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A New Local Control Strategy for Control of Discrete-Time Piecewise Affine Systems

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Abstract— **In this paper we propose an enhanced local control strategy for the control of discrete-time piecewise affine systems on full-dimensional polytopes.**

The control strategy is divided into a local control and a supervisory control problem. The local control problem is to reach and cross one selected facet of a polytope ensuring that the next sample of the time-discrete trajectory is picked up in the adjacent polytope.

The procedure is based on conditions, given as inequalities, for the discrete-time gradient of the system, evaluated in the vertices of the polytopes. Solving an optimization problem with respect to the inequality conditions, a performance index is minimized. The performance index represents a minimal retention period of the trajectory in the polytope or the minimal quadratic sum of the input signal. The control law is then obtained by a simple matrix inversion.

The supervisory control problem is to find a suitable combination of polytopes and local control strategies that transfers the trajectory to the polytope that contains the operating point.

I. Introduction

The piecewise linear approach for nonlinear systems was proposed in [1]. This approach has been generalized and finally led to hybrid systems theory. In [2] the hybrid phenomena were formulated. After this milestone a lot of research has been done, especially in modeling of hybrid systems. The analysis of this class of systems and the synthesis of controllers are still sophisticated, even nowadays.

There are two different trends in literature for the design of controllers for piecewise affine systems that represent a sub class of hybrid systems. In [3] and [4] continuous-time controllers and in [5], [6] and [7] discrete-time controllers are designed.

Discrete-time affine systems can be obtained from measurements of technical processes by identification of the hybrid system [8], [9] or by theoretical modeling [10], [11]. In the theoretical modeling approach, given continuous piecewise affine models are discretized, calculating the discrete-time affine model in each domain from the respective continuous model.

As already presented in [12], the control strategy consists of a local part and a supervisory part. In this paper, a new approach for the local control strategy for the control of discrete-time piecewise affine systems will be presented.

In section II some basic notions as polytope, facet, affine system, and affine control law are given. Using these terms, the problem of dynamical systems restricted to a polytope is presented. The simplex as a special polytope is introduced. The aim of the local controller design is formulated.

Sufficient conditions for the discrete-time gradient in the vertices of the polytope are given in section III.

In section IV, both, the local control part and the supervisory control part of the strategy, are presented.

At the beginning, the considered state space is partitioned into a simplex structure. Each simplex has a corresponding vertex in a graph.

The local control strategy completes the graph by adding selected edges. After the local control is explored, the completed graph structure is used for supervisory control. The task of the supervisory control is to find an optimal path through this graph structure. The local and the supervisory control strategy are calculated off-line.

The online effort is to determine the active simplex and therefore, the active control law, solving a linear search problem. This approach is able to track very fast dynamics because of the reduced on-line effort.

Finally, the strategy is applied to a two-tank system in section V. The two-tank system can be modeled as a piecewise affine system with discrete and continuous input signals. A performance index will determine which piecewise affine dynamic assigned to a combination of discrete inputs is the best. The plant in the example is taken from [12]. The start-state vector and the end-state vector are equal, too. A comparison with the results in [12] clarifies the advantages of the new method in a short discussion.

The contribution is finished with a conclusion in section VI.

II. Problem Statement

The considered state space $\underline{x} \in \mathbb{R}^N$, $N \in \mathbb{N}$ is bounded on the set Θ. Assuming there are M points $\underline{v}_1, ..., \underline{v}_M$, with $M \geq N + 1$, in state space $\mathbb{R}^N$, such that there exists no hyperplane of $\mathbb{R}^N$, containing all these M points.

The full-dimensional polytope P is defined as the convex hull of $\underline{v}_1, ..., \underline{v}_M$. If a point $\underline{v}_i$, $i = 1, ..., M$ cannot be written as convex combination of the points

$\underline{v}_1, ..., \underline{v}_{i-1}, \underline{v}_{i+1}, ..., \underline{v}_M$, it is called a vertex of the polytope P.

The polytope is completely characterized by its set of vertices. A full-dimensional polytope with $M = N + 1$ is called a full-dimensional *simplex*.

The intersection of a finite number of half spaces can also describe a polytope. If an integer $K \geq N + 1$, non-zero vectors $\underline{n}_1, ..., \underline{n}_K \in \mathbb{R}^N$, and scalars $\alpha_1, ..., \alpha_K \in \mathbb{R}$ exists, such that

$$P = \{\underline{x} \in \mathbb{R}^N | \forall i = 1, ..., K : \underline{n}_i^T \underline{x} \leq \alpha_i\} \tag{1}$$

is valid, then (1) is the implicit description of a polytope.

