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A tight small gain theorem for not necessarily ISS systems

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Abstract— **A new version of the small-gain theorem is presented for nonlinear finite dimensional systems. The result provides conditions for global asymptotic stability under relaxed assumptions, in particular the two interconnected subsystems need not be Input-to-State stable in open loop.**

I. INTRODUCTION

The problem of analysing the asymptotic stability of systems composed of the interconnection of asymptotically stable systems has been widely studied in the last decades. It is nowadays clear that one of the basic tools to perform this analysis is the so-called small gain theorem. While this theorem admits a very simple formulation in the case of linear systems (see e.g. [1]), it is possible to construct several nonlinear enhancements of the small gain theorem. In particular, a small gain theorem for L_2 stable systems has been developed in the early work of Hill-Moylan [2], [3] (see also the recente monograph [12]), and a generalization of this theorem, exploiting the notion of practical L_2-stability, has been developed in [8]. Alternatively, the notion of input-to-state stability (ISS), introduced by E. Sontag [9], [10], and its generalizations (such as the notion of ISS with restriction) have been exploited to derive nonlinear small gain theorems for interconnected ISS systems [6], [5], [11]. Finally, in the recent paper [4], the notion of state dependent scaling has been used to derive a small gain theorem for a class of interconnected systems.

In this paper we provide a tight small gain theorem for not necessarily ISS interconnected systems. Similarly to the results in [4] we do not assume that the systems are ISS. However, while therein it is necessary to compute the so-called state-dependent scaling functions, and hence a Lyapunov function for the interconnected system, and the subsystems are assumed to be ISS or integral-ISS, we are able to prove our main result with very simple arguments and tools, mainly borrowed from the theory of monotone systems, and we do not need to assume that the subsystems are ISS or integral-ISS.

It must be noted that, unlike the small gain theorem presented in [5], [6], which establishes asymptotic stability of the interconnected system and ISS of the resulting system with respect to a set of external signals, the result presented in this paper is applicable only to assess asymptotic stability of a given interconnected system and does not accomodate external signals. As a result, while the result in [5] is amenable to iterative or recursive applications, this is not the case for the proposed result.

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II. BASIC DEFINITIONS

Consider the following system of differential inequalities

$$\begin{array}{rcl} \dot V_1 & \leq & -\alpha_1(V_1)+\beta_1(V_2) \\ \dot V_2 & \leq & -\alpha_2(V_2)+\beta_2(V_1) \end{array} \tag{1}$$

where α_1 and α_2 are locally Lipschitz positive definite functions and β_1, β_2 are locally Lipschitz and class $\mathcal{K}$. Inequalities such as (1) may arise when studying stability of interconnected systems in feedback exploiting the knowledge of strictly decreasing Lyapunov functions for the individual subsystems V_1, V_2. When all the functions are of class $\mathcal{K}_\infty$, then it is well-known that the inequalities imply ISS of the individual subsystems and, under a suitable small-gain condition, global asymptotic stability of the interconnection can be derived. Our aim is to derive a tight set of conditions which allow to conclude GAS of the interconnection even when some of the α_i's need not be class $\mathcal{K}_\infty$. To this end we associate to (1) the following planar system of differential equations:

$$\begin{array}{rcl} \dot W_1 & = & -\alpha_1(W_1)+\beta_1(W_2) \doteq f_1(W_1,W_2) \\ \dot W_2 & = & -\alpha_2(W_2)+\beta_2(W_1) \doteq f_2(W_1,W_2) \end{array} \tag{2}$$

which evolves in the positive orthant $\mathbb{R}^2_{\geq 0}$ (the positive orthant is therefore a positively invariant set for (2)). Our first result is a simple observation showing that the stability property of (2) are inherited by (1).

