Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Convergence of a multi-grid algorithm for the controllability of the wave equation

Mihaela Negreanu and Enrique Zuazua

Abstract— **In this paper we prove that the two-grid method proposed by Glowinski in [1] to numerically approximate the control of the wave equation converges in $1-d$. We prove this result in the context of the finite-element space semi-discretization. Our method of proof relies essentially on the particular properties of the Fourier representation of initial data of the coarse mesh when projected into the fine one. The explicit representation formula of the solutions shows that the high frequency components are modulated by some weights that diminish the effect of the spurious components. This fact, combined with discrete multipliers techniques, allows us to prove uniform observability inequalities. Classical arguments allow then proving the uniform boundedness of the controls and passing to the limit as the mesh size tends to zero.**

The controls obtained by this two-grid algorithm are not exact in the sense that they only guarantee the controllability of the projection of the solutions into the coarse mesh.

I. INTRODUCTION

Consider the $1-d$ wave equation with constant coefficients and control on one extreme:

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1, \quad 0 < t < T, \\ u(0,t) = 0, \quad u(1,t) = v(t), & 0 < t < T, \\ u(x,0) = u^0(x), \; u_t(x,0) = u^1(x), & 0 < x < 1. \end{cases} \tag{1}$$

Here (u^0, u^1) are the initial state and v is the control function that acts on the system through the extreme $x = 1$ of the space interval $(0,1)$. System (1) describes the vibrations of a controlled string and it is well posed in appropriate functional spaces.

When no control acts on the boundary $x = 1$ of the string, i.e., when $v = 0$, the energy:

$$E(t) = \frac{1}{2} \int_0^1 \left(| \, u_x(x,t) \, |^2 + | \, u_t(x,t) \, |^2 \right) dx \tag{2}$$

of the solutions is conserved in time $dE(t)/dt = E'(t) = 0$, for all $t \in [0,T]$.

The starting point of our study is the following well known exact controllability result for (1) (see [3]): *Given $T \geq 2$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$, there exists a control function $v \in L^2(0,T)$ such that the solution $u = u(x,t)$ of (1) satisfies:*

$$u(T) = u_t(T) = 0, \qquad 0 < x < 1. \tag{3}$$

Research supported by the Grant BFM2002-03345 of the Spanish MCYT and the Network "New materials, adaptive systems and their nonlinearities: modelling, control and numerical simulation" of the EU.

M. Negreanu is with Faculty of Mathematics, University Complutense of Madrid, Department of Algebra, 28040 Madrid, Spain Mihaela_Negreanu@mat.ucm.es

E. Zuazua is with Faculty of Sciences, University Autónoma of Madrid, Department of Mathematics, 28049 Madrid, Spain enrique.zuazua@uam.es

The goal in this control problem is to drive solutions to equilibrium at time $t = T$. Once the equilibrium configuration is reached at time $t = T$, by taking null control for $t \geq T$, i.e., $v = 0$ for $t \geq T$, the solution remains at rest for all $t \geq T$.

For (u^0, u^1) and $T \geq 2$ as above, there may exist infinitely many controls v with (3). The *Hilbert Uniqueness Method* (HUM) introduced by J. L. Lions in 1986 (see [3]) furnishes the one of minimal $L^2(0,T)$-norm, the so-called HUM control. Note that (3) makes sense, since in these conditions, (1) admits a unique solution $u \in \mathcal{C}([0,T]; L^2(0,1)) \cap \mathcal{C}^1([0,T]; H^{-1}(0,1))$.

Let us now introduce the homogeneous system (the adjoint of (1))

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, & 0 < x < 1, \quad 0 < t < T, \\ \phi(0,t) = \phi(1,t) = 0, & 0 < t < T, \\ \phi(x,T) = \phi^0(x), \; \phi_t(x,T) = \phi^1(x), & 0 < x < 1. \end{cases} \tag{4}$$

When $(\phi^0, \phi^1) \in H_0^1(0,1) \times L^2(0,1)$, (4) admits a unique solution $\phi = \phi(x,t) \in \mathcal{C}^0\left([0,T], H_0^1(0,1)\right) \cap \mathcal{C}^1\left([0,T], L^2(0,1)\right)$.

With the aid of the HUM, the property of exact controllability of (1) is shown to be equivalent to the following observability inequality of the adjoint system (4): *Given $T \geq 2$ there exists a positive constant $C(T) > 0$ such that*

$$E(0) \leq C(T) \int_0^T |\phi_x(1,t)|^2 dt \tag{5}$$

holds for every solution $\phi = \phi(x,t)$ of the adjoint problem (4).