The intersection of a full-dimensional polytope P with one of its supporting hyperplanes

$$F_i = \{\underline{x} \in \mathbb{R}^N | \underline{n}_i^T \underline{x} = \alpha_i\} \cap P \tag{2}$$

is called a facet F_i of P, if the dimension of the intersection is equal to $N - 1$. By convention the vector $\underline{n}_i$, that is the normal vector of the facet F_i, is of unit length and points out of the polytope.

In [4] a method is presented to partition full-dimensional polytopes into full-dimensional simplices by using e.g. the Delaunay-triangulation. Therefore, in the sequel, the full-dimensional simplex P in $\mathbb{R}^N$ with $N + 1$ facets is used. Using numbered vertices of a simplex, the facets can be named with the number of that vertex which is not a vertex of the facet.

On each full-dimensional simplex P_i a discrete-time affine system

$$\underline{x}_{k+1} = \underline{\Phi}_i \underline{x}_k + \underline{H}_i \underline{u}_k + \underline{\phi}_i \tag{3}$$

is considered, with $\underline{\Phi}_i \in \mathbb{R}^{N \times N}$, $\underline{H}_i \in \mathbb{R}^{N \times m}$ and $\underline{\phi}_i \in \mathbb{R}^N$. The state $\underline{x} \in \mathbb{R}^N$ is assumed to be contained in the simplex P_i. The input signal $\underline{u}$ takes values from the bounded set $U \subset \mathbb{R}^m$ of continuous inputs.

The inside P^i of the polytope P is $P^i = P \setminus \partial P$ where ∂P denotes the hull of the polytope P. The union of the simplices P_i^i with $P_i^i \cap P_j^i = \emptyset \;\; \forall i \neq j$ forms the subset $\Theta = \bigcup P_i = \bigcup (P_i^i \cup \partial P_i)$ of the state space within which the piecewise affine system is definded.

The aim of the local controller design is to find discrete-time affine control laws

$$\underline{u}_k = -\underline{R}_i \underline{x}_k + \underline{u}_{0,i} \tag{4}$$

defined on the simplex P_i, such that the trajectories of the controlled system

$$\underline{x}_{k+1} = (\underline{\Phi}_i - \underline{H}_i \underline{R}_i)\underline{x}_k + (\underline{\phi}_i + \underline{H}_i \underline{u}_{0,i}), \tag{5}$$

starting at any possible point in P_i

- will stay in the simplex P_i (details in [12]), or
- enter one of the $N + 1$ adjacent simplices through one specific facet of the simplex (focused in this paper).

To obtain an unique solution, if the trajectory hits the common facet F_i, while leaving simplex P_i and entering simplex P_j, the dynamic of P_j is assumed to be dominant. The controlled system in equation (5) is with $(\underline{\Phi}_i + \underline{H}_i \underline{R}_i) = \underline{\Phi}_{R,i}$ and $(\underline{\phi}_i + \underline{H}_i \underline{u}_{0,i}) = \underline{\phi}_{R,i}$ equal to an autonomous discrete-time affine system

$$\underline{x}_{k+1} = \underline{\Phi}_{R,i} \underline{x}_k + \underline{\phi}_{R,i}. \tag{6}$$

If there are discrete inputs $\underline{u}_D$ to the system, $\underline{u}_D$ is assumed to be an element of a bounded (integer) set, see details in [3] or section V, a finite number of discrete-time piecewise affine systems can be determined for each possible combination of the elements of the finite discrete input vector $\underline{u}_D$. Let each combination be numbered serially in the variable x_D.

For each combination x_D the same continuous state space Θ with the same partitioning into simplices is considered. There are $\max(x_D)$ dynamics defined for one simplex.

If the trajectory enters a simplex or is within the start simplex, this is a necessary condition that the discrete input $\underline{u}_D$ is changed by the supervisory control strategy.

In the next section, sufficient conditions for feedback control are formulated.

III. Sufficient conditions for feedback control to a facet

In [4] a method was introduced for continuous-time systems that uses conditions of the gradient $\dot{\underline{x}}$ of the system.

The gradient is evaluated at the vertices of a simplex by projecting the gradient on the normal vectors of the bounding hyperplanes. This results into a linear inequality system for the input $\underline{u}(v_i)$ in these vertices $\underline{v}_i$.

