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EFFECTS OF A PARTIAL LAG ON THE ACHIEVABLE PERFORMANCE OF THE H^∞ FIXED-LAG SMOOTHING

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Abstract— **The fixed-lag smoothing problem with a partial lag is the problem in which the presence of the smoothing lag is allowed only in a part of estimation channels. This paper studies the effect of a partial smoothing lag on the achievable H^∞ performance in the continuous-time case. In particular, the limit of the achievable performance is established and the saturation of the achievable performance for a finite smoothing lag is analyzed.**

I. INTRODUCTION

In many communication systems, even interactive ones, a small amount of delay or latency between the measurement and the estimation generation can be tolerated. Some delay is often permissible in speech coding [2], multi-target tracking of a maneuvering target [7], etc. Such problems can be formulated as estimation problems with a constant delay, called *smoothing lag*, in the estimated signal generator. This setting is referred to as *fixed-lag smoothing*, and is dual to the *preview tracking* (the feed forward control problem in which preview of the reference signal is available). It is clear that potentially smoothers can achieve a better performance than the corresponding filters. The questions are how to evaluate this potential and how to exploit it.

The interest to the fixed-lag smoothing can be traced back to [14] and in the context of the H^2 (Kalman) theory the problem is currently well understood, see [1] and the references therein. On the other hand, the H^∞ version of the fixed lag smoothing problem is less studied and until the recent time only a few results were available in the literature, all in the discrete time. Recently, the continuous-time H^∞ fixed lag smoothing was solved in [10] and [13].

The solutions in [10] and [13] assume that the delay in the estimated signal is uniform. This might be an unnecessarily restrictive assumption. Allowing different smoothing lags for different components (channels) of the estimated signal may add flexibility to the design. For example, it might be possible to obtain the required performance by introducing delay only in a part of the estimated channels. The problem where different lags in different channels are allowed is referred to as the *multi-channel fixed-lag smoothing*.

In terms of the dual preview tracking problem, the multi-channel case corresponds to the situation where different preview lengths are available for different components of the reference signal. This can occure, for example, in preview-based control of active suspension [8], if the road disturbances are meassured by more than one sensor. Another possible example is scanning tunneling microscopy [16], where as a result of a scanning pattern, preview can be available only in a part of the motion axes.

Note that in the H^2 settings the problem may be solved channel-wise and thus the extension for the multi-channel case is trivial. This, however, is not true in the H^∞ case. Indeed, the result of solving H^∞ problem for every single channel apart will not be equivalent to the solution of the entire problem.

The H^∞ multi-channel fixed-lag smoothing was recently solved independently in [4] and [5]. The former reference adopts the operator-theoretic approach, while the latter—the J-spectral factorization ideas. Yet in both these references the effect of smoothing lags on the achievable performance is not readily seen. It is, for example, unclear whether (and under what conditions) improvements of the achievable performance due to the presence of smoothing lags in certain channels saturates for finite lags as happens in the single lag case [11]. Another important question left unaddressed in the available solutions is which estimation channel is most/less sensitive to the increase of the lag?

In this paper, a first step towards understanding and quantifying the effect of multiple smoothing lag on the achievable H^∞ performance is taken. We consider the problem in which a part of the estimated signal can be delayed by a single delay, while the rest of the channels have no delay. This problem is referred to as a fixed-lag smoothing with a *partial* smoothing lag. It may be thought of as the multi-channel fixed lag smoothing problem with two channels, in one of which the smoothing lag is zero. The purpose of this note is to study the effects of this partial lag on the achievable performance. In particular we address the following issues.

1) The increase of the smoothing lag should improve the performance. What is a limit of such an improvement?
2) Does an increase of the partial lag always improve the achievable H^∞ performance? Formulating this differently, does the performance improvement saturate for some finite value of the lag?

The analysis in this paper follows the arguments of [10], where effects of a uniform lag on the achievable performance were studied.