When (5) holds, the quadratic functional

$$J(\phi^0, \phi^1) = \frac{1}{2} \int_0^T |\phi_x(1,t)|^2 \, dt + \int_0^1 (u^0 \phi_t(0) - u^1 \phi(0)) dx$$

is coercive in the space $H_0^1(0,1) \times L^2(0,1)$. Therefore, *when the observability inequality (5) holds, the functional J has an unique minimizer $(\hat{\phi}^0, \hat{\phi}^1)$ in $H_0^1(0,1) \times L^2(0,1)$. The control $v = -\hat{\phi}_x(1,t)$, with $\hat{\phi}$ the solution of (4) corresponding to $(\hat{\phi}^0, \hat{\phi}^1)$ is such that the solution of (1) satisfies (3).*

Consequently, in this way, the exact controllability problem (3) is reduced to the analysis of the so-called *inverse observability inequality* (5). Furthermore, the minimization of the functional J provides an useful way of constructing the control of minimal $L^2(0,T)$-norm.

Here we are concerned with the numerical approximation of the control of system (1). A possible procedure is to approximate the wave operator by some sequence of discrete or semi-discrete operators and to obtain this control $v(t)$ as the limit of the sequence of controls of the approximating equations: take $N \in \mathbb{N}$, set $h = 1/(N+1)$ and consider the following finite-element space semi-discretization of (1):

$$\begin{cases} M_h \bar{u}'' = -A_h \bar{u}, \\ u_0(t) = 0,\ u_{N+1}(t) = v_h(t), \qquad 0 < t < T, \\ u_j(0) = u_j^0, \quad u_j'(0) = u_j^1, \quad j = 1, ..., N. \end{cases} \tag{6}$$

By $'$ we denote the time derivative and by $\bar{u}(t) = (u_1(t), ..., u_N(t))$ the vector–valued solution at time t, M_h and A_h are tridiagonal symmetric matrices, the so-called mass and rigidity or stiffness matrices, respectively:

$$M_h = h \begin{bmatrix} 2/3 & 1/6 & 0 & \cdots & 0 & 0 & 0 \\ 1/6 & 2/3 & 1/6 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1/6 & 2/3 & 1/6 \\ 0 & 0 & 0 & \cdots & 0 & 1/6 & 2/3 \end{bmatrix}$$

$$A_h = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}.$$

The problem of boundary controllability for (6) consists in finding a control $v_h \in L^2(0,T)$ such that the solution of system (6) satisfies

$$\bar{u}(T) = \bar{u}'(T) = \bar{0}.$$

Let us define the Hilbert spaces of square summable sequences ℓ^2, $\hbar^1$ and $\hbar^{-1}$ as follows

$$\ell^2 = \{\{c_k\} \ : \ \|c_k\|^2_{\ell^2} = \sum_{k \in \mathbb{N}} \mid c_k \mid^2 < \infty\},$$

$$\hbar^{\pm 1} = \{\{c_k\} \in \ell^2 \ : \ \|c_k\|^2_{\hbar^{\pm 1}} = \sum_{k \in \mathbb{N}} (k\pi)^{\pm 2} \left|c_k\right|^2 < \infty\}.$$

Let $U, V \in \mathbb{R}^{N+1}$. We define the following scalar products

$$(U, V) := \sum_{j=0}^{N} U_j V_j, \quad \langle U, V \rangle = h\,(U, V)\,. \tag{7}$$

Following the HUM method, for a given h, the existence of a control v_h for (6) is equivalent to the existence of a *observability constant* $C_h(T) > 0$ such that

$$E_h(t) \leq C_h(T) \int_0^T \left| \frac{\phi_N(t)}{h} \right|^2 dt, \tag{8}$$

where $\bar{\phi}(t)$ is the solution of the homogeneous adjoint semi-discrete system

$$\begin{cases} M_h \bar{\phi}'' = -A_h \bar{\phi} \qquad 0 < t < T, \\ \phi_0(t) = \phi_{N+1}(t) = 0, \qquad 0 < t < T \\ \phi_j(T) = \phi_j^0, \quad \phi_j'(T) = \phi_j^1, \qquad j = 1, ..., N. \end{cases} \tag{9}$$

In (8), E_h is the energy of (9),

$$E_h = \frac{1}{2} \left[\left(M_h^{1/2} \bar{\phi}', M_h^{1/2} \bar{\phi}' \right) + \left(A_h^{1/2} \bar{\phi}, A_h^{1/2} \bar{\phi} \right) \right], \tag{10}$$

where $(\cdot, \cdot)$ is the scalar product defined by (7).

Observe that the energy E_h represents a natural discretization of the continuous one (2).

The *observability inequality* (8) is said to be *uniform*, if the constants $C_h(T)$ are bounded uniformly in h as $h \to 0$. However, in [2] it was proved that, for all $T > 0$, the best constant C_h in (8) blows-up, i.e., $C_h \to \infty$, as $h \to 0$. As a consequence of this fact, there exist initial data for the wave equation (1) such that the controls v_h of (6) are unbounded in $L^2(0,T)$ as $h \to 0$. Thus, computing v_h is not necessarily an efficient way to approximate the control of (1).