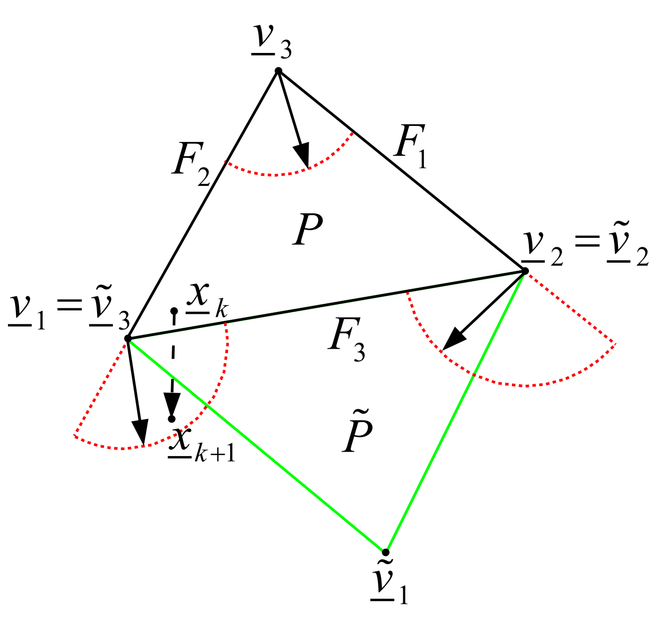

Fig. 1. Admissible range for the gradient vector

First, the continuous inequalities are presented. The index e represents the number of the facet F_e where the trajectories should leave the simplex. The vertex $\underline{v}_e$ is the only vertex that is not part of the facet F_e.

For illustrating the following equations, a two dimensional problem is given in figure 1, where $e = 3$.

$$\forall\, \underline{v}_i \in \mathcal{V} \setminus \{\underline{v}_e\}: \qquad \underline{n}_{F_e}^T \underline{\dot{x}}(\underline{v}_i, \underline{u}(\underline{v}_i)) > 0 \quad \text{(7a)}$$

$$\forall\, \underline{v}_i \in \mathcal{V} \setminus \{\underline{v}_e\}: \forall\, F_j \in \mathcal{F} \setminus \{F_e, F_i\}:$$
$$\underline{n}_{F_j}^T \underline{\dot{x}}(\underline{v}_i, \underline{u}(\underline{v}_i)) \leq 0 \quad \text{(7b)}$$

$$\forall\, F_j \in \mathcal{F} \setminus \{F_e\}: \qquad \underline{n}_{F_j}^T \underline{\dot{x}}(\underline{v}_e, \underline{u}(\underline{v}_e)) \leq 0 \quad \text{(7c)}$$

$$\sum_{\substack{j=1\\ j\neq e}}^{n+1} \underline{n}_{F_j}^T \underline{\dot{x}}(\underline{v}_e, \underline{u}(\underline{v}_e)) < 0 \quad \text{(7d)}$$

The first system of inequalities (7a) requires that the projection of the gradient on the normal vector of the facet F_e in the vertices point out of the simplex P.

The second system of inequalities (7b) requires that the projection of the gradient on the normal vector of all other facets point into simplex P. E.g. in vertex $\underline{v}_1$ in figure 1, the projection of the gradient on the normal vector of F_2 has to point into the simplex P. The admissible range of the directions of the gradients in the vertices are marked as dotted lines.

The third condition (7c) forces the gradient in vertex $\underline{v}_e$ to point into the simplex P. The last condition (7d) forces the gradient not to vanish in the vertex $\underline{v}_e$.

In the next step these four conditions (7a-7d) from [4] are modified for discrete-time systems.

The gradient $\underline{\dot{x}}$ has to be expressed as a difference $\underline{x}_{k+1} - \underline{x}_k$. With $\underline{x}_{k+1} = \underline{\Phi}\, \underline{x}_k + \underline{H}\, \underline{u}_k + \underline{\phi}$ the expressions above modify to the following equations, $\underline{I}_n$ is the unity matrix of the dimension n:

$$\forall \underline{v}_i \in \mathcal{V} \setminus \{\underline{v}_e\}: \underline{n}_{F_e}^T\big((\underline{\Phi} - \underline{I}_n)\, \underline{v}_i + \underline{H}\, \underline{u}(\underline{v}_i) + \underline{\phi}\big) > 0 \quad \text{(8a)}$$

$$\forall \underline{v}_i \in \mathcal{V} \setminus \{\underline{v}_e\}: \forall F_j \in \mathcal{F} \setminus \{F_e, F_i\}:$$
$$\underline{n}_{F_j}^T\, \big((\underline{\Phi} - \underline{I}_n)\underline{v}_i + \underline{H}\underline{u}(\underline{v}_i) + \underline{\phi}\big) \leq 0 \quad \text{(8b)}$$

$$\forall\, F_j \in \mathcal{F} \setminus \{F_e\}:$$
$$\underline{n}_{F_j}^T\, \big((\underline{\Phi} - \underline{I}_n)\underline{v}_e + \underline{H}\underline{u}(\underline{v}_e) + \underline{\phi}\big) \leq 0 \quad \text{(8c)}$$

$$\sum_{\substack{j=1\\ j\neq e}}^{n+1} \underline{n}_{F_j}^T\, \big((\underline{\Phi} - \underline{I}_n)\underline{v}_e + \underline{H}\underline{u}(\underline{v}_e) + \underline{\phi}\big) < 0 \quad \text{(8d)}$$