Lemma 2.1: Assume that $W_1(0) > V_1(0)$ and $W_2(0) > V_2(0)$. Then, for all $t \in [0, T_{\max})$, (the maximal interval of definition in forward time for the solutions of (2) corresponding to $W(0) = [W_1(0), W_2(0)]$) we have:

$$W_1(t) > V_1(t) \text{ and } W_2(t) > V_2(t).$$

□

Proof: Let τ_i be defined as the $\inf\{t \in (0, T_{\max}) : W_i(t) \leq V_i(t)\}$ and $\tau = \min\{\tau_1, \tau_2\}$. We want to show that $\tau = T_{\max}$. Assume by contradiction $\tau < T_{\max}$ and without loss of generality $0 < \tau = \tau_1 \leq \tau_2$. Hence $W_1(\tau) = V_1(\tau)$ and $W_2(\tau) \geq V_2(\tau)$. Taking derivatives at time $t \in [\tau - \epsilon, \tau)$ yields for some sufficiently small $\varepsilon > 0$:

$$\begin{array}{rcl} \dot{W}_1 - \dot{V}_1 & = & -\alpha_1(W_1) + \beta_1(W_2) + \alpha_1(V_1) - \beta_1(V_2) \\ & \geq & -L_{\alpha_1}|W_1 - V_1| + \beta_1(W_2) - \beta_1(V_2) \\ & \geq & -L_{\alpha_1}(W_1 - V_1). \end{array} \tag{3}$$

where L_α is the Lipschitz constant relative to the function α. Hence $W_1(\tau) - V_1(\tau) \geq e^{-L_\alpha \epsilon}(W_1(\tau-\varepsilon) - V_1(\tau-\varepsilon)) > 0$ which contradicts the definition of τ.

■

Thanks to Lemma 2.1 we can study the stability properties of potentially high-dimensional feedback interconnections just by looking at the planar system (2). In order to formulate our main result we need to define the following closed subsets of $\mathbb{R}^2_{\geq 0}$.

$$\begin{array}{rcl} \Omega^{+-} & := & \{[w_1, w_2] \in \mathbb{R}^2_{\geq 0} : \\ & & f_1(w_1, w_2) \geq 0 \text{ and } f_2(w_1, w_2) \leq 0\} \\ \Omega^{--} & := & \{[w_1, w_2] \in \mathbb{R}^2_{\geq 0} : \\ & & f_1(w_1, w_2) \leq 0 \text{ and } f_2(w_1, w_2) \leq 0\} \\ \Omega^{-+} & := & \{[w_1, w_2] \in \mathbb{R}^2_{\geq 0} : \\ & & f_1(w_1, w_2) \leq 0 \text{ and } f_2(w_1, w_2) \geq 0\}. \end{array} \tag{4}$$

It is well known (see [5]) that if $\Omega^{-+} \cup \Omega^{--} \cup \Omega^{+-} = \mathbb{R}^2_{\geq 0}$ and the intersections $\Gamma_2 := \Omega^{-+} \cap \Omega^{--}$ and $\Gamma_1 := \Omega^{--} \cap \Omega^{+-}$ are graphs of $\mathcal{K}_\infty$ functions whose difference is again of class $\mathcal{K}_\infty$, then the resulting closed-loop system is globally asymptotically stable (as already remarked in [5] much more is actually proved, since ISS of the closed-loop system under exogenous disturbance is proved, provided that disturbances enter in an ISS way with respect to the original open loop systems). If the main concern is however just asymptotic stability of the closed-loop system, then, the requirement that Γ_1 and Γ_2 be well-separated graphs of $\mathcal{K}_\infty$ functions γ_1 and γ_2 can be remarkably relaxed, as expressed in the following statement.

Theorem 1: Consider the system (2). Assume that

$$\Omega^{-+} \cup \Omega^{--} \cup \Omega^{+-} = \mathbb{R}^2_{\geq 0}$$

and

$$\Omega^{-+} \cap \Omega^{--} \cap \Omega^{+-} = \{0\}.$$

Then provided that any one of the following conditions be satisfied:

1) there exists positive numbers $\mathcal{L}^- < \mathcal{L}^+ \leq +\infty$ so that

$$\limsup_{w \to +\infty} \beta_1^{-1}(\alpha_1(w)) = \mathcal{L}^+$$

$$\lim_{w \to \mathcal{L}^-} \beta_2^{-1}(\alpha_2(w)) = +\infty$$

2) there exists positive numbers $\mathcal{L}^- < \mathcal{L}^+ \leq +\infty$ so that

$$\limsup_{w \to +\infty} \beta_2^{-1}(\alpha_2(w)) = \mathcal{L}^+$$

$$\lim_{w \to \mathcal{L}^-} \beta_1^{-1}(\alpha_1(w)) = +\infty$$

3)

$$\lim_{w \to +\infty} \beta_1^{-1}(\alpha_1(w)) = +\infty$$

$$\lim_{w \to +\infty} \beta_2^{-1}(\alpha_2(w)) = +\infty$$

the system (2) is globally asymptotically stable at the origin (in $\mathbb{R}^2_{\geq 0}$). ■