One way of restoring the uniform observability inequality for the homogeneous adjoint system consists in filtering the high frequency modes of the solution by considering a subclass of solutions in which the high frequency components have been truncated (see [2]-[7]).

Glowinski in [1] proposed an alternative method for efficiently computing the controls. It is a two-grid algorithm based on the use of two numerical meshes: the coarse one with step size $2h$ and the finer one corresponding to h.

There are numerical evidences of the efficiency of the method but the proof of convergence was only given recently in [6] in the case of finite-difference semi-discretizations.

This Note is devoted to present the main ideas of the proof of convergence in the present context: finite-element semi-discretizations in $1-d$. It is worth noting that the two-grid technique was introduced by Glowinski in [1] precisely in the context of finite elements in which it can be applied with more flexibility.

The key ingredient of the proof is the uniform observability inequality for any solution of (9) on the fine grid with slowly oscillating initial data. The proof of the uniform observability combines two ingredients: a) the Fourier representation of slow data on the fine grid; b) multiplier technique. This method is closely related to the filtering technique considered in [2]. In fact, it is about replacing the filtered solutions subclass by another one where we consider only the slowly oscillating initial data in the *fine grid* G_F^h, coming from a *coarse grid* G_C^{2h} with mesh size $2h$.

II. Description of the multi-grid method

Let $N \in \mathbb{N}$ be an odd number. With $h = 1/(N+1)$ we introduce the *coarse grid* G_C^{2h}, an equidistant partition of the interval $(0,1)$, $x_0 = 0 < x_1 = 2h < ... < x_{(N+1)/2} = 1$, with mesh-size $2h$ so that $x_j = 2jh$, $j = 0, ..., (N+1)/2$ and the *fine grid* G_F^h, $y_0 = 0 < y_1 = h < ... < y_N = Nh < y_{N+1} = 1$, with mesh-size h and $y_j = jh$, $j = 0, ..., N+1$.

Let us introduce the space V_h

$$V_h = \left\{ \bar{\phi}_h \in \mathbb{R}^N : \phi_{2j+1} = \frac{\phi_{2j} + \phi_{2j+2}}{2}, j = 0, ..., \frac{N-1}{2} \right\}$$

where $\phi_0 = \phi_{N+1} = 0$. Note that V_h is constituted by discrete functions defined on the fine grid G_F^h obtained by interpolation of discrete functions defined on the coarse one G_C^{2h}. Thus, V_h contains only slowly oscillatory functions.

Let us denote by $c_k \in \mathbb{R}$ the coefficients of $\bar{\phi}_h \in G_F^h$ in the orthogonal bases $\{\bar{\varphi}^k\}$ of eigenvectors of the system

$$\begin{cases} M_h^{-1} A_h \bar{\varphi} = \mu \bar{\varphi}, \\ \varphi_0 = \varphi_{N+1} = 0. \end{cases} \tag{11}$$

with $k = 1, ..., N$, $\bar{\varphi}^k = (\varphi_{k,1}, ..., \varphi_{k,N})$, $\varphi_{k,j} = \sin(jk\pi h)$ and

$$\mu_k := \mu_k(M_h^{-1} A_h) = \frac{6}{h^2} \left[\frac{1 - \cos(k\pi h)}{2 + \cos(k\pi h)} \right].$$

That is,

$$\bar{\phi}_h = \sum_{k=1}^{N} c_k \bar{\varphi}^k. \tag{12}$$

The following holds:

Proposition 2.1: For every discrete function $\bar{\phi}_h \in V_h$ the coefficients $(c_k)_k$ satisfy:

$$c_{N+1-k} = -c_k \frac{\lambda_k(h)}{\lambda_{N+1-k}(h)}, \qquad k = 1, ..., N, \tag{13}$$

$$c_{\frac{N+1}{2}} = 0, \qquad k = 1, ..., N, \tag{14}$$

where $\lambda_k(h)$ are the eigenvalues of the symmetric matrix $A = A_h/h$, given by $\lambda_k(h) = 4/h^2 \sin^2(k\pi h/2)$.

We consider the following decomposition in low and high frequencies of an arbitrary function $\bar{\phi}_h \in G_F^h$:

$$\bar{\phi}_h = \bar{\phi}_L + \bar{\phi}_H, \quad \bar{\phi}_L = \sum_{k=1}^{\frac{N-1}{2}} c_k \bar{\varphi}^k, \quad \bar{\phi}_H = \sum_{\frac{N+1}{2}}^{N} c_k \bar{\varphi}^k. \tag{15}$$

Theorem 2.2: For any discrete function $\bar{\phi}_h \in V_h$ we have

$$\bar{\phi}_L = \sum_{k=1}^{\frac{N-1}{2}} c_k \bar{\varphi}^k, \ \bar{\phi}_H = -\sum_{k=1}^{\frac{N-1}{2}} c_k \tan^2\left(\frac{k\pi h}{2}\right) \bar{\varphi}^{N+1-k}. \tag{16}$$