As illustrated in figure 1 the next sample is not necessarily picked up in the neighbor simplex $\tilde{P}$, if the conditions (8a-8d) are met. I.e. a trajectory starts in $\underline{x}_k$ in simplex P and in the next sample $\underline{x}_{k+1}$ the neighbor simplex $\tilde{P}$ is overleaped, although the conditions from [4] are met. Therefore, the discrete-time versions (8a-8d) of the continuous conditions (7a-7d) are only necessary to solve the problem. By adding the conditions (9- 11) the conditions turn out to be sufficient.

The additional conditions

$$\forall \underline{v}_i \in \mathcal{V} \setminus \{\underline{v}_e\}: \forall \tilde{F}_j \in \tilde{\mathcal{F}} \setminus \{\tilde{F}_e\}:$$
$$\underline{\tilde{n}}_{\tilde{F}_j}^T\, \big((\underline{\Phi} - \underline{I}_n)\, \underline{v}_i + \underline{H}\, \underline{u}(\underline{v}_i) + \underline{\varphi}\big) \leq 0 \quad \text{(9)}$$

keep the direction of the difference with respect to the dynamic in simplex P in the neighbor simplex $\tilde{P}$. The vertices of the neighbor simplex are $\underline{\tilde{v}}_i \in \tilde{\mathcal{V}}$.

The ranges of the direction of the differences in the vertices of the facet F_e are narrowed to the neighbor simplex.

In figure 2 the narrowed ranges are sketched.

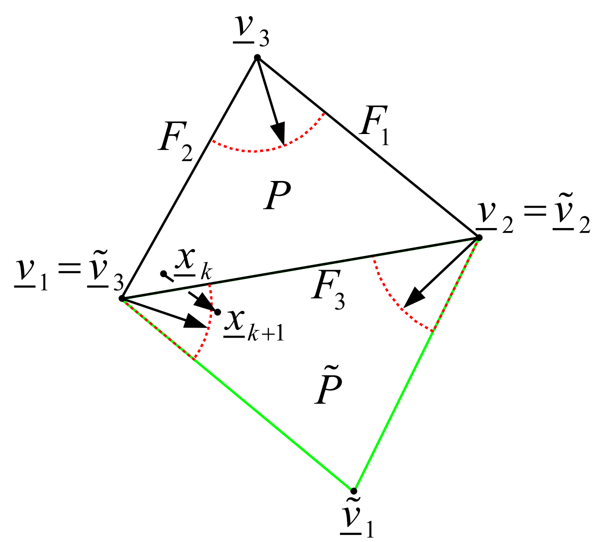

Fig. 2. Range narrowed to the neighbor simplex by additional conditions

If the trajectory crosses the facet F_e, the dynamic of the neighbor simplex $\tilde{P}$ is valid for the rest of the sampling time. If the dynamic in simplex P and the dynamic in simplex $\tilde{P}$ are equal, the next sample is guaranteed to be picked up in the neighbor simplex $\tilde{P}$ by satisfying the conditions (8a-8d,9). In general equal dynamics will not be the case. This means that the trajectory could evolve that way, without the further additional conditions (10,11), that the next sample is picked up outside the neighbor simplex $\tilde{P}$. If these two conditions are added, the next sample is picked up in the neighbor simplex, even if the dynamics in simplex P and simplex $\tilde{P}$ are different:

$$\forall \underline{v}_i \in \mathcal{V} \setminus \{\underline{v}_e\}: \forall \tilde{F}_j \in \tilde{\mathcal{F}} \setminus \{\tilde{F}_e\}:$$
$$\underline{\tilde{n}}_{\tilde{F}_j}^T\, \Big(\Big(\underline{\tilde{\Phi}} - \underline{I}_n\Big)\, \underline{v}_i + \underline{\tilde{H}}\, \underline{u}(\underline{v}_i) + \underline{\tilde{\varphi}}\Big) \leq 0 \quad \text{(10)}$$

$$\forall \underline{v}_i \in \mathcal{V} \setminus \{\underline{v}_e\}: \tilde{F}_e:$$
$$\underline{\tilde{n}}_{\tilde{F}_e}^T\, \Big(\Big(\underline{\tilde{\Phi}} - \underline{I}_n\Big)\, \underline{v}_i + \underline{\tilde{H}}\, \underline{u}(\underline{v}_i) + \underline{\tilde{\varphi}}\Big) \leq 0 \quad \text{(11)}$$

The condition (10) is the correspondency to condition (9) with the dynamic $\underline{\tilde{\Phi}}$, $\underline{\tilde{H}}$ and $\underline{\tilde{\phi}}$, valid in the neighbor simplex $\tilde{P}$. Condition (11) forces the trajectory to enter the neighbor simplex $\tilde{P}$.

