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Asymptotic Solution of Singularly Perturbed Nonlinear Discrete Periodic Optimal Control Problem

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Abstract—**The asymptotic expansion of the solution of a singularly perturbed nonlinear discrete periodic optimal control problem is constructed as series of non-negative integer powers of a small parameter. The estimates are obtained for closeness of the approximate solutions to the exact one and it is proved that the values of the minimized functional do not increase when higher-order approximations to the optimal control are used.**

I. INTRODUCTION

FOR the last thirty years many works devoted to the study of singularly perturbed optimal control problems have been published (see, for example, [1]-[3]). The most part of these publications deals with continuous systems. In [4], [5], the formal asymptotic solution of singularly perturbed linear-quadratic discrete problem with the fixed left point and the free right point has been constructed using the asymptotic expansion of the solution of the system following from the control optimality condition. The method of constructing an asymptotic expansion of the solution of an optimal control problem by substituting the postulated asymptotic expansion into the condition of the problem, and then determining a series of optimal control problems to find the expansion terms, has been used in [6] for singularly perturbed continuous systems without restrictions on the values of the controls; this method has been called the "direct scheme". The direct scheme has been also used for discrete optimal control problems with small step-size [7], [8], discrete optimal control problems for weakly controllable systems [9] and discrete periodic singularly perturbed linear-quadratic problem [10].

In the present paper, the asymptotic expansion of the solution of a singularly perturbed nonlinear discrete periodic optimal control problem is constructed as series of non-negative integer powers of a small parameter by means of the direct scheme. The estimates are obtained for the closeness of the approximate solutions to the exact one in terms of the control, the trajectory and the functional. It is also proved that the values of the minimized functional do not increase when higher-order approximations to the optimal control are used.

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II. PROBLEM FORMULATION

The following singularly perturbed nonlinear discrete periodic optimal control problem is considered

$$P_\varepsilon : J_\varepsilon(u) = \sum_{k=0}^{N-1} F_k(y(k), \varepsilon z(k), u(k)) \to \min_u, \tag{1}$$

$$\begin{aligned} y(k+1) &= f_k(y(k), \varepsilon z(k), u(k)), \\ z(k+1) &= g_k(y(k), \varepsilon z(k), u(k)), \end{aligned} \tag{2}$$

$$y(0) = y(N), z(0) = z(N), \tag{3}$$

where $y(k) \in R^n$, $z(k) \in R^m$ $(k = \overline{0, N})$, $u(k) \in R^r$ $(k = \overline{0, N-1})$, F_k is a scalar function, f_k is a function with values in R^n, g_k is a function with values in R^m, $\varepsilon > 0$ is a small parameter, the number of steps N is fixed. The functions F_k, f_k, g_k occurring in (1) and (2) are assumed to be continuously differentiable a sufficient number of times with respect to their arguments.

III. DEGENERATE PROBLEM

Let us consider the degenerate problem $\overline{P}$ when $\varepsilon = 0$ in problem (1)-(3). Namely, we have

$$\overline{P} : J_0(u) = \sum_{k=0}^{N-1} F_k(y(k), 0, u(k)) \to \min_u, \tag{4}$$

$$\begin{aligned} y(k+1) &= f_k(y(k), 0, u(k)), \\ z(k+1) &= g_k(y(k), 0, u(k)), \end{aligned} \tag{5}$$

$$y(0) = y(N),\ z(0) = z(N). \tag{6}$$

Problem $\overline{P}$ may be reduced to the optimal control problem with respect to u, y.

Assume that the following condition 1^0 holds.

Assumption 1^0. The system (5), (6) has the unique solution for any control u.

The Hamiltonian $\overline{H}(k)$ for the Problem $\overline{P}$ is given by the formula

$$\overline{H}(k) = -F_k(y(k), 0, u(k)) + p'(k+1) f_k(y(k), 0, u(k)) + \\ + q'(k+1) g_k(y(k), 0, u(k)),$$

where the adjoint variables $p(k)$, $q(k)$ $(k = \overline{0, N})$ is the solution of the problem

$$p(k)=\left(\frac{\partial \overline{H}(k)}{\partial y}\right)'=-(F_k(y(k),0,u(k)))_y'+$$
$$+(f_k(y(k),0,u(k)))_y' p(k+1)+$$
$$+(g_k(y(k),0,u(k)))_y' q(k+1), \quad (7)$$
$$q(k)=\left(\frac{\partial \overline{H}(k)}{\partial z}\right)'=0, \quad k=\overline{0,N-1},$$
$$p(0)=p(N), q(0)=q(N). \quad (8)$$

Throughout this paper, the prime denotes the transposition and the expressions of the forms $(f)_y$, $(f)_z$, $(f)_u$ stand for the derivatives of the function f with respect to the first, second and third arguments respectively.

It follows from (7), (8) that $q(k)=0$, $k=\overline{0,N}$.

The control optimality condition for the Problem $\overline{P}$ has the form

$$-(F_k(y(k),0,u(k)))_u'+(f_k(y(k),0,u(k)))_u' p(k+1)+$$
$$+(g_k(y(k),0,u(k)))_u' q(k+1)=0. \quad (9)$$

We shall assume that the following condition is satisfied.

Assumption 2^0. Problem $\overline{P}$ has the unique solution.