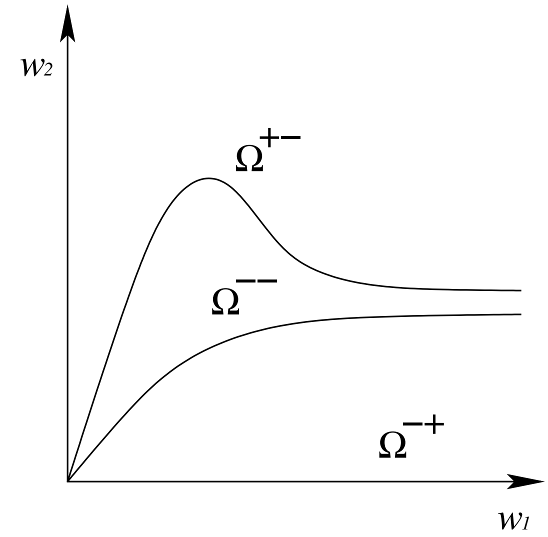

Fig. 1. Typical shapes of the Ω regions

The curves Γ_1 and Γ_2 in the (w_1, w_2)-plane may for instance look like in Fig. 1.

In order to prove the result we will need the following Lemmas.

Lemma 2.2: The region Ω^{--} is positively invariant for system (2). Moreover, if the origin is the only equilibrium of (2), then, for all $W(0)$ in Ω^{--}, we have for the corresponding solution: $\lim_{t \to +\infty} W(t) = 0$. □

Proof: Let $W = [w_1, w_2] \in \partial\Omega^{--}$. If $\dot{w}_1 = 0$ and $\dot{w}_2 = 0$, then W is an equilibrium and invariance trivially holds. If, on the other hand, $\dot{w}_1 = 0$ and $\dot{w}_2 < 0$, we may take any sequence $W_n = [w_1, w_2 - \epsilon_n]$ with $\epsilon_n \to 0$ a positive decreasing sequence. For sufficiently small ϵ_n one has $\dot{w}_2(W_n) < 0$, because of continuity, moreover

$$\begin{array}{rcl} \dot{w}_1(W_n) & = & -\alpha_1(w_1) + \beta_1(w_2 - \epsilon_n) \\ & < & -\alpha_1(w_1) + \beta_1(w_2) = 0. \end{array}$$

Thus W_n belongs to int(Ω^{--}). Hence[1]

$$[0, -1] = \lim_{n \to +\infty} (W_n - W)/\epsilon_n \in TC_W(\Omega^{--}).$$

[1] The notation $TC_W(S)$ is standard and denotes the Bouligand Tangent cone of S at W. For a set $S \subset \mathbb{R}^n$ this can be defined as follows:

$$TC_W(S) = \{\xi : \exists \xi_n \in S, \xi_n \to W, \exists\, t_n \to 0 : (\xi_n - W)/t_n \to \xi\}$$

A symmetric argument applies to the case $\dot{w}_2 = 0$. This completes the proof of invariance since we have shown that for all $W \in \partial\Omega^{--}$ we have $\dot{W} \in TC_W(\Omega^{--})$. We are only left to show convergence. By invariance of Ω^{--}, both components of W are monotone non-increasing. Since the positive orthant is invariant we have

$$\lim_{t\to+\infty} w_i(t) = \bar{w}_i \geq 0 \quad i = 1, 2,$$

viz. the limit exists and are nonnegative. By continuity then $[\bar{w}_1, \bar{w}_2]$ is an equilibrium. Hence, by unicity of equilibria in the nonnegative orthant we necessarily have $[\bar{w}_1, \bar{w}_2] = [0, 0]$.

■

We are now ready to prove our main result.