There will be no, one or an infinite number of solutions that satisfy these conditions. In the last case the degree of freedom can be used to perform an optimization. This will be discussed in the next section in detail.

IV. Control Strategy

Given a discrete-time piecewise affine system in state space Θ, partitioned into simplices, the relation between the numbered simplices can be stored in a graph structure. Each simplex P_i is represented by a vertex in this graph. The control strategy consists of two parts.

First, the local control strategy will be investigated. Using the conditions presented in section III, the local strategy tries to find a discrete-time piecewise affine control law, ensuring that every trajectory, starting in simplex P_i will leave the simplex through a specific facet $F_j, j \in 1, ..., N+1$ of the simplex P_i and the sample picked up next is within the adjacent simplex $\tilde{P}$. The simplex P_i and $\tilde{P}$ share the same facet F_j.

If a valid control law is found, an oriented edge is added to the graph, connecting simplex P_i with $\tilde{P}$. This edge is used to store the two determined parameters $\underline{R}$ and $\underline{u}_0$ of the control law (4).

If discrete inputs $\underline{u}_D$ exist, there is a third parameter x_D. To determine these parameters the best possibility will be chosen minimizing a performance index in such a way that

- the retention time in the simplex P is minimized or
- the control energy is minimized.

It is assumed that a change in x_D is only allowed, if the first sample is picked up after the trajectory crossed a facet.

A. Local control

The local control problem, as mentioned in section II, is to enter the adjacent simplex $\tilde{P}$ through a defined facet F_e if the state $\underline{x} \in \Theta \subset \mathbb{R}^N$ is not contained in the final simplex P_F. The sequence of simplices is determined by the supervisory control.

The control law that solves the problem, consists of two parameters $\underline{R}$ and $\underline{u}_0$. The aim of this procedure is to find a matrix $\underline{R}$ and the vector $\underline{u}_0$, such that constraints in the input signal $\underline{u}$ are satisfied.

For each facet of each simplex, the local control algorithm tries to find a control law (4). If there are discrete inputs $\underline{u}_D$, the calculations are done for each x_D. The effort for exploring all simplices is then multiplied with $\max(x_D)$.

A point $\underline{x}$ in the simplex can be written as a convex combination of the vertices

$$\underline{x} = \ell_1\, \underline{v}_1 + \ldots + \ell_{n+1}\, \underline{v}_{n+1}\,, \ell_i \in [0,1]\,,$$

i.e. for the inputs

$$\underline{u}(\underline{x}) = \ell_1\, \underline{u}(\underline{v}_1) + \ldots + \ell_{n+1}\, \underline{u}(\underline{v}_{n+1})$$

is valid [4].

If inputs $u(\underline{v}_i)(i = 1, \ldots, n+1)$ are found that satisfy the inequalities and the input limitations an affine feedback law can be determined.

Choosing the inputs in the vertices that way that they fulfill the inequality conditions (8a-8d,9-11), then the feedback law is

$$\underline{u}(\underline{v}_i) = -\underline{R}\, \underline{v}_i + \underline{u}_0\,, \quad i = 1, \ldots, n+1.$$

Rewriting this law, by transposing the equations, results in

$$\underline{u}(\underline{v}_i)^T = \begin{bmatrix} \underline{v}_i^T & 1 \end{bmatrix} \begin{bmatrix} -\underline{R}^T \\ \underline{u}_0^T \end{bmatrix}, \quad i = 1, \ldots, n+1.$$

After merging these $n+1$ equations, the matrix equation

$$\begin{bmatrix} \underline{u}(\underline{v}_1)^T \\ \vdots \\ \underline{u}(\underline{v}_{n+1}^T) \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{v}_1^T & 1 \\ \vdots & \vdots \\ \underline{v}_{n+1}^T & 1 \end{bmatrix}}_{=(\underline{T}_{\mathrm{B}}^{-1})^T} \begin{bmatrix} -\underline{R}^T \\ \underline{u}_0^T \end{bmatrix}$$

is obtained.