Proof: Taking into account that

$$\frac{\lambda_k(h)}{\lambda_{N+1-k}(h)} = \frac{\sin^2\left(\frac{k\pi h}{2}\right)}{\sin^2\left(\frac{(N+1-k)\pi h}{2}\right)} = \tan^2\left(\frac{k\pi h}{2}\right),$$

the proof is an immediate consequence of Proposition 2.1 and relations (15). ■

Taking into account the definitions in Fourier of the discrete norms ℓ^2 and $\hbar^1$, for every function $\bar{\phi} \in G_h^F$ we have:

$$|\bar{\phi}|_{\ell^2}^2 = \langle \bar{\phi}, \bar{\phi} \rangle = h \sum_{j=0}^{N} |\phi_j|^2, \tag{17}$$

$$\|\bar{\phi}\|_{\hbar^1}^2 = \left(A_h^{1/2} \bar{\phi}, A_h^{1/2} \bar{\phi}\right) = h \sum_{j=0}^{N} \left| \frac{\phi_{j+1} - \phi_j}{h} \right|^2. \tag{18}$$

We also deduce the following additional properties:

Proposition 2.3: For every function $\bar{\phi}_h \in V_h$ as in (16), with $\bar{\phi}_h = \bar{\phi}_L + \bar{\phi}_H$, the following inequalities hold:

$$\|\bar{\phi}_L\|_{\hbar^1}^2 \geq \|\bar{\phi}_H\|_{\hbar^1}^2, \tag{19}$$

$$|A_h \bar{\phi}_L|_{\ell^2}^2 \geq |A_h \bar{\phi}_H|_{\ell^2}^2, \tag{20}$$

where $|\cdot|_{\ell^2}$ and $\|\cdot\|_{\hbar^1}$ are the ℓ^2 and $\hbar^1$ discrete norms, defined by (17) and (18), respectively.

Proof: These inequalities are obtained directly using relations (13) and (14) and taking into account the expressions of the discrete norms (17) and (18). ■

III. Uniform observability for the 2-grid approximation scheme

This section is devoted to the proof of the uniform observability property within the class $V_h \times V_h$, one of the main results of this paper.

Theorem 3.1: Let $h > 0$ and $T > 4$. Then, there exist constants $C_j(T) > 0$, $j = 1, 2$, independent of h, such that

$$C_1(T) E_h \leq \int_0^T \left[\left| \frac{\phi_N(t)}{h} \right|^2 + \frac{|\phi_N'(t)|^2}{6} \right] dt \leq C_2(T) E_h \tag{21}$$

for all solution of (9) with initial data in the space $V_h \times V_h$ and all $h > 0$.

Remark 3.2: In the observability inequality (21) the term $\int_0^T |\phi_N'(t)|^2 / 6dt$ enters. It does not appear in the context of finite-difference approximations (see [2], [4], [5]). Filtering the high frequencies by truncating the Fourier series of the solution and using Ingham's inequality, this extra term can be absorbed by the term $\int_0^T |\phi_N(t)|^2 dt$ by increasing the observability constant $C_1(T)$ (see [2]).

We shall see that, the uniform inequality (21) is sufficient to guarantee the convergence of the sequence of discrete controls v_h.

Proof: (*Sketch of the proof*). The proof of this theorem uses discrete multiplier techniques similar to those in [2]. For simplicity, we divide the proof in several steps.

Step 1. We have the following identity for all the solutions of system (9) (see formula (3.26) in [2]):

$$\begin{aligned} T E_h + X_h(t)\Big|_0^T &= \frac{1}{2} \int_0^T \left[\left| \frac{\phi_N(t)}{h} \right|^2 + \frac{|\phi_N'(t)|^2}{6} \right] dt \\ &\quad + \frac{h}{12} \sum_{j=1}^{N} \int_0^T \left| \phi_j'(t) - \phi_{j+1}'(t) \right|^2 \end{aligned} \tag{22}$$

with

$$X_h(t) = h \sum_{j=1}^{N} j(\phi_{j+1} - \phi_{j-1}) \left(\frac{1}{3} \phi_j' + \frac{1}{12} \phi_{j+1}' + \frac{1}{12} \phi_{j-1}' \right). \tag{23}$$

Observe that, as indicated in [2], this identity is very close to the one one gets for the $1-d$ wave equation (see [3])

$$T E(0) + X(t)\Big|_0^T = \frac{1}{2} \int_0^T |\phi_x(1,t)|^2 dt \tag{24}$$

with

$$X(t) = \int_0^1 x \phi_x \phi_t dx.$$

Note however that, in (22), in addition to the discrete analogue of the terms entering in the identity (24), we also have the extra term

$$\frac{h}{12}\sum_{j=1}^{N}\int_0^T \left|\phi_j'(t)-\phi_{j+1}'(t)\right|^2. \tag{25}$$

As indicated in [2], when estimating this term one loses completely the observability inequality if the class of solutions under consideration is not restricted by filtering the high frequencies.