Proof: We show first the result under assumption 3). Let $W(0) \in \mathbb{R}^2_{\geq 0}$. If $W(0) \in \Omega^{--}$ there is nothing to prove because of Lemma 2.2. Assume without loss of generality $W(0) \in \Omega^{-+}$ (a symmetric argument applies to the case of $W \in \Omega^{+-}$). We will show that necessarily $W(t)$ enters Ω^{--} in finite time or it approaches the origin. In fact, assume that $W(t) \in \Omega^{-+}$ for all $t \in [0, T_{\max})$, the maximal interval of definitions of solutions. Then $w_1(t)$ is bounded and non-increasing, and therefore $w_1(t) \to \bar{w}_1 \geq 0$ as $t \to T_{\max}$. However, since $\lim_{w\to+\infty} \beta_2^{-1}(\alpha_2(w)) = +\infty$ there exists r so that $\beta_2^{-1}(\alpha_2(w)) \geq w_1(0)$ for all $w \geq r$. Hence, for all $w_2 \geq r$ we have $\dot{w}_2 = -\alpha_2(w_2) + \beta_2(w_1(t)) \leq -\alpha_2(w_2) + \beta_2(w_1(0)) \leq 0$. Since we assumed $\dot{w}_2(t) \geq 0$ for all $t \in [0, T_{\max})$, we can conclude that $w_2(t)$ is also bounded, and therefore it admits a limit $\bar{w}_2$ (and obviously $T_{\max} = +\infty$). Thus $[\bar{w}_1, \bar{w}_2]$ is an equilibrium and, by unicity of equilibria in the nonnegative orthant, $[\bar{w}_1, \bar{w}_2] = [0, 0]$.

We are only left to show the result under assumption 1). In fact a symmetric argument would apply to the case of assumption 2). Let $W(0) \in \Omega^{+-}$, (the case $W(0) \in \Omega^{-+}$ can be treated similarly to the previous paragraph). We want to show that necessarily $W(t)$ enters Ω^{--} in finite time or it approaches the origin. In fact assume that $W(t) \in \Omega^{+-}$ for all $t \geq 0$ for which the solution of (2) is defined; then $w_2(t) \leq w_2(0)$ and consequently, $w_1(t) \leq w_1(0) + t\beta_1(w_2(0))$. Hence the nonlinear system (2) is forward complete. Now, $w_2(t)$ is a non-increasing function, hence it admits a finite limit $\bar{w}_2$; on the other hand $w_1(t)$ also admits a limit, though possibly infinite. In any case, since by assumption 1) $\beta_2(\infty) < +\infty$ we have that $\dot{w}_2(t) = -\alpha_2(w_2(t)) + \beta_2(w_1(t))$ admits a finite limit as well, in particular: $\dot{w}_2(t) \to -\alpha_2(\bar{w}_2) + \beta_2(w_1(+\infty)) = 0$. We want to rule out the possibility that $w_1(+\infty) = +\infty$. If this were the case, then $\lim_{w\to\bar{w}_2} \beta_2^{-1}(\alpha_2(w)) = +\infty$ which implies, by assumption 1), $\bar{w}_2 = \mathcal{L}^-$. However, $\dot{w}_1(t) \geq 0$ for all t implies: $\liminf_{t\to+\infty} -\alpha_1(w_1(t)) + \beta_1(w_2(t)) \geq 0$. Thus, $\limsup_{t\to+\infty} \alpha_1(w_1(t)) \leq \beta_1(\bar{w}_2)$ and by monotonicity of β_1^{-1},

$$\mathcal{L}^+ = \limsup_{t\to+\infty} \beta_1^{-1}(\alpha_1(w_1(t))) \leq \bar{w}_2 = \mathcal{L}^-$$

This is clearly a contradiction; hence we can only conclude that $w_1(\infty) := \bar{w}_1$ is also bounded and therefore $[\bar{w}_1 \bar{w}_2]$ is an equilibrium, viz. it is the origin. ■

III. Is the small-gain theorem tight?

In this section we show by means of examples that the conditions in Theorem 1 cannot be relaxed. In particular we will show that if condition 1) (or symmetrically 2)) fails, then stability or instability cannot be concluded on the basis of geometric conditions involving the nullclines of system (2). The analysis of boundedness of solutions will be based on an explicit computation of the center-manifold of the system under a suitable change of coordinates centered at (∞, L). This is very similar in spirit to what done in [7].

