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Electric Station Keeping of Geostationary Satellites: a Differential Inclusion Approach

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Abstract— **The aim of this paper is to consider the modelling and control issues arising in the design of a station keeping system for geostationary satellites based on electric propulsion. In particular, a linear time-varying model for the dynamics of a geostationary satellite affected by perturbations is derived and the longitude and latitude station keeping problem is then formulated as a constrained linear quadratic optimal control problem. A direct method based on the so-called differential inclusion approach is then proposed. Simulation results showing the feasibility of the control task on a spacecraft equipped with electric thrusters are also presented and discussed.**

I. INTRODUCTION

The operation of geostationary satellites requires that their latitude and longitude remain confined during the whole spacecraft life [1]. To this purpose, a suitable station keeping strategy is implemented, the objectives of which are the set of manoeuvres that have to be executed in order to thwart the effects of natural perturbing forces affecting the spacecraft position. The strategy is decided by predicting the changes of the orbital parameters on the basis of models for the spacecraft dynamics which take into account only the main natural perturbing forces: the luni-solar attraction force, the solar radiation pressure and the Earth gravitational force. Nowadays, in order to achieve the objectives of a station keeping strategy most geostationary satellites are equipped with chemical propulsion systems: in order to compensate changes in the orbit parameters, chemical thrusters are typically fired once every two weeks during a time interval T_{on} of few tens of minutes, providing forces of some tens of Newton [2]. Given the small ratio between T_{on} and the geostationary orbital period, chemical thrusts can be considered with good approximation as impulsive. It thus makes sense to define station keeping strategy excluding from the dynamics equations the non conservative forces (i.e., the thrusters forces).

More recently, however, the use of electric propulsion systems is being considered as a viable alternative to the classical chemical actuators and is rapidly becoming the baseline on new telecom satellite platforms. Compared to chemical technology, this type of propulsion allows for significant improvements in the overall platform performances both in term of mass and/or lifetime. This is achieved thanks to the increase in specific impulse of a factor between 5 and 10 which makes it possible to reduce by the same factor the propellant mass needed for station keeping throughout the life of the satellite.

Replacing chemical thrusters with electric ones, however, is not without implications from the control point of view. Since electric thrusters can only provide a very low thrust level (of the order of milliNewtons), in order to achieve the same station keeping objectives that would be given for chemical propulsion it is necessary to fire the electric thrusters for some hours every day. It is then important to re-think the control strategy as a continuous process to be optimised.

Therefore, starting from the derivation of a complete nonlinear model for the dynamics of a geostationary satellite, the goal of this paper is to solve the station keeping problem (formulated as a constrained optimal control problem) by a direct method based on the differential inclusion approach [3].

The paper is organized as follows. In Section II the nonlinear model for the orbit dynamics of a geostationary satellite is described and a time-varying linearised model is derived. The station keeping problem by longitude and latitude control is formulated as a constrained optimal control problem in Section III, while in Section IV the linear model is advisably handled in order to apply the differential inclusion approach for the solution of the problem. Finally, in Section V some simulation results obtained by testing the proposed approach on a spacecraft equipped with electric thrusters are presented and discussed.

II. MATHEMATICAL MODEL

A. Coordinate Frames and Orbit Parameters

For the purpose of the present analysis, the following reference systems are adopted.

- $\vec{X}\vec{Y}\vec{Z}$, Earth Centered Inertial (ECI) coordinate frame. The $\vec{Z}$ axis is the axis of rotation of the Earth. The $\vec{X}$ axis coincides with the line of intersection of the Earth's equatorial plane and of the Earth's orbit around the Sun. The $\vec{Y}$ axis completes a right-handed orthogonal coordinate system.
- $\vec{R}\vec{T}\vec{N}$ coordinate frame. The $\vec{R}\vec{T}\vec{N}$ frame is centered in the satellite center of mass (see Fig. 1). $\vec{R}$ is the unit vector along the radial vector direction, $\vec{T}$ is the unit

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vector perpendicular to $\vec{R}$ in the orbit plane and with direction toward the satellite movement, $\vec{N}$ is the unit vector normal to the orbit plane with direction of the satellite angular momentum. In the following, a generic acceleration vector $\vec{A}$ induced from a propulsive force acting on the spacecraft will be expressed as

$$\vec{A} = u_R \vec{R} + u_T \vec{T} + u_N \vec{N} \tag{1}$$

where u_R, u_T and u_N are the acceleration components along the radial, tangential and normal directions.