Notations: Given a matrix M, $\|M\|$ denotes its spectral norm, and M' denotes the transpose of M. Given a transfer matrix $G(s)$, its conjugate is defined as $G(s)^\sim = G'(-s)$

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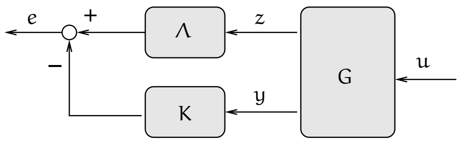

Fig. 1. Estimation setup with partial smoothing lag

and, when $G(s)$ is stable, $\|G(s)\|_2$ and $\|G(s)\|_\infty$ denote its H^2 and H^∞ norms, respectively. The acronyms ARE and DRE stand for the "Algebraic Riccati Equation" and the "Differential Riccati Equation," respectively. A $2n \times 2n$ matrix M is said to be *Hamiltonian* if $\hat{J}^{-1}M'\hat{J} = -M$ and *symplectic* if $\hat{J}^{-1}M'\hat{J} = M^{-1}$ (where $\hat{J} \doteq \mathrm{diag}(I, -I)$). A $2n \times 2n$ Hamiltonian matrix H is said to belong to $\mathrm{dom(Ric)}$ if it has no imaginary axis eigenvalues and its stable (corresponding to the open left half plane eigenvalues) co-spectral subspace is complementary to $\ker\begin{bmatrix}0 & I_n\end{bmatrix}$. In other words, $H \in \mathrm{dom(Ric)}$ if there exist an $n \times n$ matrix Y such that $\begin{bmatrix}I & Y\end{bmatrix}H = A_s\begin{bmatrix}I & Y\end{bmatrix}$ for some Hurwitz A_s. The matrix Y above is unique and thus the function $Y = \mathrm{Ric}(H)$ is well defined. The definitions above are actually dual to those used conventionally [15].

II. Problem Formulation and Solution

Consider the block diagram in Fig. 1, where y and z are a measurement signal and an estimated signal, respectively, generated by a signal generator G from an exogenous signal u. The signal generator G is a dynamic system given in terms of its state space representation

$$G(s) = \begin{bmatrix} G_z(s) \\ G_y(s) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_z & 0 \\ C_y & D_y \end{array}\right], \tag{1}$$

where the partitioning corresponds to that of $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$. The following assumptions are standard for this setting:

- $\boldsymbol{\mathcal{A}_1}$: (C_y, A) is detectable;
- $\boldsymbol{\mathcal{A}_2}$: $\begin{bmatrix} A - j\omega I & B \\ C_y & D_y \end{bmatrix}$ has full row rank $\forall \omega \in \mathbb{R}$;
- $\boldsymbol{\mathcal{A}_3}$: $D_y\begin{bmatrix}B' & D_y'\end{bmatrix} = \begin{bmatrix}0 & I\end{bmatrix}$.

Assumption $\boldsymbol{\mathcal{A}_3}$ is made to simplify the exposition. In principle, it can be relaxed to the assumption $D_y D_y' > 0$ by standard coordinate transformations.

The delay operator in the estimation channel is assumed to have the following form:

$$\Lambda(s) = \begin{bmatrix} I_{p_1} & 0 \\ 0 & e^{-sh} I_{p_2} \end{bmatrix}, \tag{2}$$

which implies that only a part of the signal z is allowed to be estimated with a delay. For this reason, in the sequel we refer to this Λ as the *partial smoothing lag*. The partitioning in (2) induces the following partitioning of the matrix C_z:

$$C_z = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \tag{3}$$

The H^∞ suboptimal smoothing problem with a partial fixed lag is then formulated as follows.

$\mathbf{PSP_h}$: Given $G(s)$ as in (1) and the partial smoothing lag $\Lambda(s)$ as in (2), determine whether there exists a proper $K(s) \in H^\infty$, which guarantees

$$\|\Lambda\, G_z - K G_y\|_\infty < \gamma$$

for a given $\gamma > 0$.