In [4] it is shown, that the matrix $\underline{T}_{\mathrm{B}}$ is always invertible for a simplex. Therefore the unique solution

$$\begin{bmatrix} -\underline{R}^T \\ \underline{u}_0^T \end{bmatrix} = \underline{T}_{\mathrm{B}}^T \begin{bmatrix} \underline{u}(\underline{v}_1)^T \\ \vdots \\ \underline{u}(\underline{v}_{n+1})^T \end{bmatrix}$$

is found.

As already mentioned in section III, the sufficient conditions (8a-8d,9-11) can result in an infinite number of solutions. To obtain an appropriate control law, the inputs $\underline{u}_i(\underline{v}_i)$ have to be fixed to a defined value. This degree of freedom can be used to find the best solution $\underline{u}^*(\underline{v}_i)^T$ with respect to the chosen performance index

$$J(\underline{v}_i) := J(\underline{v}_i, \underline{u}(\underline{v}_i)) = \underline{n}_e^T((\underline{\Phi} - \underline{I}_n)\underline{v}_i + \underline{H}\underline{u}(\underline{v}_i) + \underline{\phi})$$

in each vertex $\underline{v}_i$. The optimization problem results in the max-min-Problem

$$\max\left(\min_{\underline{v}_i \in \mathcal{V}} J(\underline{v}_i)\right)$$

that can easily transformed into a min-max-Problem

$$-\min\left(\max_{\underline{v}_i \in \mathcal{V}}\big(-J(\underline{v}_i)\big)\right).$$

Min-max-problems are standard optimization problems, here a so called constrained min-max-problem has to be solved due to the inequality constraints. A solver for this problem is e.g. *fminimax*, which is a part of the MATLAB OPTIMIZATION TOOLBOX.

The stabilizing control in the final simplex P_F can be determined with the methods presented in [12].

B. Supervisory control

The local control provides a graph with several weighted edges. The weight represent the performance index $J(\underline{u}^*)$. The supervisory control reduces to a problem to find an allowed path in this graph. From a start vertex to the end vertex the shortest path with the smallest sum of the weights of the used edges has to be found by the supervisory control strategy. This is also a standard problem and can be solved with Dijkstra's algorithm [13].

V. EXAMPLE

The system under investigation is the two-tank system shown in figure 3. For each tank there is an influx.

The left tank is labeled with 1 and the right tank with 3, and the influxes are labeled with q_1 and q_3, respectively.

In table I, the discrete states x_D as combinations of the discrete input vector $\underline{u}_D = (V1, V3, V13u)$ are given. A

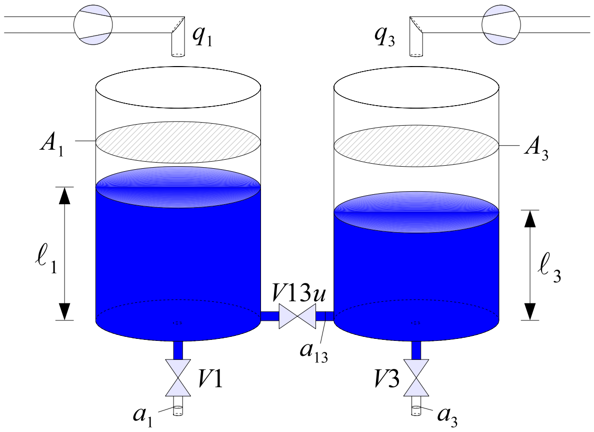

Fig. 3. Two-tank system

TABLE I
COMBINATIONS OF THE VALVES AND DISCRETE STATE x_D

x_D	1	2	3	4	5	6	7	8
V1	0	0	0	0	1	1	1	1
V3	0	0	1	1	0	0	1	1
V13u	0	1	0	1	0	1	0	1

valve is open if the value of the binary variable is one, e.g. $V1 = 1$.

The dynamic for the two-tank system for $x_D = 8$ is

$$\begin{aligned}
&\dot{\ell}_1(t) = \\
&-\operatorname{sign}(\ell_1(t) - \ell_3(t))\frac{a_{13}}{A_1}\sqrt{2g}\sqrt{|\ell_1(t) - \ell_3(t)|} \\
&-\frac{a_1}{A_1}\sqrt{2g}\sqrt{\ell_1(t)} + \frac{q_{1,max}}{A_1}u_{q,1}(t) \\
&\dot{\ell}_3(t) = \\
&\operatorname{sign}(\ell_1(t) - \ell_3(t))\frac{a_{13}}{A_3}\sqrt{2g}\sqrt{|\ell_1(t) - \ell_3(t)|} \\
&-\frac{a_3}{A_3}\sqrt{2g}\sqrt{\ell_3(t)} + \frac{q_{3,max}}{A_3}u_{q,3}(t).
\end{aligned}$$

The input $\underline{u}(t) = (u_{q,1}(t), u_{q,3}(t))^T$ is added linearly to the nonlinear state equation. The state vector is $\underline{x}(t) = (\ell_1(t), \ell_3(t))^T$.

