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Uniform observability of the wave equation via a discrete Ingham inequality

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Abstract— **In this paper we prove a discrete version of the classical Ingham inequality for nonharmonic Fourier series whose exponents satisfy a gap condition. Time integrals are replaced by discrete sums on a discrete mesh. We prove that, as the mesh becomes finer and finer, the limit of the discrete Ingham inequality is the classical continuous one. This analysis is partially motivated by control-theoretical applications. As an application we analyze the observation properties of numerical approximation schemes of the 1-d wave equation. The discrete Ingham inequality provides observability (and controllability) results which are uniform with respect to the mesh size in suitable classes of numerical solutions in which the high frequency components have been filtered. We also discuss the optimality of these results in connection with the dispersion diagrams of the considered numerical schemes.**

I. INTRODUCTION

Families of 'nonharmonic' exponentials $\left\{e^{i\lambda_k t}\right\}$ appear in various fields of mathematics and signal processing. One of the central problems arising in all of these applications is the question of the Riesz basis property.

The following inequality for nonharmonic Fourier series due to Ingham is well known (see [5] and [11], p. 162): *Assume that the strictly increasing sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ of real numbers satisfies the 'gap' condition*

$$\lambda_{k+1} - \lambda_k \geq \gamma, \text{ for all } k \in \mathbb{Z}, \tag{1}$$

for some $\gamma > 0$. Then, for all $T > 2\pi/\gamma$ there exist two positive constants C_1, C_2 depending only on γ and T such that

$$C_1(T,\gamma) \sum_{k=-\infty}^{\infty} |a_k|^2 \leq \int_0^T \left| \sum_{k=-\infty}^{\infty} a_k e^{it\lambda_k} \right|^2 dt \leq C_2(T,\gamma) \sum_{k=-\infty}^{\infty} |a_k|^2 \tag{2}$$

for every complex sequences $(a_k)_{k\in\mathbb{Z}} \in \ell^2$, where

$$C_1(T,\gamma) = \frac{2T}{\pi}\left(1 - \frac{4\pi^2}{T^2\gamma^2}\right) > 0, \tag{3}$$

$$C_2(T,\gamma) = \frac{8T}{\pi}\left(1 + \frac{4\pi^2}{T^2\gamma^2}\right) > 0 \tag{4}$$

and ℓ^2 is the Hilbert space of square summable sequences,

$$\ell^2 = \{\{a_k\} \,:\, \|a_k\|_{\ell^2}^2 = \sum_{k\in\mathbb{N}} |a_k|^2 < \infty\}. \tag{5}$$

This result shows that the sequence of exponentials $\left\{e^{i\lambda_k t}\right\}$ forms a Riesz basis of its span for $T > 2\pi/\gamma$ (see [11], Chapter 3, p. 112). The problem of observability consists in guarantee that the whole energy of the waves which propagate according to a given equation with suitable boundary conditions can be estimated in terms of the energy concentrated on a given subregion of the domain (or its boundary), where propagation occurs in a given time interval.

In the context of partial differential equations, this Ingham theorem has been used to prove observability inequalities for the solutions of 1-d evolution equations for which the sequence of eigenfrequencies has an uniform gap ([7]). Attempts to extend this technique to the case when there is not uniform gap have led to far reaching generalizations of the Ingham theorem for some sequences satisfying weakened gap conditions (see [2], [3], [4], [6]). On the other hand, in the numerical analysis of those observability inequalities the need of a discrete version of this inequality arises naturally.

In this paper we prove a discrete version of (2). More precisely, given $\Delta t = T/(M+1)$, with $M \in \mathbb{N}$ we replace in (2) the integral by a discrete sum, $\Delta t \sum_{n=0}^{M}$, and we analyze the existence of two positive constants C_1, C_2 such that the resulting discrete inequality holds. Obviously, we are interested on results that remain uniform as the mesh-size Δt tends to zero.

II. THE DISCRETE INGHAM INEQUALITY

The main result of this paper is as follows:

Theorem 2.1: (Discrete Ingham inequality) Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be an increasing sequence of real numbers satisfying for some $\gamma > 0$ the 'gap' condition

$$\lambda_{k+1} - \lambda_k \geq \gamma > 0, \quad \text{for all } k \in \mathbb{Z}. \tag{6}$$

Let $T > 0$ and $0 < \Delta t \leq 1$. Assume that the sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ satisfies for some $0 \leq p < 1/2$ the additional condition

$$|\lambda_k - \lambda_l| \leq \frac{2\pi - (\Delta t)^p}{\Delta t}, \quad \text{for all } |k| \leq N, \;\; |l| \leq N, \tag{7}$$

where $2N \leq M$ and $M = [T/\Delta t - 1]$. Then, there exists a positive number $\epsilon(\Delta t)$ such that, for all $T > T_0(\Delta t) := 2\pi/\gamma + \epsilon(\Delta t)$, there exist two positive constants

