Dioids. *IEEE Trans. On Automatic Control*, 48(12):2284–2287, Dec. 2003.

[15] E. Menguy, J.-L. Boimond, L. Hardouin, and J.-L. Ferrier. Just in Time Control of Timed Event Graphs: Update of Reference Input, Presence of Uncontrollable Input. *IEEE Trans. on Automatic Control*, 45(11):2155–2158, November 2000.

[16] SW2001. Software Tools for Manipulating Periodic Series. `http://www.istia.univ-angers.fr/~hardouin/`, `http://amadeus.inria.fr/gaubert/PAPERS/MAX.html`, `http://www.maxplus.org/`, 2001.

[17] W. Wonham. *Linear Multivariable Control : A Geometric Approach, 3rd edition.* Springer Verlag, 1985.