The root function in the state equation is approximated with discrete-time affine systems valid on corresponding simplices for the controller design, simulation later on is done on the nonlinear system.

In figure 4, the partitioned state space $\Theta, \underline{x} \in \Theta = \{\mathbb{R}^2 | 0 \leq x_1 = \ell_1 \leq 60[cm], 0 \leq x_2 = \ell_3 \leq 60[cm]\}$, into simplices is given.

The simplices are numbered. The trajectory moves from the initial state $\underline{x}_0 = (3, 20)[cm]$ to final state $\underline{x}_F = (53, 40)[cm]$ crossing the simplices composed in table II.

The information which valves are closed or opened, is given by x_D marked with big digits in these twelve used simplices in figure 4.

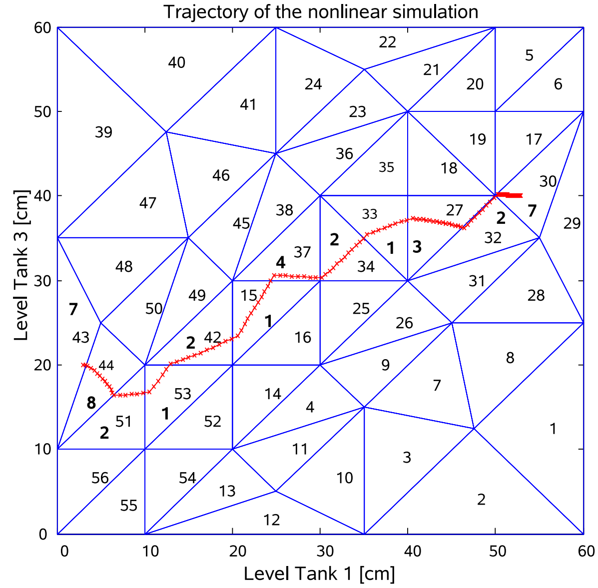

Fig. 4. The trajectory of the two-tank system, starting in $\underline{x}_0 = (3, 20)[cm]$ moving to $\underline{x}_F = (53, 40)[cm]$.

TABLE II
TRAJECTORY FROM $\underline{x}_0$ TO $\underline{x}_F$ SEQUENTIALLY MOVES OVER THE SIMPLICES P_i

Step	$\underline{x}_0$	2	3	4	5	6
i	43	44	51	52	42	15
x_D	7	8	2	1	2	1
Step	7	8	9	10	11	$\underline{x}_F$
i	37	34	33	27	32	30
x_D	4	2	1	3	2	7

For simplex '44', details are given:

$$\underline{\Phi}_{44}^{8} = \begin{bmatrix} 0.9743 & 0.0044 \\ 0.0079 & 0.9831 \end{bmatrix}, \underline{\phi}_{44}^{8} = \begin{bmatrix} 0.1686 \\ -0.2591 \end{bmatrix}$$
$$\text{and } \underline{H}_{44}^{8} = \begin{bmatrix} 0.6412 & 0.0014 \\ 0.0026 & 0.6441 \end{bmatrix}$$

The upper index denotes the x_D to indicate the necessary combination of the valves. The two parameters of a control law are given:

$$\underline{R}_{44\to 51}^{8} = \begin{bmatrix} 0.0624 & -0.0126 \\ 0.0711 & -0.0042 \end{bmatrix}, \underline{u}_{0,44\to 51}^{8} = \begin{bmatrix} 0.3752 \\ 0.6272 \end{bmatrix}$$

The lower index denotes, where the trajectories start and where they will lead to, e.g. $44 \to 51$ means the control law is valid in simplex '44' and will force the trajectory to enter simplex '51'.

There was also a control law found for $\underline{R}_{44\to 50}^{8}$, but no law was found for $\underline{R}_{44\to 43}^{8}$ that considers the limitation of the input vector $\underline{u}$.

Other possible control laws were found for $\underline{R}_{44\to 50}^{1}$, $\underline{R}_{44\to 50}^{3}$, $\underline{R}_{44\to 51}^{3}$, $\underline{R}_{44\to 50}^{5}$, $\underline{R}_{44\to 43}^{5}$, $\underline{R}_{44\to 50}^{6}$, $\underline{R}_{44\to 51}^{6}$, $\underline{R}_{44\to 50}^{7}$, $\underline{R}_{44\to 51}^{7}$ and $\underline{R}_{44\to 43}^{7}$.