Consider again system (2) under what we call the *no gap condition*:

$$\alpha_1(+\infty) = \beta_1(L) \quad \alpha_2(L) = \beta_2(\infty). \tag{5}$$

Let the functions α_1, α_2, β_1 and β_2 satisfy:

$$\begin{array}{rcl} \alpha_1(W_1) & = & \alpha_1(\infty) + \alpha_1'(\infty)\frac{1}{W_1} + o(1/W_1) \\ \beta_2(W_1) & = & \beta_2(\infty) + \beta_2'(\infty)\frac{1}{W_1} + o(1/W_1) \\ \alpha_2(W_2) & = & \alpha_2(L) + \alpha_2'(L)(W_2 - L) + o(|W_2 - L|) \\ \beta_1(W_2) & = & \beta_1(L) + \beta_1'(L)(W_2 - L) + o(|W_2 - L|). \end{array} \tag{6}$$

Rewrite now equations (2) in the new coordinates: $\eta_1 = 1/W_1$ and $\eta_2 = W_2 - L$. By making use of the series expansions in (6) and the no gap condition (5) one obtains:

$$\begin{array}{rcl} \dot{\eta}_1 & = & -\dfrac{\dot{W}_1}{W_1^2} = \alpha_1'(\infty)\eta_1^3 - \beta_1'(L)\eta_1^2\eta_2 + o(\eta_1^3) \\ \dot{\eta}_2 & = & -\alpha_2'(L)\eta_2 + \beta_2'(\infty)\,\eta_1. \end{array} \tag{7}$$

Notice that the set $\{(\eta_1, \eta_2), \eta_1 = 0\}$ is invariant for system (7); however, such solutions do not have physical meaning in the original coordinates and correspond to solutions starting at infinity. Since $W_2(t)$ is bounded (under our set of assumptions), the presence of unbounded solutions for the initial subsystem is equivalent to the existence of a non-trivial stable manifold at 0 in the quadrant $\{(\eta_1, \eta_2) : \eta_1 > 0, \eta_2 > -L\}$ for subsystems (7). The Jacobian of the vector field evaluated at $(\eta_1, \eta_2) = 0$ is:

$$J = \left[\begin{array}{cc} 0 & 0 \\ \beta_2'(\infty) & -\alpha_2'(L) \end{array}\right];$$

therefore, under the assumption that $\alpha_2'(L) \neq 0$, there exists a one dimensional center manifold. As a matter of fact the only case of practical relevance is when $\alpha_2'(L) > 0$; however, for the sake of completeness we carry out the analysis also in the case $\alpha_2'(L) < 0$. By the implicit function theorem, $\beta_2'(\infty) \neq 0$ yields a series expansion of the center manifold of the following type:

$$\eta_2 = \pi(\eta_1) = \frac{\beta_2'(\infty)}{\alpha_2'(L)}\eta_1 + O(\eta_1^3) \tag{8}$$

The dynamics restricted to the center manifold are

$$\dot{\eta}_1 = \frac{\alpha_1'(\infty)\alpha_2'(L) - \beta_1'(L)\beta_2'(\infty)}{\alpha_2'(L)}\eta_1^3 + o(\eta_1^3). \tag{9}$$

Notice that the quantity

$$D := \frac{\alpha_1'(\infty)\alpha_2'(L) - \beta_1'(L)\beta_2'(\infty)}{\alpha_2'(L)}$$

determines with its sign the stability or instability of the center manifold dynamics and consequently the local phase portrait in the η_1, η_2 plane. We can proceed to the classification below.

$\alpha_2'(L)$	D	Phase plane (η_1, η_2)	w-coordinates
+	+	Saddle	GAS
+	-	Stable node	Unbounded
-	+	Antistable node	GAS
-	-	Saddle	Unbounded

The corresponding portraits are shown in Fig. 2.

IV. Examples

In this section we illustrate the theoretical results with two simple examples, which show the simplicity of the proposed approach.