The position and velocity of any satellite in space can be established by means of the classical six orbital parameters (see, e.g., [4], [5]). The satellite orbit plane is defined via the inclination i and the right ascension Ω, while the satellite trajectory on the orbit plane is defined by the semi-major axis a, the eccentricity e and the argument of perigee ω. Finally, the position of the satellite along its orbit is given by the mean anomaly $M = n(t - t_p)$, where t_p is the passage time from the perigee and $n = \sqrt{\mu/a^3}$ is the mean motion, function of the Earth's gravitational coefficient μ and of the semi-major axis.

Geostationary orbits, however, are almost circular, so the eccentricity e and the inclination i are both very small and the right ascension of the ascending node Ω and the argument of perigee ω are not conveniently defined. Therefore, it is frequently useful to operate with the so-called equinoctial parameters (see Fig. 1) [5]. The satellite orbit plane is defined thanks to the components of the inclination vector $\vec{i}$ (with modulus $\tan(i/2)$) directed along the line of nodes and pointing towards the ascending node

$$i_y = \tan(i/2)\sin\Omega \tag{2}$$

$$i_x = \tan(i/2)\cos\Omega. \tag{3}$$

The satellite trajectory on its orbit is defined thanks to the semi-major axis a and, supposing the parameters Ω and ω in the same plane, to the components of the eccentricity vector $\vec{e}$ directed along the line of apsis and pointing towards the perigee

$$e_y = e\sin(\omega + \Omega) \tag{4}$$

$$e_x = e\cos(\omega + \Omega). \tag{5}$$

Finally, the position of the satellite along its orbit is represented by the mean longitude

$$l = \omega + \Omega + M - \Theta, \tag{6}$$

where Θ is the Greenwich sidereal angle (i.e., the angle in the equatorial plane between the $\vec{X}$ axis and the Greenwich meridian).

B. Dynamics of a Geostationary Satellite

The motion of a geostationary satellite can be described by the rate of change of the equinoctial orbital parameters under the influence of the forces acting on the satellite. The geostationary dynamics results in the following nonlinear time-varying system

$$\dot{\mathrm{x}}(t) = f_K(\mathrm{x}(t)) + f_L(t, \mathrm{x}(t)) + f_G(t, \mathrm{x}(t))\mathrm{u}(t) \tag{7}$$

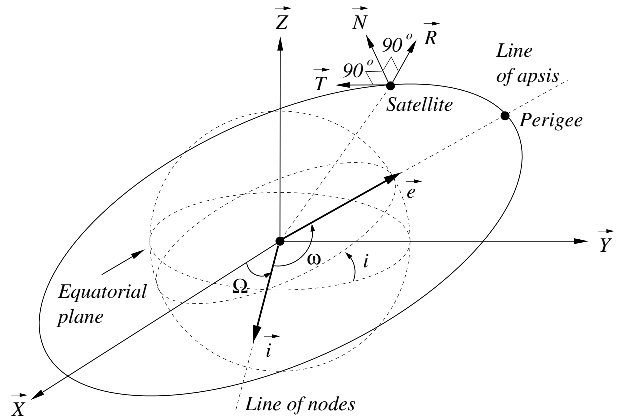

Fig. 1. Coordinate frames and orbit parameters.