IV. Formalism of Asymptotic Expansions Construction

We will seek a solution of the perturbed problem (1)-(3) in the series form

$$y(k)=\sum_{j\ge 0}\varepsilon^j y_j(k), \ z(k)=\sum_{j\ge 0}\varepsilon^j z_j(k),$$
$$u(k)=\sum_{j\ge 0}\varepsilon^j u_j(k). \quad (10)$$

We substitute relations (10) into expressions (1)-(3), expand the right-hand sides of (1), (2) in series in powers of ε, and then equate the coefficients of like powers of ε in equations (2), (3). Then the functional to be minimized may be written in the form

$$J_\varepsilon(u)=\sum_{j\ge 0}\varepsilon^j J_j, \quad (11)$$

and relations (2), (3) yield the equations for determining terms in the decompositions (10)

$$y_0(k+1)=\bar{f}_k,$$
$$z_0(k+1)=\bar{g}_k, \quad (12)$$

$$y_j(k+1)=(\bar{f}_k)_y y_j(k)+(\bar{f}_k)_u u_j(k)+[\tilde{f}_k]_j,$$
$$z_j(k+1)=(\bar{g}_k)_y y_j(k)+(\bar{g}_k)_u u_j(k)+[\tilde{g}_k]_j, j\ge 1. \quad (13)$$

$$y_j(0)=y_j(N), z_j(0)=z_j(N), j\ge 0. \quad (14)$$

Here and throughout this paper, the bar means that the values of appropriate functions and their derivatives are calculated at $y(k)=y_0(k)$, $\varepsilon z(k)=0$, $u(k)=u_0(k)$, the tilde over a function means that the function is evaluated at $y(k)=\tilde{y}_{j-1}(k)$, $z(k)=\tilde{z}_{j-1}(k)$, $u(k)=\tilde{u}_{j-1}(k)$, where

$$\tilde{y}_{j-1}(k)=\sum_{i=0}^{j-1}\varepsilon^i y_i(k), \ \tilde{z}_{j-1}(k)=\sum_{i=0}^{j-1}\varepsilon^i z_i(k),$$
$$\tilde{u}_{j-1}(k)=\sum_{i=0}^{j-1}\varepsilon^i u_i(k). \quad (15)$$

The following notations are also used for the expansion of an arbitrary function $\varphi=\varphi(\varepsilon)$ in powers of ε

$$\varphi(\varepsilon)=\sum_{j\ge 0}\varepsilon^j\varphi_j=\{\varphi\}_{n-1}+\varepsilon^n[\varphi]_n+\alpha(\varepsilon^{n+1}),$$

where

$$[\varphi]_n=\varphi_n, \ \{\varphi\}_{n-1}=\sum_{i=0}^{n-1}\varepsilon^i\varphi_i,$$

and $\alpha(\varepsilon^{n+1})$ denotes the sum of terms of the expansion of the order ε^{n+1} and higher. In particular, we have

$$[\tilde{f}_k]_1=(\bar{f}_k)_z z_0(k),$$
$$[\tilde{f}_k]_2=(\bar{f}_k)_z z_1(k)+\frac{1}{2}(\bar{f}_k)_{yy}y_1^2(k)+\frac{1}{2}(\bar{f}_k)_{uu}u_1^2(k)+$$
$$+\frac{1}{2}(\bar{f}_k)_{zz}z_0^2(k)+(\bar{f}_k)_{zy}z_0(k)y_1(k)+$$
$$+(\bar{f}_k)_{zu}z_0(k)u_1(k)+(\bar{f}_k)_{yu}y_1(k)u_1(k).$$

The analogous formulas take place for $[\tilde{g}_k]_1$, $[\tilde{g}_k]_2$.

Let us write down the first coefficients of expansion (11).

$$J_0(u)=\sum_{k=0}^{N-1}\overline{F}_k,$$

$$J_1(u)=\sum_{k=0}^{N-1}((\overline{F}_k)_y y_1(k)+(\overline{F}_k)_z z_0(k)+(\overline{F}_k)_u u_1(k)),$$

$$J_2(u)=\sum_{k=0}^{N-1}((\overline{F}_k)_y y_2(k)+(\overline{F}_k)_u u_2(k)+(\overline{F}_k)_z z_1(k)+$$
$$+\frac{1}{2}(\overline{F}_k)_{yy}y_1^2(k)+(\overline{F}_k)_{yz}y_1(k)z_0(k)+\frac{1}{2}(\overline{F}_k)_{zz}z_0^2(k)+$$
$$+\frac{1}{2}(\overline{F}_k)_{uu}u_1^2(k)+(\overline{F}_k)_{zu}z_0(k)u_1(k)+(\overline{F}_k)_{yu}y_1(k)u_1(k)).$$

To determine the triplet of functions u_0, y_0, z_0, we consider the Problem P_0 which consists of the minimization of functional J_0 on periodic trajectories of system (12). It is not difficult to see that Problem P_0 coincides with degenerate Problem $\overline{P}$.