A. The case of finite gap

Consider the system [4]

$$\begin{aligned} \dot{x}_1 &= -\frac{2x_1}{x_1+1} + \frac{x_2}{(x_1+1)(x_2+1)} \\ \dot{x}_2 &= -\frac{2x_2}{x_2+1} + x_1 \end{aligned} \tag{10}$$

and assume that $(x_1, x_2) \in R_+^2$, *i.e.* the system is defined in the positive orthant. For such a system, as done in [4], select $V_1(x_1) = x_1$ and $V_2(x_2) = x_2$ and note that inequalities (1) hold with

$$\begin{aligned} \alpha_1(s) = 2\frac{s}{s+1} \quad & \beta_1(s) = \frac{s}{s+1} \\ \alpha_2(s) = \frac{2s}{s+1} \quad & \beta_2(s) = s. \end{aligned} \tag{11}$$

Note now that

$$\beta_1^{-1}(\alpha_1(w)) = 2\frac{w}{1-w}$$

and that

$$\beta_2^{-1}(\alpha_2(w)) = 2\frac{w}{1+w}.$$

Hence, case 2) of Theorem 1 applies with $\mathcal{L}^- = 1$ and $\mathcal{L}^+ = 2$, implying that the interconnected system (10) is globally asymptotically stable.

B. The no-gap case

Consider now the system:

$$\begin{aligned} \dot{W}_1 &= -\frac{W_1}{1+W_1} + W_2 \\ \dot{W}_2 &= -W_2^n + \left(\frac{W_1^2}{1+W_1+W_1^2}\right)^n. \end{aligned} \tag{12}$$

This is an example satisfying the no-gap condition. Moreover for all $k \geq 1$ the nullclines satisfy $\Omega^{-+} \cup \Omega^{--} \cup \Omega^{+-} = \mathbb{R}^2$. The expansion of α_1 reads:

$$\frac{W_1}{1+W_1} = 1 - \frac{1}{W_1} + \frac{1}{W_1^2} - \ldots$$

Expanding β_2 at infinity yields:

$$\left(\frac{W_1^2}{1+W_1+W_1^2}\right)^n = 1 - n\frac{1}{W_1} + [n^2 - n]\frac{1}{W_1}^2 + O\left(\frac{1}{W_1^3}\right).$$

Moreover, $\beta_1'(1) = 1$ and $\alpha_2'(1) = 1$. Therefore, the system in η coordinates has parameter $D = n - 1$. As a result, if $n > 1$ we are in case (a) and GAS follows. If $0 < n < 1$ we are in case (b) and therefore there exist unbounded trajectories. If instead $n = 1$ first order analysis is not enough to conclude stability or instability.

Note however that, if $n = 1$, then

$$\dot{W}_1 + \dot{W}_2 = -\frac{W_1}{(1+W_1)(1+W_1+W_1^2)} \leq 0.$$

Therefore, as the function $W_1 + W_2$ is radially unbounded in the positive orthant, we conclude global asymptotic stability of system (12) for $n = 1$.

V. Conclusions

A novel version of the nonlinear small gain theorem, under relaxed assumptions, has been developed. It is shown that the proposed conditions cannot be further relaxed, *i.e.* if the conditions fail it is not possible to decide (asymptotic) stability of the considered interconnected system by means of gain conditions alone. However, it is shown that in such case it is possible to study the behavior of the interconnected system by means of the center manifold theory.

The extension of the proposed result to more complex interconnected systems is under investigation, together with the development of input-output properties in the presence of external signals.

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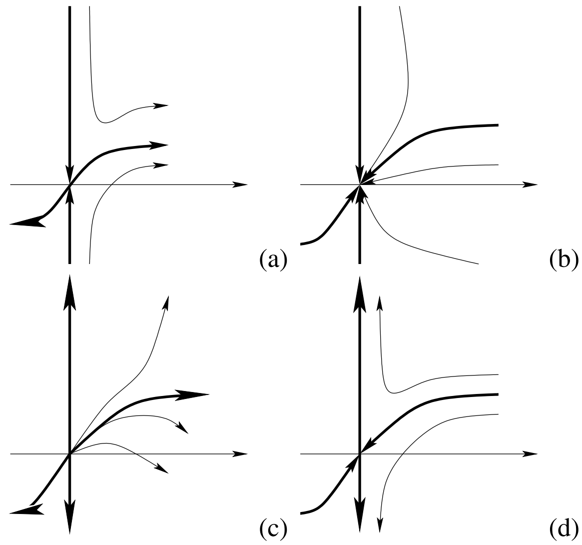

Fig. 2. Phase portraits: (a) $\alpha_2' > 0$, $D > 0$; (b) $\alpha_2' > 0$, $D < 0$; (c) $\alpha_2' < 0$, $D > 0$; (d) $\alpha_2' < 0$, $D < 0$

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