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$C_j(\Delta t, T, \gamma) > 0$, $j = 1, 2$, such that

$$C_1(\Delta t, T, \gamma) \sum_{k=-N}^{N} |a_k|^2 \leq \Delta t \sum_{n=0}^{M} \left| \sum_{k=-N}^{N} a_k e^{in\Delta t \lambda_k} \right|^2 \leq C_2(\Delta t, T, \gamma) \sum_{k=-N}^{N} |a_k|^2 \tag{8}$$

for every complex sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2$.

Moreover, if γ and p in (6) and (7) are kept fixed, then $\epsilon(\Delta t) = o(\Delta t)^{1-2p}$ and the constants in (8) satisfy

$$C_j(\Delta t, T, \gamma) = C_j(T, \gamma) \mp \delta_j(\Delta t), \ \ \delta_j(\Delta t) \geq 0, \ j = 1, 2, \tag{9}$$

where $C_j(T, \gamma)$, $j = 1, 2$, are the Ingham constants (3), (4) and $\lim_{\Delta t \to 0} \delta_j(\Delta t) = 0$, $j = 1, 2$.

The proof of the discrete inequality (8) follows the scheme used in [11] (p. 162-163) to prove the classical Ingham inequality (2). It is easy to see that for every $N \in \mathbb{N}$ fixed, if we pass to limit with $\Delta t \to 0$ in (8) we get the classical Ingham inequality (2).

Remark 2.2: In the original paper by Ingham (see [5], p. 368) it is pointed out that, the following L^1 analogue of inequality (2) holds, for every increasing sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ of real numbers satisfying the 'gap' condition (1):

$$C_1(T, \gamma) |a_k| \leq \int_0^T \left| \sum_{k=-\infty}^{\infty} a_k e^{it\lambda_k} \right| dt \leq C_2(T, \gamma) |a_k|, \tag{10}$$

for all $T > 2\pi/\gamma$ and $\forall\ k \in \mathbb{Z}$. Under the hypotheses of our discrete Ingham's Theorem 2.1 we also have a discrete version of (10).

Remark 2.3: In both the continuous and discrete cases, the sequence $\{\lambda_k\}_k$ is required to satisfy (6), the so-called gap condition. The restriction (7) imposed on $\{\lambda_k\}_k$ in Theorem 2.1 is not needed in the classical continuous Ingham inequality (2).

The restriction $2N \leq M$ with $M = [T/\Delta t - 1]$ is sharp. Indeed, when $2N > M$ one can find non-trivial values of the coefficients $\{a_k\}_k$ such that

$$\sum_{k=-N}^{N} a_k e^{in\Delta t \lambda_k} = 0, \ \ 0 \leq n \leq M \tag{11}$$

and $\sum_{k=-N}^{N} |a_k|^2 \neq 0$. Condition (11) consists on a system of homogeneous linear equations in a_k with $2N + 1$ unknown quantities and $M + 1$ equations. If $2N > M$ this system has non trivial solutions. This is in agreement with common sense. Indeed, in view of the fact that we only make $M + 1$ measurements for $n = 0, ..., M$ one can not expect to recover more than $M + 1$ coefficients of the solution. If $2N \leq M$, in general, the above situation (11) does not happen. Notice that, $e^{in\Delta t(\lambda_k - \lambda_l)} = 1$ if $\lambda_k - \lambda_l = 2\pi m/\Delta t$, with $m \in \mathbb{Z}$. Consequently, the function $v(\lambda_k - \lambda_l) := \Delta t \sum_{n=0}^{M} a_k \bar{a}_l e^{in\Delta t(\lambda_k - \lambda_l)}$ is $2\pi/\Delta t$ periodic on $\mathbb{R}$ and therefore it is enough to analyze it only for the increasing sequence of real numbers $(\lambda_k)_k$ with $\lambda_{k+1} - \lambda_k \in [2\pi m/\Delta t, 2\pi(m+1)/\Delta t]$, $m \in \mathbb{Z}$.

However, if $\lambda_k - \lambda_l \in 2\pi\mathbb{Z}/\Delta t$, for certain values of k and l with $k \neq l$, the sequence such that $a_k = -a_l = 1$, $a_n = 0$, $n \neq k$, l satisfies (11). Then, an inequality of type (8) is impossible. So, it is natural to impose the condition $\lambda_k - \lambda_l \notin 2\pi\mathbb{Z}/\Delta t$ for a discrete Ingham inequality (8) to hold. In our theorem this latter condition is implied by the stronger one (7).