Using relations (7)-(9), (13), (14), let us transform the terms in the expression for J_1 which remain unknown after Problem P_0 has been solved. Denoting adjoint variables in Problem P_0 by p_0, q_0, we have

$$\sum_{k=0}^{N-1}((\overline{F}_k)_y y_1(k)+(\overline{F}_k)_u u_1(k))=\sum_{k=0}^{N-1}((p_0^{'}(k+1)(\bar{f}_k)_y-$$

$$-p_0^{'}(k))y_1(k)+p_0^{'}(k+1)(\bar{f}_k)_u u_1(k))=$$

$$=\sum_{k=0}^{N-1}(p_0^{'}(k+1)((\bar{f}_k)_y y_1(k)+(\bar{f}_k)_u u_1(k))-p_0^{'}(k)y_1(k))=$$

$$=-\sum_{k=0}^{N-1}p_0^{'}(k+1)(\bar{f}_k)_z z_0(k).$$

Thus, the coefficient J_1 in expansion (11) depends only on the solution of Problem P_0.

Carrying out similar transformations for part of the terms in the expression for J_2, we have

$$\sum_{k=0}^{N-1}((\overline{F}_k)_y y_2(k)+(\overline{F}_k)_u u_2(k))=$$

$$=-\sum_{k=0}^{N-1}p_0^{'}(k+1)((\bar{f}_k)_z z_1(k)+\frac{1}{2}(\bar{f}_k)_{yy}y_1^2(k)+$$

$$+(\bar{f}_k)_{yz}y_1(k)z_0(k)+\frac{1}{2}(\bar{f}_k)_{zz}z_0^2(k)+\frac{1}{2}(\bar{f}_k)_{uu}u_1^2(k)+$$

$$+(\bar{f}_k)_{yu}y_1(k)u_1(k)+(\bar{f}_k)_{zu}z_0(k)u_1(k)).$$

Taking the last relation into account, let $\widetilde{J}_1$ denotes the sum of those terms in the expression for J_2 that are still unknown after Problem P_0 has been solved. To determine the triplet of functions u_1, y_1, z_1, we consider the following linear-quadratic problem

$$P_1:\widetilde{J}_1(u)=\sum_{k=0}^{N-1}(\frac{1}{2}y_1^{'}(k)W_k y_1(k)+\frac{1}{2}u_1^{'}(k)R_k u_1(k)+$$

$$+y_1^{'}(k)S_k u_1(k)+a_k^{'}y_1(k)+b_k^{'}z_1(k)+c_k^{'}u_1(k))\to\min_{u_1},$$

$$\begin{aligned}y_1(k+1)&=(\bar{f}_k)_y y_1(k)+(\bar{f}_k)_u u_1(k)+[\widetilde{f}_k]_1,\\ z_1(k+1)&=(\bar{g}_k)_y y_1(k)+(\bar{g}_k)_u u_1(k)+[\widetilde{g}_k]_1,\end{aligned}\qquad(16)$$

$$y_1(0)=y_1(N), z_1(0)=z_1(N),\qquad(17)$$

where

$$y_1^{'}(k)W_k y_1(k)=((\overline{F}_k)_{yy}-p_0^{'}(k+1)(\bar{f}_k)_{yy})y_1^2(k),$$

$$y_1^{'}(k)S_k u_1(k)=((\overline{F}_k)_{yu}-p_0^{'}(k+1)(\bar{f}_k)_{yu})y_1(k)u_1(k),$$

$$u_1^{'}(k)R_k u_1(k)=((\overline{F}_k)_{uu}-p_0^{'}(k+1)(\bar{f}_k)_{uu})u_1^2(k),$$

$$a_k^{'}y_1(k)=((\overline{F}_k)_{yz}-p_0^{'}(k+1)(\bar{f}_k)_{yz})y_1(k)z_0(k),$$

$$b_k^{'}z_1(k)=((\overline{F}_k)_z-p_0^{'}(k+1)(\bar{f}_k)_z)z_1(k),$$

$$c_k^{'}u_1(k)=((\overline{F}_k)_{zu}-p_0^{'}(k+1)(\bar{f}_k)_{zu})z_0(k)u_1(k).$$

The expressions for W_k, S_k, R_k, a_k, b_k, c_k depend on the Problem P_0 solution .

We shall assume that the following condition is satisfied.

Assumption 3^0. The operators $\begin{pmatrix}W_k & S_k\\ S_k^{'} & R_k\end{pmatrix}$ are non-negative definite, the operators R_k are positive definite, $k=\overline{0,N-1}$, and the system

$$y(k+1)=(A_k-B_k R_k^{-1}S_k^{'})y(k),\ k=\overline{0,N-1},$$

$$y(0)=y(N),$$

where $A_k=\begin{pmatrix}(\bar{f}_k)_y & 0\\ (\bar{g}_k)_y & 0\end{pmatrix}$, $B_k=\begin{pmatrix}(\bar{f}_k)_u\\ (\bar{g}_k)_u\end{pmatrix}$, has the unique solution.

Then the linear-quadratic problem P_1 is uniquely solvable.

It is not difficult to see that Problem P_1 may be reduced to the optimal control problem of less dimension with respect to u_1, y_1.