Note that the goal of this paper is to account for the effects of h on the achievable performance. For this reason $\mathbf{PSP_h}$ is only concerned with the achievable performance and not with the construction of the estimator K.

Remark 1: The partial delay as in (2) is sometimes called the *adobe delay*. Delay transfer matrices of this form play an important role in the solutions of the H^∞ problems, that involve multi-channel delays (see [9]).

A. Solvability conditions

The $\mathbf{PSP_h}$ is a special case of the H^∞ multi-channel fixed-lag smoothing problem solved in [5], therefore its formal solution consists merely of the reformulation of the corresponding result in that reference. Define Hamiltonian matrices

$$H_{\gamma,z_1} \doteq \begin{bmatrix} A & -BB' \\ \frac{1}{\gamma^2}C_1'C_1 - C_y'C_y & -A' \end{bmatrix}$$

and

$$H_{\gamma,z} \doteq \begin{bmatrix} A & -BB' \\ \frac{1}{\gamma^2}C_z'C_z - C_y'C_y & -A' \end{bmatrix}.$$

It is readily seen that $H_{\gamma,z}$ is associated with the H^∞ suboptimal filtering of z in the sense that the problem is solvable iff $H_{\gamma,z} \in \mathrm{dom(Ric)}$ and $Y_{\gamma,z} \doteq \mathrm{Ric}(H_{\gamma,z}) \geqslant 0$. Similarly, H_{γ,z_1} is associated with the H^∞ suboptimal filtering of the first estimated channel z_1 and we define $Y_{\gamma,z_1} \doteq \mathrm{Ric}(H_{\gamma,z_1})$ whenever it exists. We also need the Hamiltonian matrix

$$H_\gamma(h) \doteq \Sigma^{-1}\, H_{\gamma,z}\, \Sigma, \tag{4}$$

where $\Sigma \doteq e^{-H_{\gamma,z_1} h}$ is symplectic. The following result, which is a special case of [5, Theorem 1], constitutes a complete solution of $\mathbf{PSP_h}$.

Theorem 1: $\mathbf{PSP_h}$ is solvable iff $H_\gamma(h) \in \mathrm{dom(Ric)}$ and $Y_\gamma(h) \doteq \mathrm{Ric}(H_\gamma(h)) \geqslant 0$.

Theorem 1 provides solvability conditions for $\mathbf{PSP_h}$ in terms of one ARE and turns the Hamiltonian $H_\gamma(h)$ and the Riccati solution $Y_\gamma(h)$ into the central objects of the following study.

III. The Performance Limit

Intuitively, the increase of a smoothing lag may lead, in some cases, to the improvement in the achievable performance. The question that we address in this section is "What is the limit of this improvement ?" More precisely, we are looking for a performance limit:

$$\gamma_{\min} \doteq \lim_{h\to\infty} \gamma_o(h) \tag{5}$$

where

$\gamma_o(h)$ is the minimal constant such that $\mathbf{PSP_h}$ is solvable $\forall\gamma > \gamma_o(h)$ for a given h.

The main difficulty in analyzing the solvability conditions of $\mathbf{PSP_h}$ and, in particular, for finding $\gamma_{\min}$ is the involved dependence of $H_\gamma(h)$ on γ and h. Yet, it turned out that the structure of $H_\gamma(h)$ in (4) can be used to overcome this obstacle.

Roughly speaking, it turned out that for a relevant values of γ, $Y_\gamma(h)$ and $Y_\gamma^{-1}(h)$ as functions of h satisfy a specific DRE and that the analysis can be done using its properties. This kind of relation between ARE solution and DRE was used in [10], and the analysis in this section will be done in a similar fashion but for the case of a *partial* lag.

The solvability conditions for $\mathbf{PSP_h}$ in Theorem 1 are expressed in terms of the Riccati solution $Y_\gamma(h)$. The use of the latter for further analysis is complicated by the fact that it is a discontinuous function of h. To resolve this difficulty we assume that

$\boldsymbol{\mathcal{A}_4}$: (A, B) has no stable uncontrollable modes .