The technical choice of the parameter $0 \leq p < 1/2$ in (7) is sufficient to obtain the asymptotically optimal condition $T > 2\pi/\gamma + \epsilon(\Delta t)$, with $\epsilon(\Delta t) \to 0$ as $\Delta t \to 0$ since $\epsilon(\Delta t) = o(\Delta t)^{1-2p}$.

Condition $T > 2\pi/\gamma$ is optimal for the classical Ingham inequality (see [11], p. 163). In this sense, the condition $T > 2\pi/\gamma + \epsilon(\Delta t)$ in Theorem 2.1 is asymptotically optimal since $\epsilon(\Delta t) \to 0$ as $\Delta t \to 0$.

It is easy to see that, for every $N \in \mathbb{N}$ fixed, if we pass to the limit $\Delta t \to 0$ in (8) we get the classical Ingham inequality (2). Indeed, for (2) to be true for all sequences $(a_k)_{k\in\mathbb{Z}} \in \ell^2$ it is sufficient, by density, to prove it for sequences with only a finite number of non-zero components.

In that case (2) is the limit of (8) because of the convergence of the minimal time T and the constants C_j, $j = 1, 2$, in (8) to those of (2).

III. Application to the uniform observability of the full discretizations of the wave equation

The content of this section is motivated by the classical problem of control of waves. More precisely, it is related with the controllability of the 1-d wave equation: given $T > 0$ and $(u^0, u^1) \in L^2(0, 1) \times H^{-1}(0, 1)$, the problem is to find a control function $v \in L^2(0, T)$ such that the solution of the system

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad 0 < t < T, \\ u(0, t) = 0, \quad u(1, t) = v(t), \quad 0 < t < T, \\ u(x, 0) = u^0(x), \ u_t(x, 0) = u^1(x), \ 0 < x < 1, \end{cases} \tag{12}$$

satisfies

$$u(T) = u_t(T) = 0, \qquad 0 < x < 1. \tag{13}$$

This property is well known to be true for $T \geq 2$. This problem has been studied and solved in a much more general setting and, in particular, for multi-dimensional wave equations ([7]). Several approaches to the problem have been developed. In particular, the Hilbert Uniqueness Method (HUM) introduced by Lions in [7] offers a general way of reducing the problem to the so-called *observability problem* for the adjoint (up to an inversion in time) wave equation in the absence of control:

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, \quad 0 < x < 1, \quad 0 < t < T, \\ \phi(0, t) = \phi(1, t) = 0, \quad 0 < t < T, \\ \phi(x, 0) = \phi^0(x), \ \phi_t(x, 0) = \phi^1(x), \ 0 < x < 1. \end{cases} \tag{14}$$

It is well known that the energy

$$E(t) = \frac{1}{2} \int_0^1 \left(| \phi_x(x, t) |^2 + | \phi_t(x, t) |^2 \right) dx \tag{15}$$

of the solutions of (14) satisfies $dE(t)/dt = E'(t) = 0$, $t \in [0,T]$ and therefore it is conserved in time.

The observability problem is as follows: *To find* $T > 0$ *such that there exists a constant* $C(T) > 0$ *for which*

$$E(0) \leq C(T) \int_0^T \mid \phi_x(1,t) \mid^2 dt \tag{16}$$

holds for every solution of (14). HUM allows showing that, once the observability (16) is satisfied for the adjoint system (14), the system (12) is controllable in time T. Moreover, HUM provides a systematic method to build the control $v = \phi_x(1,t)$ of minimal $L^2(0,T)$- norm.

In the context of the $1-d$ wave equation (14), inequality (16) can be easily proved by several methods including *Fourier series, D'Alembert Formula*, *multiplier techniques* and *Ingham's theorem* (2), provided $T \geq 2$.

In order to prove (16) applying the classical Ingham inequality, one uses Fourier series techniques. Indeed, the solution of (14) admits the Fourier development

$$\phi(x,t) = \sum_{k \in \mathbb{Z}\setminus\{0\}} a_k e^{i\lambda_k t} \varphi_k(x), \tag{17}$$

with $(\lambda_k)_k$, $\lambda_k = k\pi = -\lambda_{-k}$, $k > 0$, being the sequence of eigenvalues of the system, $\varphi_k(x) = \sin(k\pi x)$, the corresponding eigenfunctions and $a_k \in \mathbb{C}$ the Fourier coefficients, which can be computed explicitly in terms of the initial data in (14).