Taking into account the maximum principle, we have for the Problem P_1 solution the relations

$$p_1(k)=-W_k y_1(k)-S_k u_1(k)-a_k+(\bar{f}_k)_y^{'}p_1(k+1)+$$

$$+(\bar{g}_k)_y^{'}q_1(k+1),\qquad(18)$$

$$q_1(k)=-b_k,$$

$$p_1(0)=p_1(N), q_1(0)=q_1(N),\qquad(19)$$

$$\begin{aligned}&-R_k u_1(k)-S_k^{'}y_1(k)+(\bar{f}_k)_u^{'}p_1(k+1)+\\ &+(\bar{g}_k)_u^{'}q_1(k+1)=0.\end{aligned}\qquad(20)$$

Let us write down the Hamiltonian for Problem P_ε

$$H(k)=-F_k(y(k),\varepsilon z(k),u(k))+$$

$$+p^{'}(k+1)f_k(y(k),\varepsilon z(k),u(k))+$$

$$+q^{'}(k+1)g_k(y(k),\varepsilon z(k),u(k)),k=\overline{0,N-1},$$

where the adjoint variables $p(k)$, $q(k)$ $(k=\overline{0,N})$ satisfy the equations

$$p(k)=\left(\frac{\partial H(k)}{\partial y}\right)^{'}=-(F_k(y(k),\varepsilon z(k),u(k)))_y^{'}+$$

$$+(f_k(y(k),\varepsilon z(k),u(k)))_y^{'}p(k+1)+$$

$$+(g_k(y(k),\varepsilon z(k),u(k)))_y^{'}q(k+1),$$

$$\begin{aligned}q(k)&=\left(\frac{\partial H(k)}{\partial z}\right)^{'}=\varepsilon(-(F_k(y(k),\varepsilon z(k),u(k)))_z^{'}+\\ &+(f_k(y(k),\varepsilon z(k),u(k)))_z^{'}p(k+1)+\\ &+(g_k(y(k),\varepsilon z(k),u(k)))_z^{'}q(k+1)),\end{aligned}\qquad(21)$$

$$p(0)=p(N),q(0)=q(N).\qquad(22)$$

Since the problem under consideration involves no restrictions on the control, a necessary condition for an optimal control in Problem P_ε is that

$$\frac{\partial H(k)}{\partial u} = -(F_k(y(k), \varepsilon z(k), u(k)))_u + + p'(k+1)(f_k(y(k), \varepsilon z(k), u(k)))_u + + q'(k+1)(g_k(y(k), \varepsilon z(k), u(k)))_u = 0,\ k = \overline{0, N-1}. \tag{23}$$

Consider Problems P_j ($j \geq 0$). For $j = 0,1$, Problems P_0 and P_1 have already been defined; for $j > 1$, Problems P_j are linear-quadratic problems of the following form

$$P_j : \tilde{J}_j(u_j) = \sum_{k=0}^{N-1} (\frac{1}{2} y_j'(k) W_k y_j(k) + \frac{1}{2} u_j'(k) R_k u_j(k) + + y_j'(k) S_k u_j(k) + [(\tilde{F}_k)_z - \tilde{p}_{j-1}'(k+1)(\tilde{f}_k)_z - - \tilde{q}_{j-1}'(k+1)(\tilde{g}_k)_z]_{j-1} z_j(k) + [(\tilde{F}_k)_y - \tilde{p}_{j-1}'(k+1)(\tilde{f}_k)_y - - \tilde{q}_{j-1}'(k+1)(\tilde{g}_k)_y]_j y_j(k) + [(\tilde{F}_k)_u - \tilde{p}_{j-1}'(k+1)(\tilde{f}_k)_u - - \tilde{q}_{j-1}'(k+1)(\tilde{g}_k)_u]_j u_j(k)) \to \min_{u_j}, \tag{24}$$

$$y_j(k+1) = (\bar{f}_k)_y y_j(k) + (\bar{f}_k)_u u_j(k) + [\tilde{f}_k]_j, \quad z_j(k+1) = (\bar{g}_k)_y y_j(k) + (\bar{g}_k)_u u_j(k) + [\tilde{g}_k]_j, \tag{25}$$

$$y_j(0) = y_j(N), z_j(0) = z_j(N). \tag{26}$$

Recall that a tilde over the symbols for the functions F_k, f_k, g_k and their derivatives means that they are evaluated at $y = \tilde{y}_{j-1}$, $z = \tilde{z}_{j-1}$, $u = \tilde{u}_{j-1}$, where $\tilde{y}_{j-1}(k)$, $\tilde{z}_{j-1}(k)$, $\tilde{u}_{j-1}(k)$ are defined by (15) and the functions $\tilde{p}_{j-1}$, $\tilde{q}_{j-1}$ are defined by the equalities

$$\tilde{p}_{j-1}(k) = \sum_{i=0}^{j-1} \varepsilon^i p_i(k), \tilde{q}_{j-1}(k) = \sum_{i=0}^{j-1} \varepsilon^i q_i(k), \tag{27}$$

where p_i, q_i are the adjoint variables in Problem P_i.

Problem P_j may be reduced to the optimal control problem with respect to u_j, y_j.