If $\boldsymbol{\mathcal{A}_3}$ were relaxed, $\boldsymbol{\mathcal{A}_4}$ would be formulated in terms of stable invariant zeros of G_y, which can clearly be canceled by the estimator. Thus, assumption $\boldsymbol{\mathcal{A}_4}$, in fact, rules out the solution redundancy. It is worth stressing, however, that it can be omitted as it was done in [10, Section 4].

Technically, the purpose of $\boldsymbol{\mathcal{A}_4}$ is to eliminate the singular part of $Y_\gamma(h)$. This enables us to perform the analysis in terms of $Y_\gamma^{-1}(h)$, which turns out to be a continuous function of both γ and h. To this end define Hamiltonian matrices

$$\tilde{H}_{\gamma,z_1} \doteq \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] H_{\gamma,z_1} \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \begin{bmatrix} -A' & \frac{1}{\gamma^2}C_1'C_1 - C_y'C_y \\ -BB' & A \end{bmatrix},$$

$$\tilde{H}_{\gamma,z} \doteq \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] H_{\gamma,z} \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \begin{bmatrix} -A' & \frac{1}{\gamma^2}C_z'C_z - C_y'C_y \\ -BB' & A \end{bmatrix},$$

and

$$\tilde{H}_\gamma(h) \doteq \tilde{\Sigma}^{-1}\tilde{H}_{\gamma,z}\tilde{\Sigma},$$

where $\tilde{\Sigma} \doteq e^{-\tilde{H}_{\gamma,z_1}h}$ is symplectic. It is easy to verify that the Hamiltonian matrices above define an inverse counterparts for the Riccati equations discussed in the previous section.

We also define some constants associated with the Hamiltonian matrices above:

- γ_∞ is the maximal γ for which $\tilde{H}_{\gamma,z}$ has eigenvalues on the $j\omega$-axis
- γ_1 is the infimal γ for which $\tilde{H}_{\gamma,z_1} \in \text{dom}(\text{Ric})$ and $\tilde{Y}_{\gamma,z_1} > 0$

It can be shown [10] that γ_∞ is the optimal performance of the H^∞ fixed interval smoothing of z and consequently the performance limit of the H^∞ smoothing with a fixed *uniform* lag. The other constant, γ_1, is related to the H^∞ filtering of a first channel, z_1. It is known [12], [3] that the null space of Y_{γ,z_1} does not depend on γ. It is also readily verifiable that

$$Y_{\infty,z_1} = \text{Ric}\left(\begin{bmatrix} A & -BB' \\ -C_y'C_y & -A' \end{bmatrix}\right) > 0$$

Hence, $Y_{\gamma,z_1} \geqslant 0 \iff \tilde{Y}_{\gamma,z_1} > 0$ and, as a result, the H^∞ suboptimal filtering of z_1 is solvable iff $\gamma > \gamma_1$.

Before we start considering $\tilde{H}_\gamma(h)$ and $\tilde{Y}_\gamma(h) \doteq \text{Ric}(\tilde{H}_\gamma(h))$ (the subjects of our primary interest), some preliminary results regarding the properties of ARE associated with $\tilde{H}_{\gamma,z_1}$ and $\tilde{H}_{\gamma,z}$ are required. These properties will play an important role in the sequel.

Lemma 1: $\tilde{H}_{\gamma,z} \in \text{dom}(\text{Ric})$ and $\tilde{Y}_{\gamma,z} \doteq \text{Ric}(\tilde{H}_{\gamma,z})$ iff $\gamma > \gamma_\infty$

Proof: See the proof of Claim 1 in [10]. □

Lemma 2: If $\gamma > \gamma_\infty$, then $\tilde{H}_{\gamma,z_1} \in \text{dom}(\text{Ric})$, moreover $\tilde{Y}_{\gamma,z_1} \doteq \text{Ric}(\tilde{H}_{\gamma,z_1}) \geqslant \tilde{Y}_{\gamma,z}$.