The performance index in this example was defined to describe the retention time of the trajectory in the investigated simplex. This is the time the trajectory needs to leave the simplex through the facet F_e, if it starts in the vertex $\underline{v}_e$.

The control law $\underline{R}^8_{44\rightarrow51}$ has the performance index $J^8_{44\rightarrow51} = 19.32$. This performance index is smaller than the performance index $J^3_{44\rightarrow51} = J^7_{44\rightarrow51} = 32.84$ of the control law $\underline{R}^3_{44\rightarrow51}$ and $\underline{R}^7_{44\rightarrow51}$, and it is smaller than the performance index $J^6_{44\rightarrow51} = 46.46$ of the control law $\underline{R}^6_{44\rightarrow51}$.

Therefore, the supervisory control chose a path through the graph the way, that simplex '44' has to be left through the facet that is adjacent to simplex '51', using valve position '8'.

The influxes q_1 and q_3 represent the continuous inputs $\underline{u}$ of the system. The needed continuous inputs $\underline{u}$ are guaranteed to stay within their upper bounds $\underline{c}_u$ and lower bounds $\underline{c}_l$

$$\underline{c}_l \leq \underline{u} = \underline{R}\,\underline{x} + \underline{u}_0 \leq \underline{c}_u \text{ with } \underline{c}_l = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \underline{c}_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In figure 5, the two states, i.e. the levels in tank 1 and 3, the influxes to tank 1 and 3 and the binary state of the discrete inputs for the valves are plotted over time.

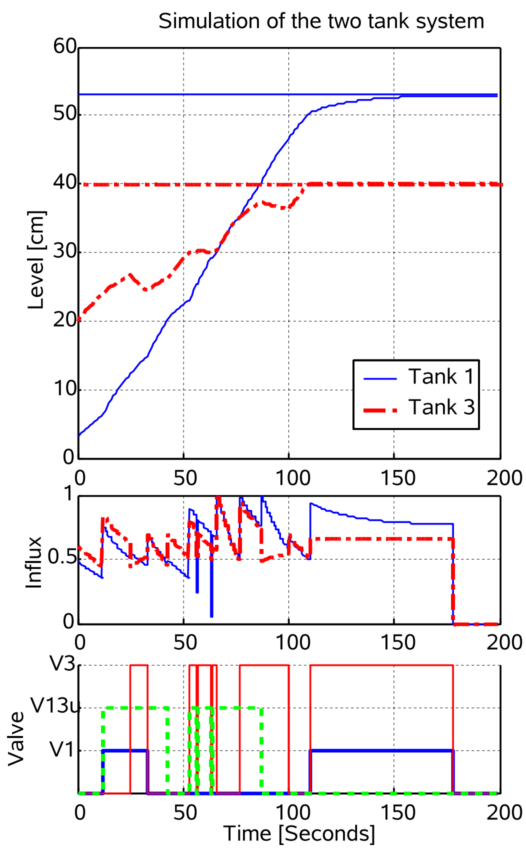

Fig. 5. The level of the two-tank system and the binary state of the valves.

For non-zero values the valves $V1$, $V2$ and $V13u$ are open. If there is a function value for $V1$, the valve $V1$ is open. There is no steady state error. The calculation on a 1.5 GHz Athlon for the local control was done in about 10 minutes, using MATLAB R12.1

To compare the results in this paper with the results in [12], figure 4 is investigated. The difference is that the method in [12] forces the trajectory to move on a straight line. In simplex '44' it is obvious that this is not the case in this paper. Also remember that the simulation is done with the nonlinear system, not with the affine one.

If the sampling period is too large, than it might happen that the neighbor simplex $\tilde{P}$ is overleaped by the trajectory. A found controller has to be checked about that. If this problem occurs, it indicates that the sampling time or the locations of the vertices is unbalanced.

VI. Conclusion

A new approach for the design of controllers for piecewise affine hybrid systems was presented. Based on a discrete-time model, the design of the controller is divided into a local control and a supervisory control.

The supervisory control is performed in a graph, done by path planning, the local control is done by solving an optimization problem, fulfilling inequality conditions.

The resulting control law is an affine feedback law, valid on the corresponding simplex. Linear search algorithms reduce the on-line computational effort for the control action, determining which control law is active. For that reason, the presented method is suitable even for more complex industrial applications.

Future work will focus on finding affine control laws for a general polytope structure and will investigate systems with less inputs than states.

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