The Hamiltonian for Problem P_j ($j > 1$) is

$$H_j(k) = -\frac{1}{2} y_j'(k) W_k y_j(k) - \frac{1}{2} u_j'(k) R_k u_j(k) - - y_j'(k) S_k u_j(k) - [(\tilde{F}_k)_z - \tilde{p}_{j-1}'(k+1)(\tilde{f}_k)_z - - \tilde{q}_{j-1}'(k+1)(\tilde{g}_k)_z]_{j-1} z_j(k) - [(\tilde{F}_k)_y - \tilde{p}_{j-1}'(k+1)(\tilde{f}_k)_y - - \tilde{q}_{j-1}'(k+1)(\tilde{g}_k)_y]_j y_j(k) - [(\tilde{F}_k)_u - \tilde{p}_{j-1}'(k+1)(\tilde{f}_k)_u - - \tilde{q}_{j-1}'(k+1)(\tilde{g}_k)_u]_j u_j(k) + p_j'(k+1)((\bar{f}_k)_y y_j(k) + + (\bar{f}_k)_u u_j(k) + [\tilde{f}_k]_j) + q_j'(k+1)((\bar{g}_k)_y y_j(k) + + (\bar{g}_k)_u u_j(k) + [\tilde{g}_k]_j),$$

where the adjoint variables $p_j(k)$, $q_j(k)$ $(k = \overline{0, N})$ satisfy the equations

$$p_j(k) = (H_j(k))'_{y_j} = -W_k y_j(k) - S_k u_j(k) - - [(\tilde{F}_k)'_y - (\tilde{f}_k)'_y \tilde{p}_{j-1}(k+1) - (\tilde{g}_k)'_y \tilde{q}_{j-1}(k+1)]_j + + (\bar{f}_k)'_y p_j(k+1) + (\bar{g}_k)'_y q_j(k+1), \tag{28}$$

$$q_j(k) = (H_j(k))'_{z_j} = -[(\tilde{F}_k)'_z - (\tilde{f}_k)'_z \tilde{p}_{j-1}(k+1) - - (\tilde{g}_k)'_z \tilde{q}_{j-1}(k+1)]_{j-1},$$

$$p_j(0) = p_j(N), q_j(0) = q_j(N). \tag{29}$$

The condition for optimality of the control in Problem P_j ($j > 1$) is the equality

$$(H_j(k))_{u_j} = -R_k u_j(k) - S_k' y_j(k) - - [(\tilde{F}_k)'_u - (\tilde{f}_k)'_u \tilde{p}_{j-1}(k+1) - (\tilde{g}_k)'_u \tilde{q}_{j-1}(k+1)]_j + + (\bar{f}_k)'_u p_j(k+1) + (\bar{g}_k)'_u q_j(k+1) = 0. \tag{30}$$

Let us substitute relations (10) and

$$p(k) = \sum_{j \geq 0} \varepsilon^j p_j(k), q(k) = \sum_{j \geq 0} \varepsilon^j q_j(k) \tag{31}$$

into expressions (2), (3), (21)-(23). Equating the coefficients of like powers of ε in the obtained relations, we have equations (5)-(9) when $j = 0$, equations (16)-(20) when $j = 1$.

Theorem 1. The equations for the state, control and adjoint variable, obtained from the condition for optimality of the control in Problem P_m, are identical with the equations for y_m, z_m, u_m, p_m, q_m from asymptotic expansions (10) and (31) of the solution of problem (2), (3), (21)-(23) obtained using the condition for optimality of the control in Problem P_ε.

Proof. For $m = 0,1$, the statement of the theorem has already been proved. Suppose it is true for $m < j$. For $j > 1$ we introduce the notations

$$\Delta y(k) = y(k) - \tilde{y}_{j-1}(k) = \varepsilon^j y_j(k) + \alpha(\varepsilon^{j+1}),$$
$$\Delta z(k) = z(k) - \tilde{z}_{j-1}(k) = \varepsilon^j z_j(k) + \alpha(\varepsilon^{j+1}),$$
$$\Delta p(k) = p(k) - \tilde{p}_{j-1}(k) = \varepsilon^j p_j(k) + \alpha(\varepsilon^{j+1}), \tag{32}$$
$$\Delta q(k) = q(k) - \tilde{q}_{j-1}(k) = \varepsilon^j q_j(k) + \alpha(\varepsilon^{j+1}),$$

where $\tilde{y}_{j-1}(k)$, $\tilde{z}_{j-1}(k)$, $\tilde{u}_{j-1}(k)$, $\tilde{p}_{j-1}(k)$, $\tilde{q}_{j-1}(k)$ are given by formulae (15) and (27).

Replacing y, z, u, p, q in (2), (3) and (21)-(23) by their representations in (32), and transforming, we obtain

$$\Delta y(k+1) + \tilde{y}_{j-1}(k+1) = \tilde{f}_k + (\tilde{f}_k)_y \Delta y(k) + \varepsilon (\tilde{f}_k)_z \Delta z(k) + + (\tilde{f}_k)_u \Delta u(k) + \varepsilon^{2j} (\frac{1}{2} (\bar{f}_k)_{yy} y_j^2(k) + \frac{1}{2} (\bar{f}_k)_{uu} u_j^2(k) + + (\bar{f}_k)_{uy} u_j(k) y_j(k)) + \alpha(\varepsilon^{2j+1}),$$