By definition (15) of the conserved energy of the solution ϕ of (14) given by (17), we have

$$E_\phi = \frac{1}{2} \sum_{k \in \mathbb{Z}\setminus\{0\}} k^2\pi^2 \left|a_k\right|^2. \tag{18}$$

Then, inequality (16) may be written as:

$$\sum_{k \in \mathbb{Z}\setminus\{0\}} k^2\pi^2 \left|a_k\right|^2 \leq C(T) \int_0^T \left| \sum_{k \in \mathbb{Z}\setminus\{0\}} (-1)^k k\pi a_k e^{i\lambda_k t} \right|^2 dt. \tag{19}$$

According to Ingham's inequality (2), inequality (19) holds for $T > 2$, since the gap of the sequence $(\lambda_k)_k$ is constant, $\gamma = \pi$, and consequently $2\pi/\gamma = 2$. In this particular case the inequality holds also for the minimal time $T = 2$. This is due to the orthogonality properties of the trigonometric polynomials. But, in general, i.e., for a general sequence $(\lambda_k)_{k\in\mathbb{Z}}$ under the gap condition (1), it is well known that the Ingham inequality (2) may fail for the minimal time $T = 2\pi/\gamma$ (see [11], p. 163).

As a further step towards a complete theory of numerical approximation of controls it is natural to address the same issue for full space-time discretizations.

The main ingredients for the discrete analogue of (16) for a finite-difference full discretization of a homogeneous 1-d wave equation (14) are the Fourier representation of solutions and our discrete Ingham inequality in Theorem 2.1.

In this section we study an application of the discrete Ingham inequality (8) to a finite-difference full discretization of a homogeneous 1-d wave equation.

Given $M, N \in \mathbb{N}$ we set $\Delta x = 1/(N+1)$ and $\Delta t = T/(M+1)$ and introduce the nets $0 = x_0 < ... < x_{N+1} = 1$, $0 = t_0 < ... < t_{M+1} = T$ with $x_j = j\Delta x$ and $t_n = n\Delta t$, $j = 0, ..., N+1$, $n = 0, ..., M+1$.

We consider the following finite-difference discretization of (12):

$$\begin{cases} u_j^{n+1} - 2u_j^n + u_j^{n-1} = \mu^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right), \\ u_0^n = 0,\ u_{N+1}^n = v_{\Delta t}^n, \\ u_j^0 = u_{0j},\ u_j^1 = \Delta t u_{1j} + u_{0j}, \end{cases} \tag{20}$$

with $j = 1, ..., N$, $n = 1, ..., M$ and $\mu = \Delta t/\Delta x$. We shall denote by $\bar{u}^n = (u_1^n, ..., u_N^n)$ the solution at the time step n. As in the context of the continuous wave equation above, we consider the uncontrolled system

$$\begin{cases} \phi_j^{n+1} - 2\phi_j^n + \phi_j^{n-1} = \mu^2\left(\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n\right), \\ \phi_0^n = \phi_{N+1}^n = 0, \\ \phi_j^0 = \phi_{0j},\ \phi_j^1 = \phi_{0j} + \Delta t \phi_{1j}, \end{cases} \tag{21}$$

$j = 1, 2, ..., N$, $n = 1, 2, ..., M$, a central finite difference discretization of (14).

Under the stability condition $\mu = \Delta t/\Delta x \leq 1$ (μ is the Courant number), the solutions $\phi_j^n(\Delta t, \Delta x)$ of the finite dimensional system (21) converge towards the solutions ϕ of (14), when Δx and Δt, the space and the time mesh sizes, respectively, go to zero. The error of convergence is of order $\mathcal{O}((\Delta t)^2 + (\Delta x)^2)$ (order 2) (see [8]). Obviously, ϕ_j^n is an approximation of $\phi(x,t)$, ϕ being the solution of (14), provided the initial data (ϕ_j^0, ϕ_j^1), $j = 0, ..., N+1$ are an approximation of the initial data in (14).

The energy of (21) is

$$\begin{aligned} E_n \quad &= \Delta x/2 \sum_{j=0}^{N} \Big[\left((\phi_j^1 - \phi_j^0)/\Delta t\right)^2 \\ &+ (\phi_{j+1}^1 - \phi_j^1)(\phi_{j+1}^0 - \phi_j^0)/(\Delta x)^2\Big] \geq 0, \end{aligned} \tag{22}$$

which is a discretization of the continuous energy E in (15), and it is conserved in all the time steps $E_n = E_0$, $n = 1, ..., M$, for the solution of (21) (see [8]).