In our case, without filtering, we are able to obtain an uniform, respect to h, estimation for (25) in the class of the solutions with initial data in $V_h \times V_h$.

Step 2. (Equipartition of energy) We have:

Proposition 3.3: For any $h > 0$ and $\bar{\phi}$ solution of system (9), the following identity holds:

$$\frac{h}{12}\int_0^T \sum_{j=0}^{N}\left|\frac{\phi_{j+1}(t)-\phi_j(t)}{h}\right|^2 = \frac{1}{12}TE_h - \frac{1}{24}Y_h(t)\Big|_0^T, \tag{26}$$

where

$$Y_h(t) = h\sum_{j=1}^{N}\left(\frac{2}{3}\phi_j' + \frac{1}{6}\phi_{j+1}' + \frac{1}{6}\phi_{j-1}'\right)\phi_j. \tag{27}$$

Proof: This identity is verified by every solution $\bar{\phi}$ of system (9) (see the formula (3.33) in [2]). It is obtained multiplying equation (9) by $\bar{\phi}$ and integrating by parts, taking into account the expression of the energy (10). ■

In view of (26), we have the following identity for the time derivative ϕ' of $\bar{\phi}$:

$$\frac{h}{12}\int_0^T \sum_{j=0}^{N}\left|\phi_{j+1}'(t)-\phi_j'(t)\right|^2 = \frac{h^2}{12}TE_{\bar{\phi}'} - \frac{h^2}{24}Y_h(t)\Big|_0^T, \tag{28}$$

where

$$Y_h(t) = h\sum_{j=1}^{N}\left(\frac{2}{3}\phi_j'' + \frac{1}{6}\phi_{j+1}'' + \frac{1}{6}\phi_{j-1}''\right)\phi_j', \tag{29}$$

and $E_{\bar{\phi}'}$ is the conservative energy of the derivative $\bar{\phi}'$ of the solution $\bar{\phi}$.

Step 3. (Some key estimations) The following holds:

Proposition 3.4: Let $\bar{\phi} = \bar{\phi}_L + \bar{\phi}_H$ be the solution of (9) with given data $(\bar{\phi}^0, \bar{\phi}^1)$ in time $t = T$ in $V_h \times V_h$. For every $t \in [0, T]$ we have:

$$E_h(t) = E_{\bar{\phi}_L}(t) + E_{\bar{\phi}_H}(t) \;\; \textit{and} \;\; E_{\bar{\phi}_L}(t) \geq E_{\bar{\phi}_H}(t), \tag{30}$$

where $E_{\bar{\phi}_L}$ and $E_{\bar{\phi}_H}$ are the conservative energies of the solutions $\bar{\phi}_L$ and $\bar{\phi}_H$, respectively.

Proof: This identity is obtained directly using relations (19) and (20) and the conservation of energy. ■

This result is the key point in the proof of (21).

Proposition 3.5: For any $h > 0$ and any solution $\bar{\phi}$ of (9) with initial data in the space $V_h \times V_h$, the following relation holds:

$$\frac{h^2}{12}E_{\bar{\phi}'} \leq \frac{1}{2}E_{\bar{\phi}_L} \leq \frac{1}{2}E_h. \tag{31}$$

Proof: The time derivative of every solution ϕ of (9) is a solution of (9), too, with data $\bar{\phi}'(T) = \bar{\phi}^1$ and $\bar{\phi}''(T) = M_h^{-1}A_h\bar{\phi}^0$. In virtue of Proposition 3.4 we have

$$\frac{h^2}{12}E_{\bar{\phi}'} = \frac{h^2}{12}(E_{\bar{\phi}_L'} + E_{\bar{\phi}_H'}) \leq \frac{h^2}{6}E_{\bar{\phi}_L'} \leq \frac{h^2}{6}\Lambda E_{\bar{\phi}_L}, \tag{32}$$

where Λ is the largest eigenvalue entering in the Fourier development of the solution $\bar{\phi}_L$ of (9), corresponding to the low frequencies, i.e.,

$$\bar{\phi}_L(t) = \sum_{k=1}^{\frac{N-1}{2}} c_k\bar{\varphi}^k. \tag{33}$$

That is

$$\Lambda = \max_{1\leq k\leq (N-1)/2}\mu_k(M_h^{-1}A_h) = \frac{3}{h^2}. \tag{34}$$

Thus, using estimation (34) of Λ in (32), we find exactly inequality (31) and the proof is complete. ■

Step 4. Now, we are able to obtain the following sharp estimate for the extra term (25) entering in the identity (24) in the class V_h of slowly varying initial data.