$$\Delta z(k+1)+\tilde z_{j-1}(k+1)=\tilde g_k+(\tilde g_k)_y\Delta y(k)+\varepsilon(\tilde g_k)_z\Delta z(k)+$$
$$+(\tilde g_k)_u\Delta u(k)+\varepsilon^{2j}(\frac{1}{2}(\bar g_k)_{yy}y_j^2(k)+\frac{1}{2}(\bar g_k)_{uu}u_j^2(k)+$$
$$+(\bar g_k)_{uy}u_j(k)y_j(k))+\alpha(\varepsilon^{2j+1}),$$
$$\tilde y_{j-1}(0)+\Delta y(0)=\tilde y_{j-1}(N)+\Delta y(N),$$
$$\tilde z_{j-1}(0)+\Delta z(0)=\tilde z_{j-1}(N)+\Delta z(N), \tag{33}$$
$$\varepsilon^j p_j(k)+\tilde p_{j-1}(k)=-(\tilde F_k)'_y+$$
$$+(\tilde f_k)'_y\tilde p_{j-1}(k+1)+(\tilde g_k)'_y\tilde q_{j-1}(k+1)+$$
$$+\varepsilon^j((\bar f_k)'_y p_j(k+1)+(\bar g_k)'_y q_j(k+1)-W_k y_j(k)-$$
$$-S_k u_j(k))+\alpha(\varepsilon^{j+1}), \tag{34}$$
$$\varepsilon^j q_j(k)+\tilde q_{j-1}(k)=\varepsilon(-(\tilde F_k)'_z+$$
$$+(\tilde f_k)'_z\tilde p_{j-1}(k+1)+(\tilde g_k)'_z\tilde q_{j-1}(k+1))+\alpha(\varepsilon^{j+1}),$$
$$\tilde p_{j-1}(0)+\Delta p(0)=\tilde p_{j-1}(N)+\Delta p(N),$$
$$\tilde q_{j-1}(0)+\Delta q(0)=\tilde q_{j-1}(N)+\Delta q(N),$$
$$-(\tilde F_k)_u+\tilde p'_{j-1}(k+1)(\tilde f_k)_u+\tilde q'_{j-1}(k+1)(\tilde g_k)_u+$$
$$+\varepsilon^j(-R_k u_j(k)-S'_k y_j(k)+(\bar f_k)'_u p_j(k+1)+ \tag{35}$$
$$+(\bar g_k)'_u q_j(k+1))+\alpha(\varepsilon^{j+1})=0.$$

Equating the coefficients of ε^j in (33)-(35), we obtain relations (25), (26) and (28) - (30), which follow from the condition for optimality of the control in Problem P_j. This established the statement of the theorem for $m=j$, and thereby proves Theorem 1.

Supposing that Assumption 1^0 and 2^0 are satisfied the following theorem holds.

Theorem 2. The coefficient J_{2m-1} in expansion (11) is known after Problems P_i ($i=\overline{0,m-1}$, $m\ge 1$) have been solved, from which one finds y_i, z_i, u_i. The transformed expression for the coefficient J_{2m}, omitting terms known after Problems P_i ($i=\overline{0,m-1}$, $m\ge 1$) have been solved, is identical with the performance criterion $\tilde J_m(u_m)$ in Problem P_m.

Proof. If $m=1$, this theorem has already been proved. Suppose the statement of the theorem is true for $1\le m<n$.

If the solutions of Problems P_j ($j=\overline{0,n-1}$) have been found, then $\tilde y_{n-1}(k)$, $\tilde z_{n-1}(k)$, $\tilde u_{n-1}(k)$, $\tilde p_{n-1}(k)$, $\tilde q_{n-1}(k)$, defined by formulae (15) and (27) with $j=n$, are known functions.

Let us transform the expression for $J_\varepsilon(u)$ from (1) replacing y, z, u by their representations according to (32) with $j=n$. Then we have

$$J_\varepsilon(u)=\sum_{k=0}^{N-1}(\tilde F_k+(\tilde F_k)_y\Delta y(k)+\varepsilon(\tilde F_k)_z\Delta z(k)+$$
$$+(\tilde F_k)_u\Delta u(k)+\varepsilon^{2n}(\frac{1}{2}(\bar F_k)_{yy}y_n^2(k)+\frac{1}{2}(\bar F_k)_{uu}u_n^2(k)+$$
$$+(\bar F_k)_{uy}u_n(k)y_n(k)))+\alpha(\varepsilon^{2n+1}), \tag{36}$$

where the tilde over the symbols for the functions and their derivatives means that they are evaluated at $y(k)=\tilde y_{n-1}(k)$, $z(k)=\tilde z_{n-1}(k)$, $u(k)=\tilde u_{n-1}(k)$.

Using the introduced notations, we deduce from (36) and (34), (35) with $j=n$ that