Solutions of (21) admit the Fourier development

$$\bar{\phi}^n = \sum_{k=-N, k\neq 0}^{N} a_k e^{i\lambda_k n\Delta t} \bar{\varphi}_{|k|}, \tag{23}$$

with $a_k \in \mathbb{C}$, $\bar{\varphi}_k = (\sin(k\pi\Delta x), ..., \sin(Nk\pi\Delta x))$ and

$$\lambda_k = \operatorname{sgn}(k) \frac{2}{\Delta t} \arcsin\left(\frac{\Delta t}{\Delta x} \sin\frac{k\pi\Delta x}{2}\right), \tag{24}$$

$\lambda_k = -\lambda_{-k}$ for $k > 0$, being the eigenvalues of the system (21) (see [8]).

Our goal is to analyze the discrete version of the observability inequality (16):

$$E_0 \leq C\left[\Delta t \sum_{n=0}^{M} \left|\frac{\phi_N^n}{\Delta x}\right|^2\right], \tag{25}$$

where E_0 is the conserved energy of the solutions of the discrete system (21). This inequality implies by HUM a controllability property of the discrete analogue (20) of the control system (12). We seek for a positive constant $C > 0$, independent on Δt and Δx such that (25) holds. This will yield a family of controls that will be bounded as $\Delta t \to 0$, which constitutes a natural candidate to converge to the control of (12).

According to Theorem 2.1, the spectral gap between two consecutive eigenvalues plays a very important role in the analysis of the uniform observability inequality (25).

It is important to distinguish two cases:

• In the particular case where $\Delta t = \Delta x := h$ ($\mu = 1$) we have $\lambda_k = 2\text{sgn}(k)/h \arcsin(\sin(k\pi h/2)) = \text{sgn}(k)k\pi$. Thus, $\lambda_{k+1} - \lambda_k = \gamma = \pi$. But the condition (7),

$$|\lambda_k - \lambda_l| \leq \frac{2\pi - (\Delta t)^p}{\Delta t}, \quad p < 1/2$$

does not hold, because $\max_{k,l} |\lambda_k - \lambda_l| = (2\pi - 2\pi\Delta t)/\Delta t$. Note however, that in this particular case, due to the orthogonality properties of the family of complex discrete exponentials involved in the Fourier representation of solutions, $\sum_{n=0}^{M} e^{in\Delta t\pi(k-l)} = (M+1)\delta_{k,l}$, where $\delta_{k,l}$ is Kronecker's delta, an inequality of type (8) holds immediately and the discrete Ingham inequality is not needed.

Indeed, denoting by $m_k = (-1)^k a_k \sin(k\pi\Delta x)/\Delta x$, the energy of the solutions (21) concentrated on the extreme $x = 1$ can be written as

$$\Delta t \sum_{n=0}^{M} \left|\frac{\phi_N}{\Delta x}\right|^2 = \Delta t \sum_{n=0}^{M} \left|\sum_{k=-N}^{N} m_k e^{in\Delta t\pi k}\right|^2 \qquad (26)$$

and the total energy of the solutions is

$$E_0 = \frac{1}{2} \sum_{k=-N}^{N} |m_k|^2 \qquad (27)$$

(see [8] for more details). Then, for $T = 2$ we have

$$h \sum_{n=0}^{M} \left|\frac{\phi_N}{h}\right|^2 = 2 \sum_{k=-N}^{N} |m_k|^2,$$

and therefore

$$E_0 = \frac{1}{4}\left[h \sum_{n=0}^{M} \left|\frac{\phi_N^n}{h}\right|^2\right].$$

A similar identity holds for the continuous wave equation (14) in the minimal observability time $T = 2$. Namely

$$E = \frac{1}{4}\int_0^T |\phi_x(1,t)|^2$$

for every solution ϕ of (14), where E is the energy of the solutions $\phi = \phi(x,t)$.

• In the case when $\Delta t < \Delta x$ ($\mu < 1$), and, in particular, in the semi-discrete case ($\Delta t = 0$), the gap between two consecutive eigenfrequencies decreases at high frequencies and it is of the order of Δx when $\Delta x \to 0$. Indeed, the gap for the highest frequencies satisfies

$$|\lambda_N - \lambda_{N-1}| \leq \frac{\pi^2}{2}\left(\frac{\pi\Delta x}{4} + \frac{\pi\Delta x}{2}\right) = \frac{3\pi^3\Delta x}{8} \to 0,$$

when $\Delta x \to 0$. Therefore, the lack of spectral gap may produce the degeneracy of the observability constant.