In view of (31), in (28), we have:

$$\frac{h}{12}\int_0^T \sum_{j=0}^{N}\left|\phi_{j+1}'(t)-\phi_j'(t)\right|^2 \leq \frac{T}{2}E_h - \frac{h^2}{24}Y_h(t)\Big|_0^T, \tag{35}$$

for the solution of (9) in the class of slowly oscillating data in $V_h \times V_h$.

In view of this relation, combining (22), (26) and (35) we deduce that

$$\frac{T}{2}E_h + Z_h(t)\Big|_0^T \leq \frac{1}{2}\int_0^T \left[\left|\frac{\phi_N(t)}{h}\right|^2 + \frac{|\phi_N'(t)|^2}{6}\right]dt, \tag{36}$$

with

$$Z_h(t) = X_h(t) - \frac{h^2}{24}Y_h(t). \tag{37}$$

The following provides an estimate on the reminder term Z_h given by (37):

Proposition 3.6: For any $h > 0$ and any solution $\bar{\phi}$ of (9) it follows that

$$|Z_h(t)| \leq \sqrt{1 - \frac{h^2}{8} + \frac{15h}{16\mu_1}}E_h, \;\; \forall\, 0 < t < T, \tag{38}$$

where $\mu_1 = 6/h^2 (1-\cos(\pi h))/(2+\cos(\pi h))$ is the first eigenvalue of matrix $M_h^{-1}A_h$.
The proof of this relation is similar to the proof of formula (3.43) in [2] and it is given in detail in [4].

Now, by (36) and (38) we have:

$$C_1(T)E_h \leq \int_0^T \left[\left| \frac{\phi_N}{h} \right|^2 + \frac{|\ \phi_N'\ |^2}{6} \right] dt \tag{39}$$

and (21) holds for every $T > 4$.

This concludes the proof of the first (so-called *inverse*) inequality in (21).
The proof of the second one (so-called *direct*) in (21) is similar to the above proof and it is given in detail in ([4]).

This completes the proof of Theorem 3.1 ■

Remark 3.7: It is important to underline that the inequality (39) is uniform in h precisely because of considering slowly oscillating initial data in the fine grid. Relation (31) plays a fundamental role in the above proof.
The time $T > 4$ is twice the observability time for the continuous wave equation. The analysis of the dispersion diagram of filtered solutions predicts a propagation velocity equal to $1/2$ in the class of solutions under consideration and confirms the observability time $T = 4$.

IV. Construction and convergence of the controls

We now briefly describe the algorithm for building an efficient approximation of the control v of (1) which is inspired in Glowinski [1]. To simplify the notations we define

$$f(\bar{u}_h, \bar{\phi}_h) := \left(M_h \bar{u}_h^1, \bar{\phi}_h(0)\right) + \left(-M_h \bar{u}_h^0, \bar{\phi}_h'(0)\right), \tag{40}$$

where $\bar{u}_h$ and $\bar{\phi}_h$ are the solutions of systems (6) and (9), respectively and $(\cdot,\cdot)$ is the usual scalar product (7) in $\mathbb{R}^N$.

Let us denote by $\mathcal{F}_h$ the subspace of $\mathbb{R}^N \times \mathbb{R}^N$, $\mathcal{F}_h = V_h \times V_h$ endowed with the norm

$$\|(U,V)\|^2_{\mathcal{F}_h} := \|U\|^2_{\hbar^1} + |M^{1/2}V|^2_{\ell^2}, \quad U,\ V \in V_h,$$

where $M = M_h/h$.

Set $T > 0$. The partial controllability problem of system (6) in the space $\ell^2 \times \hbar^{-1}$ consists in finding a control $v_h \in L^2(0,T)$ such that the solution $\bar{u}_h$ of (6) satisfies

$$\mathcal{R}\left(\bar{u}_h'(T), \bar{u}_h(T)\right) = (0,0) \tag{41}$$

where $\mathcal{R}$ is the projection operator from $\hbar^{-1} \times \ell^2$ into $\mathcal{F}_h^*$ (the dual space of $\mathcal{F}_h$, considered as a subspace of the euclidian space $\hbar^{-1} \times \ell^2$)

$$\mathcal{R}\left(\bar{u}_h'(T), \bar{u}_h(T)\right) = (\bar{U}'(T), \bar{U}(T)),$$

with

$$U_j = \frac{u_{2j-2}/2 + 3u_{2j-1} + u_{2j} + 3u_{2j+1} + u_{2j+2}/2}{12} \tag{42}$$

and

$$U_j' = \frac{u_{2j-2}'/2 + 3u_{2j-1}' + u_{2j}' + 3u_{2j+1}' + u_{2j+2}'/2}{12}, \tag{43}$$

$j = 1, ..., (N+1)/2$.