$$J_\varepsilon(u)=\sum_{k=0}^{N-1}(\{\tilde F_k\}_{2n}+\{(\tilde F_k)_y\}_{n-1}\Delta y(k)+\{\varepsilon(\tilde F_k)_z\}_{n-1}\Delta z(k)+$$
$$+\{(\tilde F_k)_u\}_{n-1}\Delta u(k)+\varepsilon^{2n}([(\tilde F_k)_y]_n y_n(k)+[\varepsilon(\tilde F_k)_z]_n z_n(k)+$$
$$+[(\tilde F_k)_u]_n u_n(k)+\frac{1}{2}(\bar F_k)_{yy}y_n^2(k)+\frac{1}{2}(\bar F_k)_{uu}u_n^2(k)+$$
$$+(\bar F_k)_{uy}u_n(k)y_n(k)))+\alpha(\varepsilon^{2n+1}), \tag{37}$$
$$\{(\tilde F_k)_y\}_{n-1}=-\tilde p'_{n-1}(k)+\{\tilde p'_{n-1}(k+1)(\tilde f_k)_y+$$
$$+\tilde q'_{n-1}(k+1)(\tilde g_k)_y\}_{n-1},$$
$$\{\varepsilon(\tilde F_k)_z\}_{n-1}=-\tilde q'_{n-1}(k)+\{\varepsilon(\tilde p'_{n-1}(k+1)(\tilde f_k)_z+$$
$$+\tilde q'_{n-1}(k+1)(\tilde g_k)_z)\}_{n-1},$$
$$\tilde p_{n-1}(0)=\tilde p_{n-1}(N),\ \tilde q_{n-1}(0)=\tilde q_{n-1}(N),$$
$$\{(\tilde F_k)_u\}_{n-1}=\{\tilde p'_{n-1}(k+1)(\tilde f_k)_u+\tilde q'_{n-1}(k+1)(\tilde g_k)_u\}_{n-1}.$$

Taking the last four equalities and the relations obtained from (33) with $j=n$ into consideration we deduce from (37) that

$$J_\varepsilon(u)=\sum_{k=0}^{N-1}(\{\tilde F_k+\tilde p'_{n-1}(k+1)(\tilde y_{n-1}(k+1)-\tilde f_k)+$$
$$+\tilde q'_{n-1}(k+1)(\tilde z_{n-1}(k+1)-\tilde g_k)\}_{2n}+\varepsilon^{2n}([(\tilde F_k)_y-$$
$$-\tilde p'_{n-1}(k+1)(\tilde f_k)_y-\tilde q'_{n-1}(k+1)(\tilde g_k)_y]_n y_n(k)+$$
$$+[(\tilde F_k)_z-\tilde p'_{n-1}(k+1)(\tilde f_k)_z-\tilde q'_{n-1}(k+1)(\tilde g_k)_z]_{n-1}z_n(k)+$$
$$+[(\tilde F_k)_u-\tilde p'_{n-1}(k+1)(\tilde f_k)_u-\tilde q'_{n-1}(k+1)(\tilde g_k)_u]_n u_n(k)+$$
$$+\frac{1}{2}y'_n(k)W_k y_n(k)+\frac{1}{2}u'_n(k)R_k u_n(k)+y'_n(k)S_k u_n(k)))+$$
$$+\alpha(\varepsilon^{2n+1}).$$

It is obvious from this expression that J_{2n-1} is known after Problems P_i ($i=\overline{0,n-1}$) have been solved. If we take the sum of the terms in J_{2n} (the coefficient of ε^{2n}), which depend on the unknowns $y_n(k)$, $z_n(k)$, $u_n(k)$, it is identical with the performance criterion $\tilde J_n(u_n)$ in Problem P_n (see (24) with $j=n$).

This completes the proof of Theorem 2.

V. Estimates of Approximate Solution

Let us assume that solutions have been found for problems P_j, $j=\overline{0,n}$: the functions $y_j(k)$, $z_j(k)$, $u_j(k)$. We shall estimate the approximate solution of the problem P_ε: $\tilde{y}_n(k)$, $\tilde{z}_n(k)$, $\tilde{u}_n(k)$.

Theorem 3. Under Assumptions 1^0 - 3^0 and sufficiently small $\varepsilon>0$ Problem P_ε is uniquely solvable in the neighbourhood of the control u_0 and its solution u^*, y^*, z^* satisfies the estimates

$$u^*(k)-\tilde{u}_n(k)=O(\varepsilon^{n+1}),\ y^*(k)-\tilde{y}_n(k)=O(\varepsilon^{n+1}), \quad (38)$$
$$z^*(k)-\tilde{z}_n(k)=O(\varepsilon^{n+1}),\ J_\varepsilon(u^*)-J_\varepsilon(\tilde{u}_n)=O(\varepsilon^{2n+2}).$$

Proof. The proof of this theorem is similar to the proof of Theorem 3 in [9].

First of all, it is proved that under assumptions 1^0 - 3^0 and sufficiently small $\varepsilon>0$ problem (2), (3), (21)-(23) has an unique solution (y^*,z^*,u^*,p^*,q^*) in the neighbourhood ($\tilde{y}_n(k)$, $\tilde{z}_n(k)$, $\tilde{u}_n(k)$, $\tilde{p}_n(k)$, $\tilde{q}_n(k)$) and the following estimates hold $\| y^*(k)-\tilde{y}_n(k)\|\le c\varepsilon^{n+1}$, $\| z^*(k)-\tilde{z}_n(k)\|\le c\varepsilon^{n+1}$, $\| u^*(k)-\tilde{u}_n(k)\|\le c\varepsilon^{n+1}$, $\| p^*(k)-\tilde{p}_n(k)\|\le c\varepsilon^{n+1}$, $\| q^*(k)-\tilde{q}_n(k)\|\le c\varepsilon^{n+1}$, where the constant c is independent of k and ε.