So the uniform gap condition (6) is not satisfied and we cannot apply directly Theorem 2.1 to prove inequality (25). We need to introduce a subclass of solutions of system (21) where the high frequency components have been filtered. To do that, given $\alpha \in (0,1)$, the so-called filtering parameter, we consider the class of solutions involving the eigenvalues $\{\lambda_k\}_{k\in[-\alpha N,\alpha N]}$, $k \neq 0$:

$$\bar{\phi}^n = \sum_{k=-\alpha N, k\neq 0}^{\alpha N} a_k e^{i\lambda_k n\Delta t} \bar{\varphi}_{|k|}. \qquad (28)$$

Let us first check the gap condition. We have

$$\lambda_{k+1} - \lambda_k = \frac{\pi \cos\frac{\xi\Delta x}{2}}{\sqrt{1 - \left(\frac{\Delta t}{\Delta x}\sin\frac{\xi\Delta x}{2}\right)^2}} := \gamma_k, \qquad (29)$$

for every $k \in [-\alpha N, \alpha N]$ and for some $\xi \in [k\pi, (k+1)\pi]$. Therefore

$$\lambda_{k+1} - \lambda_k \geq \pi\cos\frac{N\alpha\pi\Delta x}{2} \geq \pi(1-\alpha).$$

Consequently, for any filtering parameter $\alpha \in (0,1)$, the gap condition (6) holds with

$$\gamma_\alpha := \min_k(\gamma_k) \geq \pi\cos\left(\frac{N\alpha\pi\Delta x}{2}\right) \geq \pi(1-\alpha). \qquad (30)$$

On the other hand

$$|\lambda_k - \lambda_l| < \frac{2\pi\alpha(1-\Delta t)}{\Delta t}. \qquad (31)$$

In view of (31), by choosing conveniently the filtering parameter α such that

$$\alpha \leq \alpha^*(\Delta t) := \frac{2\pi - (\Delta t)^p}{2\pi(1-\Delta t)}, \qquad (32)$$

with $0 \leq p < 1/2$, hypothesis (7) of Theorem 2.1 is verified.

In this way, for every $0 < \alpha \leq \alpha^*(\Delta t)$, the truncated sequence $\{\lambda_k\}_{|k|\leq N\alpha}$ verifies the hypotheses (6) and (7) of Theorem 2.1 with the spectral gap given by (30).

Note that $\alpha^*(\Delta t) \nearrow 1$ as $\Delta t \to 0$. Thus, the filtering parameter α may be chosen arbitrarily in the interval $\alpha \in (0,1)$.

The energy of the solutions (23) of the discrete system (21), concentred on $x = 1$ is given by (26) and the total energy (22) of the solutions is

$$E_0 = \frac{1}{2}\sum_k |\ m_k\ |^2 \frac{1}{\cos^2\frac{k\pi\Delta x}{2}}.$$

where $m_k = \sin(Nk\pi\Delta x)/\Delta x$. For all $|k| \leq \alpha N$ we have $\cos(\alpha\pi/2) \leq \cos(\alpha N\pi\Delta x/2) \leq \cos(k\pi\Delta x/2) \leq 1$ and , in this case,

$$\frac{1}{2}\sum_k |\ m_k\ |^2 \leq E_0 \leq \frac{1}{2\cos^2\frac{\alpha\pi}{2}}\sum_k |\ m_k\ |^2\ . \qquad (33)$$

Applying Theorem 2.1 and the Fourier representation (28) of the solutions we obtain that, for all $T > 2\pi/\gamma_\alpha + \epsilon(\Delta t)$, for every α as in (32), by (33), the following inequalities hold

$$2\cos^2\frac{\alpha\pi}{2}C_1(\Delta t, T, \gamma_\alpha)E_0 \le \Delta t\sum_{n=0}^{M}\left|\frac{\phi_N}{\Delta x}\right|^2, \qquad (34)$$

$$\Delta t\sum_{n=0}^{M}\left|\frac{\phi_N}{\Delta x}\right|^2 \le 2C_2(\Delta t, T, \gamma_\alpha)E_0, \qquad (35)$$

with $C_j(\Delta t, T, \gamma_\alpha)$, $j = 1, 2$, defined by relations (9), for every truncated solutions (28) of system (21) belonging to the class

$$\mathcal{C}_\alpha(\Delta x) = \{\bar{\phi}^n = \sum_{k=-\alpha N, k\neq 0}^{\alpha N} a_k e^{i\lambda_k n\Delta t}\bar{\varphi}_{|k|}\}, \qquad (36)$$

where $\bar{\phi}^n = (\phi_1^n, ..., \phi_N^n)$.

Remark 3.1: When the filtering parameter α is equal to 1, that is, when the sequence $\{\lambda_k\}_k$ is not truncated, inequality (34) degenerates.