Multiplying the first equation in (6) by $\{\bar{\phi}_h\}$, integrating by parts in $[0,T]$ and adding in j, $1 \leq j \leq N$, we deduce that

$$\begin{aligned}&\left(M_h \bar{u}_h'(T), \bar{\phi}_h^0\right) + \left(-M_h \bar{u}_h(T), \bar{\phi}_h^1\right) - f(\bar{u}_h, \bar{\phi}_h)\\ &= \frac{1}{h}\int_0^T v_h(t)\phi_N(t)dt - \frac{h}{6}\int_0^T v_h(t)\phi_N''(t)dt\end{aligned} \tag{44}$$

with $f(\bar{u}_h, \bar{\phi}_h)$ defined as in (40).

The following Proposition provides a characterization of the partial controllability property of the semi-discrete system (6):

Proposition 4.1: Set $T > 0$. Problem (6) is partially controllable in $\ell^2 \times \hbar^{-1}$ if and only if, for each initial state $(\bar{u}_h^0, \bar{u}_h^1) \in \ell^2 \times \hbar^{-1}$, there exists a function $v_h \in L^2(0,T)$ such that

$$f(\bar{u}_h, \bar{\phi}_h) = \frac{h}{6}\int_0^T v_h(t)\phi_N''(t)dt - \frac{1}{h}\int_0^T v_h(t)\phi_N(t)dt. \tag{45}$$

for any initial data $(\bar{\phi}_h^0, \bar{\phi}_h^1) \in \mathcal{F}_h$ associated to the solution $\bar{\phi}_h$ of (9).

Proof: In view of identity (44), it is immediate to see that the control v_h is characterized by the property (45). ■

Remark 4.2: In the continuous case we have the following characterization of the property of exact controllability: the control v steers the initial data (u^0, u^1) of problem (1) to $(0,0)$ if and only if $\int_0^T v(t)\phi_x(1,t)dt = \int_0^1 (u^1\phi^0 - u^0\phi^1)dx$, for every solution ϕ of the homogeneous problem (4) with initial data (ϕ^0, ϕ^1).

Let us consider the functional $J_h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$\begin{aligned}J_h\left((\bar{\phi}_h^0, \bar{\phi}_h^1)\right) &= \frac{1}{2}\int_0^T \left|\frac{\phi_N(t)}{h}\right|^2 dt + f(\bar{u}_h, \bar{\phi}_h)\\ &+ \frac{h^2}{12}\int_0^T \left|\frac{\phi'_N(t)}{h}\right|^2 dt\end{aligned} \tag{46}$$

where $\bar{\phi}_h$ is the solution of the adjoint problem (9) with initial data $(\bar{\phi}_h^0,\ \bar{\phi}_h^1) \in \mathcal{F}_h$, $(\bar{u}_h^0, \bar{u}_h^1)$ are the initial data of system (6) to be controlled.

This functional is convex and continuous. On the other hand, in view of inequality (21), it is uniformly (with respect to $0 < h < 1$) coercive in $\mathcal{F}_h = V_h \times V_h$ provided $T > 4$.

Thus, for each $0 < h < 1$ and for any $T > 4$ there is a unique minimizer $(\hat{\phi}_h^0, \hat{\phi}_h^1) \in \mathcal{F}_h$ of the functional J_h,

$$I_h = J_h\left(\hat{\phi}_h^0, \hat{\phi}_h^1\right) = \min_{(\bar{\phi}_h^0, \bar{\phi}_h^1) \in \mathcal{F}_h} J_h\left(\bar{\phi}_h^0, \bar{\phi}_h^1\right). \quad (47)$$

The corresponding Euler equation reeds

$$\int_0^T \frac{\hat{\phi}_N(t)}{h}\left[\frac{\phi_N(t)}{h} - \frac{h^2}{6}\frac{\phi_N''(t)}{h}\right] dt + f(\bar{u}_h, \bar{\phi}_h) = 0, \quad (48)$$

for every solution $\bar{\phi}_h$ of the adjoint problem (9).

Now, we define $v_h(t) = \hat{\phi}_N(t)/h$, hence, in (48) we obtain

$$\int_0^T v_h(t)\left[\frac{\phi_N(t)}{h} - \frac{h^2}{6}\frac{\phi_N''(t)}{h}\right] dt + f(\bar{u}_h, \bar{\phi}_h) = 0, \quad (49)$$

by the way it is equivalent to a partial controllability property. In view of the uniform inequality (21), the sequence of controls v_h we have obtained is bounded in $L^2(0,T)$.