Secondly, it is proved that for any $\gamma>0$ there are constants $\varepsilon_0>0$ and $c>0$ such that for $k=\overline{0,N-1}$, $0<\varepsilon\le\varepsilon_0$, $\| u^*(k)-u(k)\|\le\gamma$ following inequalities hold $\| y^*(k)-y(k)\|\le c\| u^*(k)-u(k)\|$, $\| z^*(k)-z(k)\|\le c\| u^*(k)-u(k)\|$, where ($y,z$) is the trajectory corresponding to the control u.

Thirdly, it is proved that under assumptions 1^0 - 3^0 and sufficiently small $\varepsilon>0$ the function u^* is a local optimal control for Problem P_ε.

Finally, using the results of the previous steps we obtain the statement of this theorem.

VI. Lack of Increase for Functional

Theorem 4. Under Assumptions 1^0 - 3^0 and sufficiently small $\varepsilon>0$, we have

$$J_\varepsilon(\tilde{u}_i)\le J_\varepsilon(\tilde{u}_{i-1}),\ i=\overline{1,n}, \quad (39)$$

where $\tilde{u}_i(k)=\sum_{j=0}^{i}\varepsilon^j u_j(k)$.

Proof. If $u_i(k)\equiv 0$, inequality (39) is obvious.

Let us consider the case when $u_i\ne 0$. Expand the solution of the problem (2), (3) for $u(k)=\tilde{u}_s(k)$ $(s=i-1,i)$ in a series of non-negative integer powers of ε. Then, by the algorithm for determing the terms of expansion (10), the solution will have the form

$$\sum_{j=0}^{s}\varepsilon^j y_j(k)+O(\varepsilon^{s+1}),\ \sum_{j=0}^{s}\varepsilon^j z_j(k)+O(\varepsilon^{s+1}).$$

Expanding $J_\varepsilon(\tilde{u}_s)$ $(s=i-1,i)$ in series (11) and using Theorem 2, we obtain

$$J_\varepsilon(\tilde{u}_i)=\sum_{j=0}^{2i-1}\varepsilon^j J_j+\varepsilon^{2i}(\hat{J}_{2i}+\tilde{J}_i(u_i))+O(\varepsilon^{2i+1}),$$

$$J_\varepsilon(\tilde{u}_{i-1})=\sum_{j=0}^{2i-1}\varepsilon^j J_j+\varepsilon^{2i}(\hat{J}_{2i}+\tilde{J}_i(0))+O(\varepsilon^{2i+1}), \quad (40)$$

where $\hat{J}_{2i}$ depends on y_j, z_j, u_j $(j=\overline{0,i-1})$.

Since u_i is a solution of the linear-quadratic Problem P_i, which is to minimize the functional $\tilde{J}_i(u_i)$, it follows, by the uniqueness of the optimal control when $u_i\ne 0$, that $J_i(\tilde{u}_i)<J_i(0)$. Hence, using also (40), it follows that inequality (39) is true for sufficiently small $\varepsilon>0$.

We have thus established that the values of the minimized functional do not increase with each new approximation of the optimal control.

References

[1] A.B. Vasil'eva and M.G. Dmitriev, *Singular perturbations in optimal control problems*, Advances in Science and Technology. Mathematical Analysis, VINITI, Moscow, 20, 1982, pp. 3-77.

[2] P.V. Kokotovic, H.K. Khalil and J. O'Reilly, *Singular perturbation methods in control: Analysis and Design*, Academic Press, 1986.

[3] Z. Gajic and M. Lim, *Optimal control of singularly perturbed linear systems and applications. High-accuracy techniques*, Control Engineering series, Marcel Dekker 2000.

[4] D.S. Naidu and A.K. Rao, *Singular perturbation analysis of discrete control systems*, Lect. Notes Math., 1154, 1985.

[5] D.S. Naidu, *Singular perturbation methodology in control systems*, IEE Control Engineering Series, Peter Peregrinus Ltd., London, United Kingdom, 34, 1988.

[6] S.V. Belokopytov and M.G. Dmitriev, "Direct scheme in optimal control problems with fast and slow motions", *Systems and Control Letters*, vol. 8, no. 2, pp. 129-135, 1986.

[7] M.A. Gaipov, "Asymptotic expansion of the solution of a non-linear discrete optimal control problem with small step-size without restrictions on the control (formalism) I", *Izv. Akad. Nauk TSSR, Ser. Fiz.-Tekhn., Khim. i Geol. Nauk*, no. 1, pp. 9-16, 1990 (in Russian).

[8] M.G. Dmitriev, S.V. Belokopytov and M.A. Gaipov, "Asymptotic expansion of the solution of a non-linear discrete optimal control problem with small step-size without restrictions on the control (proofs) II", *Izv. Akad. Nauk TSSR, Ser. Fiz.-Tekhn., Khim. i Geol. Nauk*, no. 2, pp. 10-18, 1990 (in Russian).

[9] G.A. Kurina, "Asymptotic expansion of solutions of optimal control problems for discrete weakly controllable systems", *J. Appl. Maths Mechs*, vol. 66, no. 2, pp. 201-213, 2002.

[10] G.A. Kurina, N.V. Nekrasova, "Asymptotic solution of discrete periodic singularly perturbed linear-quadratic problem", *Generalized Solutions in Control Problems*, in *Proc. IFAC Workshop GSCP-2004 and satellite events*, Pereslavl-Zalessky, Russia, 2004, Moscow: Fizmatlit, 2004, pp. 169-175.