As consequence of (34), it follows that the observability inequality:

$$E_0 \le \frac{1}{2\cos^2\frac{\alpha\pi}{2}C_1(T, \gamma_\alpha)}\left[\Delta t\sum_{n=0}^{M}\left|\frac{\phi_N}{\Delta x}\right|^2\right] \qquad (37)$$

holds uniformly for every solution of (21) in the class $\mathcal{C}_\alpha(h)$ as $(\Delta t, \Delta x) \to (0, 0)$ for any $T > T(\alpha) = 2\pi/\gamma_\alpha + \epsilon(\Delta t)$, with $C_1(T, \gamma_\alpha)$ given by (3). Observe that the gap γ_α (respectively the minimal time $2\pi/\gamma_\alpha$) tends to π (respectively to 2) when $\alpha \searrow 0^+$ while it converges to zero (respectively to infinity) when $\alpha \nearrow 1^-$. This coincides with the predictions one may deduce from the analysis of the dispersion diagram of the numerical scheme ([12]) as we shall see in the next section.

This allows to recover the uniform observability of the original system (14) as the limit when $(\Delta t, \Delta x) \to (0, 0)$ of the observability of the solutions of discrete system (21) belonginging to class (36) by means of Fourier filtering.

In practice, it is also possible to fix the filtering parameter $\alpha \in (0, 1)$. Then, the truncated sequence $\{\lambda_k\}_{|k|\le N\alpha}$ verifies the condition (6) with the spectral gap given by (30). In this way, $\{\lambda_k\}_{|k|\le N\alpha}$ also verifies the hypothesis (7) of Theorem 2.1. Therefore, we have an uniform inequality in the filtered class (36) for every $T > 2\pi/\gamma_\alpha$. Indeed, if $\alpha < 1$, (32) is satisfied and $\varepsilon(\Delta t) \to 0$. Consequently, every $T > 2\pi/\gamma_\alpha$ satisfies $T > 2\pi/\gamma_\alpha + \varepsilon(\Delta t)$, for Δt small enough.

The uniform observability inequality (34) implies uniform controllability results for the projection (over the subspace of unfiltered Fourier components) of solutions of the dual controlled system (20). In the limit as $\Delta t \to 0$ one may recover the sharp controllability results of the wave equation. This problem was studied in the particular case $\Delta t = \Delta x$ in [8]. We refer to [8] for the details of the proof of convergence of controls. But, as mentioned above, for this particular one, the discrete Ingham's inequality is not needed.

The usual centered finite-difference approximation of the wave equation we have considered here is only a simple example in which the discrete Ingham's theorem can be applied, together with some filtering mechanism, to get uniform observability inequalities.

IV. Discrete Ingham inequalities and dispersion diagrams

In this section we discuss the results obtained applying discrete Ingham inequalities in connection with the dispersion diagrams of the equations under consideration. We also discuss the optimality of these results. First of all, we introduce and recall some classical concepts and notations.

Any time-dependent scalar, linear partial differential equation with constant coefficients admits plane wave solutions

$$\phi(x, t) = e^{i(\omega t - \xi x)}, \quad \xi \in \mathbb{R},\ \omega \in \mathbb{C}, \qquad (38)$$

where ξ is the *wave number* and ω is the *frequency*. The relationship

$$\omega = \omega(\xi) \qquad (39)$$

is known as the *dispersion relation* for the equation. By dispersion one understands the property of a dynamical continuous or discrete (in time) system to propagate, with different velocities, the components of the solution.

Any individual *'monochromatic wave'* (involving only one Fourier component) of (38) moves at the *phase velocity*

$$c(\xi, \omega) = \frac{\omega(\xi)}{\xi}. \qquad (40)$$

When one superimposes two waves with nearby propagation velocities, there appear wave packets which can propagate with different velocities. The energy of wave packets propagates at the so-called *group velocity*

$$C(\xi, \omega) = \frac{d\omega(\xi)}{d\xi}. \qquad (41)$$

In general, the dispersion relation for a partial differential equation is a polynomial relation between ξ and ω, while a discrete model amounts to a trigonometric approximation.

• **Continuous problem.** For the continuous wave equation (14) we have $\omega(\xi) = \xi$ and therefore $c(\xi) = C(\xi) = 1$.

• **Discrete problem.** The same analysis can be developed for the fully discrete scheme. Considering numerical plane wave

$$\phi_j^n = e^{i(\omega n\Delta t - \xi j\Delta x)},$$

one obtains the dispersion relation

$$\omega(\xi) = \frac{2}{\Delta t}\arcsin\left(\frac{\Delta t}{\Delta x}\sin\frac{\xi\Delta x}{2}\right). \qquad (42)$$

This dispersion relation is $2\pi/\Delta x$-periodic in ξ and $2\pi/\Delta t$-periodic in ω.