where

$$\mathrm{x} = \begin{bmatrix} a & l & e_y & e_x & i_y & i_x \end{bmatrix}^T \tag{8}$$

$$\mathrm{u} = \begin{bmatrix} u_R & u_T & u_N \end{bmatrix}^T \tag{9}$$

and the functions f_K, f_L and f_G are the variational contributions to the equinoctial parameters coming respectively from the Kepler's, Lagrange's and Gauss planetary equations [6], [7]. f_K describes the satellite motion under the effect of the gravitational attraction of the Earth considered with spherical shape and a homogenous mass distribution. f_L takes into account the effect of the natural perturbing forces. More precisely, this contribution depends on the derivative of the sum of the main perturbing potentials [4], [8], [9]: the luni-solar attraction potential, the solar radiation pressure pseudo potential and the part of the Earth's gravity potential coming from the asymmetric mass repartition. Note that f_L is an explicit function of time because the perturbing potentials depend on the Sun's and Moon's ephemerides. The f_G contribution, on the other hand, is given by the accelerations provided by on-board thrusters.

If no perturbing forces act on the geostationary satellite, its motion is synchronized with the Earth's rotational motion, with constant angular velocity ω_E. The satellite maintains the mean station longitude l_s and its orbit has a semi-major axis a_{sk} making the mean motion n equal to the angular velocity of the Earth. Assuming that the functions f_K, f_L and f_G are continuously differentiable and expanding the right-hand side of (7) into its Taylor series up to the first order around the nominal operating point

$$\mathrm{x}_0 = \begin{bmatrix} a_{sk} & l_s & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{10}$$

$$\mathrm{u}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \tag{11}$$

we obtain the time-varying linear model

$$\dot{x}(t) = A(t)x(t) + D(t) + B(t)u(t) \tag{12}$$

where $x = \mathrm{x} - \mathrm{x}_0$ and $u = \mathrm{u} - \mathrm{u}_0$. The $A(t)$ and $D(t)$ matrices turn out to be time-varying because of the presence of periodic terms with periods equal to multiples of the periodicities of the Earth's, Sun's and Moon's motion relative to the satellite: one day, one year, one month respectively. The presence of the drift term $D(t)$ in equation (12) follows from the fact that the linearization of the nonlinear system (7) is performed in a point which is not an equilibrium point for this system. In fact $f_K(\mathrm{x}_0) = f_G(\mathrm{x}_0, \mathrm{u}_0) = 0$ but $f_L(t, \mathrm{x}_0) = D(t)$. Similarly, $B(t)$ is a periodic function with period equal to one day because it depends on sine and cosine of the Greenwich sidereal angle Θ

$$B(t) = \sqrt{\frac{a_{sk}}{\mu}} \begin{bmatrix} 0 & 2a_{sk} & 0 \\ -2 & 0 & 0 \\ -\cos K_0(t) & 2\sin K_0(t) & 0 \\ \sin K_0(t) & 2\cos K_0(t) & 0 \\ 0 & 0 & \frac{1}{2}\sin K_0(t) \\ 0 & 0 & \frac{1}{2}\cos K_0(t) \end{bmatrix} \quad (13)$$

where $K_0(t) = l_s + \Theta(t)$ and μ is the Earth's gravitational coefficient (see, e.g., [4]).

Finally, since geostationary orbits are characterized by very small values of eccentricity and inclination, the longitude λ and latitude φ can be defined with good approximation as [1]

$$\lambda = l + 2e_x \sin(l + \Theta) - 2e_y \cos(l + \Theta), \quad (14)$$
$$\varphi = 2i_x \sin(\lambda + \Theta) - 2i_y \cos(\lambda + \Theta). \quad (15)$$

Expanding the right-hand side of the above equations into its Taylor series up to the first order around x_0, we get the output equations of the linear time-varying system (12)

$$y(t) = C(t)x(t) \quad (16)$$

where

$$y = \begin{bmatrix} \lambda - l_s & \varphi \end{bmatrix}^T \quad (17)$$

and

$$C(t) = \begin{bmatrix} C_\lambda(t) & 0_{1\times 2} \\ 0_{1\times 4} & C_\varphi(t) \end{bmatrix} \quad (18)$$

with

$$C_\lambda(t) = \begin{bmatrix} 0 & 1 & -2\cos K_0(t) & 2\sin K_0(t) \end{bmatrix} \quad (19)$$
$$C_\varphi(t) = \begin{bmatrix} -2\cos K_0(t) & 2\sin K_0(t) \end{bmatrix}. \quad (20)$$

III. STATION KEEPING PROBLEM FORMULATION

The station keeping criteria for a geostationary satellite are normally expressed by imposing that the spacecraft longitude and latitude shall be confined in a rectangular box of the (λ, φ) plane centered in $(l_s, 0)$ and whose sides are equal to $2\lambda_{max}$ and $2\varphi_{max}$ (see Fig. 2).