Remark 4.3: Observe that, in this method, the functional J_h is minimized on $\mathcal{F}_h$, in contrast with the direct application of the minimization method of the functional J_h (that leads to divergent controls as $h \to 0$ for some pathological initial data, see [2] and [7]). The uniform observability inequality (21) guarantees the uniform coercivity of J_h when restricted to $\mathcal{F}_h$ and, consequently, the boundedness of the controls.
Note that (48) does not imply that the solution of (6) vanishes at time $t = T$. Rather a suitable projection of the state (u, u') on the coarse grid vanishes. More precisely, if we define the restriction (42)–(43) we have

$$\mathcal{R}(\bar{u}(T), \bar{u}'(T)) = (\bar{U}_h(T), \bar{U}_h'(T)) = (\bar{0}, \bar{0}).$$

Given an initial state $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ of the continuous system (1), we develop it in Fourier series

$$(u^0, u^1) = \sum_{k=1}^{\infty} (c_k^0, c_k^1)\varphi_k(x), \quad (50)$$

with $(c_k^0, c_k^1) \in \ell^2 \times \hbar^{-1}$, $\varphi_k(x) = \sin(k\pi x)$. Further, we construct the sequence of discrete initial states:

$$(\bar{u}_h^0, \bar{u}_h^1) = \sum_{k=1}^{N} (c_k^0, c_k^1)\bar{\varphi}_k, \quad (51)$$

with $\bar{\varphi}_k = (\varphi_{k,1}, ..., \varphi_{k,N})$, $\varphi_{k,j} = \sin(k\pi j\Delta x)$, $j, k = 1, ..., N$.

As the solution u of (1) with initial data $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ and control $v \in L^2(0,T)$ satisfies $u \in \mathcal{C}([0,T]; L^2(0,1)) \cap \mathcal{C}^1([0,T]; H^{-1}(0,1))$, then, there exist functions $c_k(t)$, $k \in \mathbb{N}$ such that

$$u(x,t) = \sum_{k \in \mathbb{N}} c_k(t)\varphi_k(x), \quad (52)$$

with $\sup_{t\in[0,T]} [\| c_k(t) \|_{\ell^2} + \| c_k'(t) \|_{\hbar^{-1}}] < \infty$. The solution of (6) may be written in a similar form as

$$\bar{u}_h(t) = \sum_{k \in \mathbb{N}} c_{k,h}(t)\bar{\varphi}_k$$

by putting $c_{k,h}(t) = 0$ for $k > N$.

The following convergence result holds:

Theorem 4.4: Let $T > 4$. Let $\bar{u}_h$ and u be the solutions of (6) and (1) as above. Then, as $h \to 0$ and $1 \le p < \infty$

$$\{c_{k,h}(\cdot)\}_{k\in\mathbb{N}} \to \{c_k(\cdot)\}_{k\in\mathbb{N}}, \quad L^p(0,T;\ell^2) \cap W^{1,p}(0,T;\hbar^{-1}) \quad (53)$$

$$v_h(\cdot) \to v(\cdot) \quad L^2(0,T), \ as\ h \to 0, \quad (54)$$

where $v_h = v_h(t)$ is the control of (6) constructed by the multi-grid algorithm above and $v = v(t)$ is the HUM control for the continuous wave equation which drives the initial data (u^0, u^1) to rest in time T.
The proof of Theorem 4.4 is given in detail in [4].

By taking limits in (49) and thanks to the convergence assumptions of the initial data, we obtain

$$0 = \int_0^1 \left[u^1(x)\phi^0(x) - u^0(x)\phi^1(x)dx\right] + \int_0^T v(t)\partial_x\phi(1,t)dt \quad (55)$$

and this condition is equivalent to the fact that v is a control for the system (1) driving initial data (u^0, u^1) to rest. One can also show by a Γ-convergence argument that v is in fact the HUM control.

Similar results can be obtained for the finite difference semi-discretization of the $1-d$ wave equation (see [4]-[5], [8]).

REFERENCES

[1] R. Glowinski, *Ensuring well-posedness by analogy; Stokes problem and boundary control of the wave equation*, J. Compt. Phys., **103**(2), 189-221, 1992.

[2] J.A. Infante, E. Zuazua, Boundary observability for the space discretization of the one-dimensional wave equation, M^2AN, **33** (2), 407-438, 1999.

[3] J. L. Lions, *Contrôlabilité exacte, stabilisation et perturbations du systemes distribués. Tome 1 Contrôlabilité exacte*, Masson, RMA8, 1988.

[4] M. Negreanu, *Métodos numéricos para el análisis de la propagación, observación y control de ondas* Doctoral Thesis, Universidad Complutense de Madrid, Departamento de Matemática Aplicada, 2004.

[5] M. Negreanu, E. Zuazua, *Convergence of a multi-grid method for the controllability of a 1-d wave equation*, C. R. Acad. Sci. Paris, Série I **338** (5), 413-418, 2004.

[6] M. Negreanu, E. Zuazua, *Uniform boundary controllability of a discrete $1-D$ wave equation*, Systems and Control Letters **48** (3-4), 261-279, 2003.

[7] E. Zuazua, *Propagation, Observation, Control and Numerical Approximation of Waves*, SIAM Review, to appear.

[8] E. Zuazua, *Optimal and Approximate Control of Finite-Difference Approximation Schemes for the 1-D Wave Equation*, Rendiconti di Matematica, to appear.