◦ When $\Delta t = \Delta x$ we obtain

$$\omega(\xi) = \xi. \qquad (43)$$

This case is particularly interesting because the dispersion relation reduces to (43), the same for the continuous wave equation. In this case, the discrete waves propagate at a constant velocity identically equal to one, like in the continuous case. Therefore $c(\xi,\omega) = C(\xi,\omega) = 1$.

∘ In the general case, when $\Delta t < \Delta x$, i.e. $\mu < 1$, the phase velocity is given by

$$c(\xi,\omega) = \frac{2}{\xi\Delta t}\arcsin\left(\frac{\Delta t}{\Delta x}\sin\frac{\xi\Delta x}{2}\right) \tag{44}$$

and the group velocity is

$$C(\xi,\omega) = \frac{d\omega(\xi)}{d\xi} = \frac{\cos\frac{\xi\Delta x}{2}}{\sqrt{1-\left(\frac{\Delta t}{\Delta x}\sin\frac{\xi\Delta x}{2}\right)^2}}. \tag{45}$$

For $\Delta t = 0$ the phase velocity and group velocity obtained in (44) and (45), which depend on ξ, coincide with that of the semi-discrete case, respectively, as expected.

Note that, as $\Delta x \to 0$, for all ξ we have

$$C(\xi,\omega) \leq \frac{\cos\frac{\xi\Delta x}{2}}{\sqrt{1-\left(\frac{\Delta t}{\Delta x}\right)^2}} \to 0$$

when $\xi = \pi/\Delta x$.

One can deduce some interesting conclusions about the property of observability in view of the expressions above of the group velocity.

In fact, the group velocity is the derivative of the eigenfrequencies λ_k and the spectral gap is, as we have seen, $\lambda_{k+1} - \lambda_k$. Both magnitudes are similar, and they become closer as $\Delta x \to 0$.

According to Theorem 2.1, the uniform gap between two consecutive eigenvalues is a sufficient (and actually also necessary) property for uniform (with respect to Δx and Δt) observability inequalities.

Thus, to efficiently observe at the point $x = 1$ a wave packet (or an initial disturbance concentrated near the extreme $x = 1$) that moves to the left (in the space variable) as t increases, starting at a point sufficiently close to $x = 1$, the time needed is

$$T \geq \frac{2}{\min\limits_{\xi}\{C(\xi,\omega)\}}. \tag{46}$$

This is the time that the wave packet needs to, after bouncing at the left extreme $x = 0$, reach the point $x = 1$.

In the continuous case, (46) reduces to the well-known condition for observability $T \geq 2$ and it is uniform for all the frequencies. According to the Ingham's theorem (2), the uniform gap ($\gamma = \pi$) between two consecutive eigenvalues is a sufficient (and actually also necessary) property that leads to uniform observability inequalities.

For the fully discrete problem (21) the time needed for observation is

$$T \geq \max_{\xi}\frac{2\sqrt{1-\left(\frac{\Delta t}{\Delta x}\sin\frac{\xi\Delta x}{2}\right)^2}}{\cos\frac{\xi\Delta x}{2}}. \tag{47}$$

Passing to the limit in (47) with $\Delta t \to 0$ for fixed Δx, one obtains the same time as in the semi-discrete case. Also, in this case, the observation time grows with the high frequencies, except for the case when $\Delta t = \Delta x$, where the time obtained in the previous section, using the orthogonality of the time exponentials, is

$$T = 2 = \frac{2\pi}{\gamma}$$

and it coincides with the observation time given by the group velocity (47).

For $\mu = \Delta t/\Delta x < 1$, observe that, the condition (47) is reduced to $T \geq 2/\min(\gamma_k)$, with (γ_k) given by (29). When the sequence of eigenvalues has not an uniform gap the observability time (47) tends to infinity. Therefore, as in the semi-discrete case, these facts confirm that a suitable filtering of the spurious numerical high frequencies is necessary. The filtering parameter α may be chosen arbitrarily in the interval $\alpha \in (0,1)$.

Therefore, the consequences of this fact are:

- In accordance with the analysis of the group velocity we check that the observation time of a waves packet is of order of (46).
- Both in the semi-discrete and in the discrete cases with $\mu < 1$, the group velocity is of order of Δx for the high frequencies and this indicates the necessity of filtering to obtain a uniform observability property.
- Ingham's inequality (2) and its discrete version that we have presented in this paper are the analytical tool to do this analysis rigorously. As the spectral gap and the group velocity are of the same order, imposing the gap condition in Ingham's inequality corresponds to filtering as suggested by the analysis of the group velocity.

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