With reference to the linear dynamics, the station keeping problem can be formulated as a constrained linear quadratic continuous-time optimal control problem: given the linear model (12)-(16) with initial condition $x(t_i) = x_i$ at the initial time t_i, the problem is to find the control sequence $u^{opt}(t)$ over a finite time horizon $t_f - t_i$ that minimizes the criterion

$$J_{ct} = \frac{1}{2}\int_{t_i}^{t_f} \left[y^T(t)Q(t)y(t) + u^T(t)R(t)u(t) \right] dt \quad (21)$$

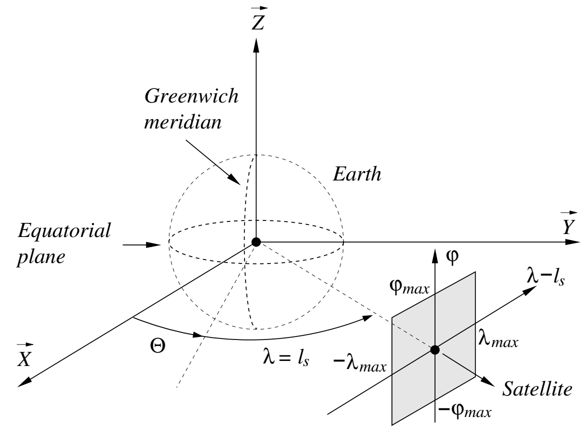

Fig. 2. The gray zone in the (λ, φ) plane is the permitted one.

subject to the conditions[1]

$$-y_{max} \leq y \leq +y_{max} \quad (22)$$

where

$$y_{max} = \begin{bmatrix} \lambda_{max} & \varphi_{max} \end{bmatrix}^T. \quad (23)$$

In addition, the control variables must satisfy technological constraints depending on the type and on the configuration of the considered thrusters. In general, with n_F thrusters, we can write the control vector u as

$$u = \Gamma F \quad (24)$$

where Γ is a 3 by n_F matrix depending on the thrusters configuration and on the inverse of the spacecraft mass and F is the vector of the thrusters time-varying forces. Constraints on F such as

$$F_{min} \leq F \leq F_{max} \quad (25)$$

can be converted into constraints on the control variable

$$F_{min} \leq \Gamma^T \left(\Gamma\Gamma^T\right)^{-1} u \leq F_{max}. \quad (26)$$

IV. DIFFERENTIAL INCLUSION APPROACH

A number of different techniques is available for the numerical solution of the problem formulated in the previous Section. In this paper we will consider the so-called direct methods for the solution of continuous optimal control problems, which can be defined as the approaches that do not explicitly employ the necessary conditions in the derivation of the solution and therefore do not lead to the classical two-point-boundary-value-problems, which are well known to be numerically critical. On the other hand, the idea behind direct methods is to discretize the control time history and/or the state variable time history [10], [11], [3]. In particular, if the problem is such that the control variables can be written explicitly as functions of the states and their time derivatives, then the control variables can be completely eliminated from the optimisation problem. This approach is know in the literature as the differential inclusion one [3]: elimination

[1] Inequalities between vectors should be interpreted componentwise, i.e., $[a_1 \quad a_2]^T \leq [b_1 \quad b_2]^T$ means $a_1 \leq b_1$ and $a_2 \leq b_2$.

of the control terms significantly reduces the dimensionality of the problem, leading to a particularly efficient formulation of the optimisation.

In the differential inclusion approach, the control inputs have to be written explicitly as functions of the state and its rate of change so that bounds on the control variables have to be translated in bounds on the attainable rates of change of the state variables. Unfortunately this cannot be done directly for the system (12), since the presence of time-varying elements in matrix $B(t)$ makes it impossible to write the control variables explicitly without introducing singularities. However, it can be shown that the linear model (12) can be written in a different form, characterized by a constant B matrix. To this purpose, we introduce the Lyapunov transformation (see [12]) in the state space defined as

$$\widetilde{x}(t) = W(t)x(t) \tag{27}$$

where

$$W(t) = \begin{bmatrix} W_\lambda(t) & 0_{4\times 2} \\ 0_{2\times 4} & W_\varphi(t) \end{bmatrix} \tag{28}$$

and

$$W_\lambda(t) = \sqrt{\frac{\mu}{a_{sk}}} \begin{bmatrix} 0 & -1 & 2\cos K_0(t) & -2\sin K_0(t) \\ -\frac{2}{a_{sk}} & 0 & 2\sin K_0(t) & 2\cos K_0(t) \\ 0 & -\frac{3}{2} & 2\cos K_0(t) & -2\sin K_0(t) \\ -\frac{3}{2a_{sk}} & 0 & 2\sin K_0(t) & 2\cos K_0(t) \end{bmatrix},$$

$$W_\varphi(t) = \sqrt{\frac{\mu}{a_{sk}}} \begin{bmatrix} 2\cos K_0(t) & -2\sin K_0(t) \\ 2\sin K_0(t) & 2\cos K_0(t) \end{bmatrix}.$$

Then, it is easy to verify that

$$W(t)B(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{29}$$

Furthermore, since the transformation $W(t)$ is nonsingular and differentiable for every t, equation (12) can be written in the new state variables $\widetilde{x}$ as

$$\dot{\widetilde{x}}(t) = \widetilde{A}(t)\widetilde{x}(t) + \widetilde{D}(t) + \widetilde{B}u(t) \tag{30}$$

where

$$\widetilde{A}(t) = \dot{W}(t)W^{-1}(t) + W(t)A(t)W^{-1}(t), \tag{31}$$

$$\widetilde{D}(t) = W(t)D(t), \tag{32}$$

and $\widetilde{B}$ is given by

$$\widetilde{B} = W(t)B(t). \tag{33}$$

At this point, we can write the control variable of the linear dynamics as a function of the state variable and its rate of change. Note that system (30) can be equivalently written as

$$u(t) = S_1\dot{\widetilde{x}}(t) - S_1\widetilde{A}(t)\widetilde{x}(t) - S_1\widetilde{D}(t) \tag{34}$$

$$S_2\dot{\widetilde{x}}(t) = S_2\widetilde{A}(t)\widetilde{x}(t) + S_2\widetilde{D}(t) \tag{35}$$

where S_1 and S_2 are defined as

$$S_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Similarly, the output equation given by (16) becomes

$$y(t) = \widetilde{C}(t)\widetilde{x}(t) \tag{36}$$

where

$$\widetilde{C}(t) = C(t)W^{-1}(t). \tag{37}$$

The station keeping problem formulated in the previous Section as a constrained continuous-time optimal control problem can be translated in a quadratic programming problem with constraints only on the state variables.

Over a finite time horizon $t_f - t_i$ discretized in N intervals of length h each, the optimal control $u^{opt}(t)$ is taken constant in the $(k+1)-th$ interval and equal to

$$u_k^{opt} \quad \text{with} \quad k = 0, \ldots, N-1. \tag{38}$$

The sequence of the u_k^{opt} samples is a function of the solution of the following quadratic programming problem.

Assuming the initial condition $\widetilde{x}_0 = \widetilde{x}(t_i) = W(t_i)x(t_i)$ and

$$\widetilde{x}_k = \widetilde{x}(t_k), \quad \widetilde{Q}_k = \widetilde{Q}(t_k), \quad \widetilde{C}_k = \widetilde{C}(t_k) \tag{39}$$

$$\widetilde{R}_k = \widetilde{R}(\bar{t}_k), \quad \widetilde{A}_k = \widetilde{A}(\bar{t}_k), \quad \widetilde{D}_k = \widetilde{D}(\bar{t}_k) \tag{40}$$

$$\text{where} \quad t_k = kh \quad \text{and} \quad \bar{t}_k = t_k + h/2,$$

the problem consists in finding the optimal sequence $\widetilde{x}_k^{opt}$ that minimizes the criterion

$$J_{dt} = \frac{1}{2}\sum_{k=1}^{N} y_k^T Q_k y_k + \frac{1}{2}\sum_{k=0}^{N-1} u_k^T R_k u_k \quad \text{with} \tag{41}$$

$$y_k = \widetilde{C}_k\widetilde{x}_k \quad \text{and} \tag{42}$$

$$u_k = S_1\frac{\widetilde{x}_{k+1} - \widetilde{x}_k}{h} - S_1\widetilde{A}_k\frac{\widetilde{x}_{k+1} + \widetilde{x}_k}{2} - S_1\widetilde{D}_k, \tag{43}$$

subject

- to the inequality constraints (35) on the output variable $y(t)$, discretized for $k = 1, \ldots, N$

$$-y_{max} \leq y_k \leq +y_{max}, \tag{44}$$

- to the inequality constraints (26) on the control variable $u(t)$, discretized for $k = 0, \ldots, N-1$

$$F_{min} \leq \Gamma^T \left(\Gamma\Gamma^T\right)^{-1} u_k \leq F_{max}, \tag{45}$$

- and to equality constraints (35) on the auxiliary state variable $\widetilde{x}(t)$, discretized for $k = 0, \ldots, N-1$

$$S_2\frac{\widetilde{x}_{k+1} - \widetilde{x}_k}{h} = S_2\widetilde{A}_k\frac{\widetilde{x}_{k+1} + \widetilde{x}_k}{2} + S_2\widetilde{D}_k. \tag{46}$$

The samples y_k and u_k are functions of $\widetilde{x}_k$ according to equations (42) and (43). The optimal control sequence u_k^{opt} is obtained replacing the optimal solution $\widetilde{x}_k^{opt}$ in (43).

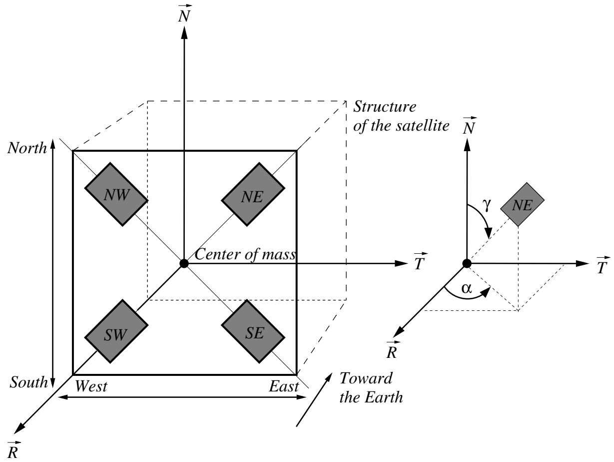

Fig. 3. Electric propulsors configuration.

V. SIMULATION STUDY

The spacecraft considered in this study is equipped with four electric thrusters mounted on its anti-nadir face. The directions of thrust pass through the satellite center of mass. This propulsor configuration has been proposed in [13]. As Fig. 3 shows, the thrust lines of the north-west (NW) and north-east (NE) thrusters and the south-west (SW) and south-east (SE) thrusters form a cant angle γ with respect to the satellite north-south axis in a northern and southern direction respectively. The thrusters are laterally separated and slewed by a slew angle α with respect to the north-south axis. With this configuration the force vector in (24) becomes

$$F = \begin{bmatrix} F_{NW} & F_{NE} & F_{SW} & F_{SE} \end{bmatrix}^T. \tag{47}$$

Matrix Γ in (24) becomes

$$\Gamma = \frac{1}{m} \begin{bmatrix} -s_\gamma c_\alpha & -s_\gamma c_\alpha & -s_\gamma c_\alpha & -s_\gamma c_\alpha \\ +s_\gamma s_\alpha & -s_\gamma s_\alpha & +s_\gamma s_\alpha & -s_\gamma s_\alpha \\ -c_\gamma & -c_\gamma & +c_\gamma & +c_\gamma \end{bmatrix} \tag{48}$$

where $s_\gamma = \sin(\gamma)$, $c_\gamma = \cos(\gamma)$, $s_\alpha = \sin(\alpha)$, $c_\alpha = \cos(\alpha)$ and m is the spacecraft mass. In particular, in this paper we consider a spacecraft with mass $m = 4500$ kg at the beginning of the station keeping mission, with cant and slew angles $\gamma = 50$ deg and $\alpha = 15$ deg realistic from a technological point of view. The considered thrusters are characterized by a maximum force modulus $F_{max} = 0.17$ N and a specific impulse $I_{sp} = 3800$ s. These values are typical of ionic propulsion, as, e.g., in the Boeing HS702 satellite.

The goal is to determine the set of manoeuvres (objectives of the station keeping strategy) that have to be executed in order to keep the satellite in a latitude and longitude box centered at the station longitude $l_s = 10$ deg with half dead-bands equal to $\lambda_{max} = 0.01$ deg and $\varphi_{max} = 0.01$ deg. These objectives are obtained over a time horizon of 28 days, starting from the date of the beginning of the mission: January 1st 2010 at 12 pm.

We proceed according to the following steps.

Step 1. We formulate the station keeping problem (see Section III) over a finite time horizon $t_f - t_i = 1$ day with $t_i = 0$. In the quadratic criterion (21), the weighting matrices are equal to identity matrices: $Q = I_{2\times2}$ and $R = I_{3\times3}$. The initial conditions of the linear model are given by

$$x(t_i) = \begin{bmatrix} a_s & l_s & 0 & 0 & i_{y_i} & i_{x_i} \end{bmatrix}^T \tag{49}$$

where a_s is the synchronous semi-major axis depending on the station longitude l_s [2], [8]. For $l_s = 10$ deg we have $a_s = 42166.279$ km. The inclination components i_{y_i} and i_{x_i} are chosen such that $i_{y_i}/i_{x_i} = \tan\Omega$ with $\Omega = 82$ deg and such that $\varphi(0) = 0.8\varphi_{max}$ in equation (16). In addition to the inequality constraints (22) and (26), we consider the following equality constraint on the final value of the output variable: $y(t_f) = y(0)$. Precisely, we impose $\lambda(t_f) = \lambda(0)$ via these equality constraints on the mean longitude and the eccentricity components:

$$l(t_f) = l_s, \quad e_y(t_f) = 0 \quad \text{and} \quad e_x(t_f) = 0. \tag{50}$$

We impose $\varphi(t_f) = \varphi(0)$ via these equality constraints on the inclination components:

$$i_y(t_f) = i_{y_i} \quad \text{and} \quad i_x(t_f) = i_{x_i}. \tag{51}$$

Step 2. We solve the above described station keeping problem by means of the differential inclusion approach (see Section IV) with a discretization step of length $h = 0.01$ day. We integrate the nonlinear model (7) over the time horizon $t_f - t_i$ with $\mathrm{u}(t) = u^{opt}(t)$ and we obtain the final value $\mathrm{x}(t_f)$.

Step 3. We repeat the previous steps for 27 consecutive days, updating the t_i value, the initial condition $x(t_i)$ and the mass of the satellite, day after day. On the second day, for example, $t_i = 1$, $x(t_i) = \mathrm{x}(1) - \mathrm{x}_0$ and the mass m is reduced by the quantity [14]

$$\Delta m = -\frac{1}{g_0 I_{sp}} \int_0^1 (F_{NW} + F_{NE} + F_{SW} + F_{SE})\, dt \tag{52}$$

where g_0 is the normalized gravity acceleration.

Fig. 4 shows the component of the optimal control u^{opt} versus time; Fig.s 5 and 8 show the time evolution of the four propulsive forces. The controlled and uncontrolled time history of the true longitude, of the latitude and of the semi-major axis are plotted in Fig.s 6, 9 and 7.

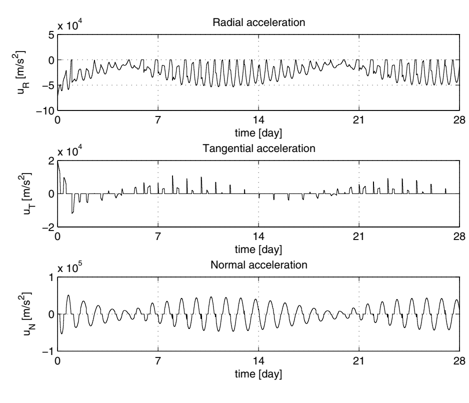

Fig. 4. Time histories of the optimal control variable components.

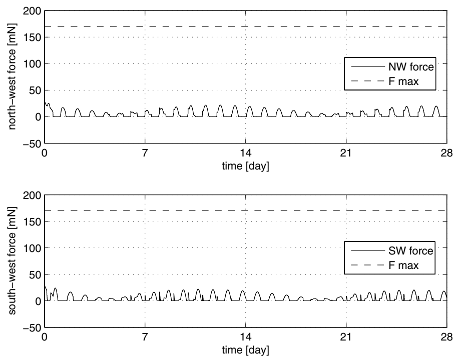

Fig. 5. Time histories of the west propulsive forces.

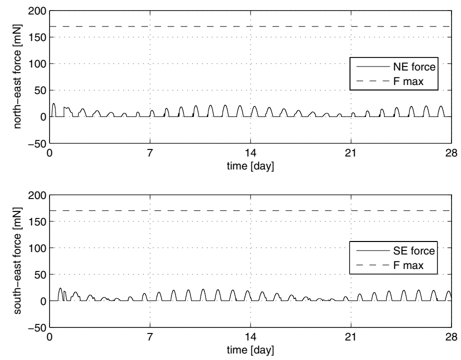

Fig. 8. Time histories of the east propulsive forces.

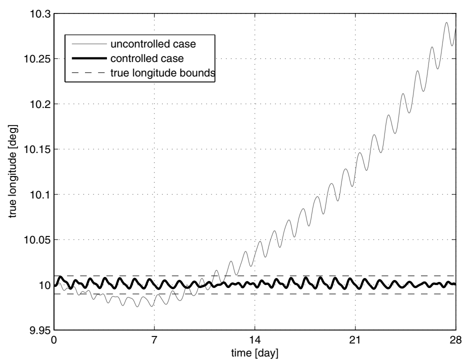

Fig. 6. Time histories of the uncontrolled and controlled true longitude.

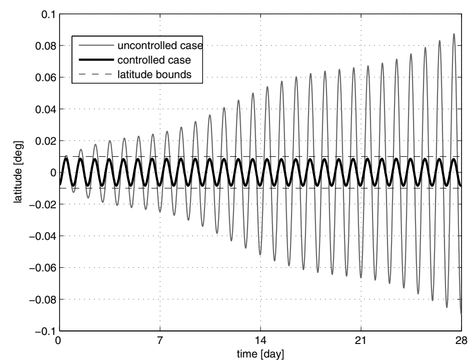

Fig. 9. Time histories of the uncontrolled and controlled latitude.

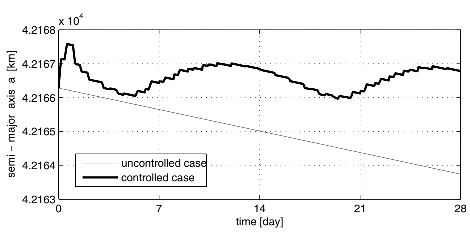

Fig. 7. Time histories of the uncontrolled and controlled semi-major axis.

VI. CONCLUSIONS

The problem of electric station keeping for geostationary satellites has been considered and a novel approach based on direct optimisation techniques of the differential inclusion type has been presented and discussed. The feasibility of the proposed technique has been validated in a detailed simulation study.

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