Proof: $\gamma > \gamma_\infty$ therefore $\tilde{Y}_{\gamma,z}$ exists and we may define a Hurwitz matrix $A_{\gamma,z} = -A - BB'\tilde{Y}_{\gamma,z}$ and Hamiltonian matrix

$$\tilde{H}_\Delta = \begin{bmatrix} A_{\gamma,z}' & -\frac{1}{\gamma^2}C_2'C_2 \\ -BB' & -A_{\gamma,z} \end{bmatrix}$$

According to [15], $\tilde{H}_\Delta \in \text{dom}(\text{Ric})$ and $\tilde{Y}_\Delta \doteq \text{Ric}(\tilde{H}_\Delta) \geqslant 0$. It can be shown by a direct substitution that $(\tilde{Y}_{\gamma,z} + \tilde{Y}_\Delta)$ is a stabilizing solution of ARE associated with $\tilde{H}_{\gamma,z_1}$. Therefore $\tilde{H}_{\gamma,z_1} \in \text{dom}(\text{Ric})$, and $\tilde{Y}_{\gamma,z_1} = \tilde{Y}_{\gamma,z} + \tilde{Y}_\Delta \geqslant \tilde{Y}_{\gamma,z}$ □

At this point we are in the position to analyze the ARE associated with $\tilde{H}_\gamma(h)$. We start with finding the relation between $\tilde{Y}_\gamma(h)$ and the solution of the DRE

$$\begin{aligned} -\dot{Q}_\gamma(t) &= A'Q_\gamma(t) + Q_\gamma(t)A - C_y'C_y + \tfrac{1}{\gamma^2}C_1'C_1 \\ &\quad + Q_\gamma(t)BB'Q_\gamma(t), \quad Q_\gamma(0) = \tilde{Y}_{\gamma,z}, \end{aligned} \tag{6}$$

which is defined for all $\gamma > \gamma_\infty$.

Lemma 3: Let $\gamma > \gamma_\infty$. Then the following statements are equivalent

1) $\tilde{H}_\gamma(h) \in \text{dom}(\text{Ric})$
2) $Q_\gamma(t)|_{t=h}$ exists.

Moreover, if either of these statements holds, then $\tilde{Y}_\gamma(h) = Q_\gamma(h)$.

Proof: Since $\gamma > \gamma_\infty$, $\tilde{H}_{\gamma,z} \in \text{dom}(\text{Ric})$ and there exists a Hurwitz matrix A_L such that

$$\begin{gathered} \begin{bmatrix} I & \tilde{Y}_{\gamma,z} \end{bmatrix}\tilde{H}_{\gamma,z} = A_L\begin{bmatrix} I & \tilde{Y}_{\gamma,z} \end{bmatrix} \\ \Downarrow \\ \begin{bmatrix} I & \tilde{Y}_{\gamma,z} \end{bmatrix}\tilde{\Sigma}\tilde{\Sigma}^{-1}\tilde{H}_{\gamma,z}\tilde{\Sigma} = A_L\begin{bmatrix} I & \tilde{Y}_\gamma^z \end{bmatrix}\tilde{\Sigma} \\ \Downarrow \end{gathered}$$

$$\begin{aligned} &\begin{bmatrix} \tilde{\Sigma}_{11} + \tilde{Y}_\gamma^z\tilde{\Sigma}_{21} & \tilde{\Sigma}_{12} + \tilde{Y}_\gamma^z\tilde{\Sigma}_{22} \end{bmatrix}\tilde{H}_\gamma(h) \\ &\qquad = A_L\begin{bmatrix} \tilde{\Sigma}_{11} + \tilde{Y}_\gamma^z\tilde{\Sigma}_{21} & \tilde{\Sigma}_{12} + \tilde{Y}_\gamma^z\tilde{\Sigma}_{22} \end{bmatrix}, \end{aligned}$$

where $\tilde{\Sigma}_{ij}$ are the corresponding sub-blocks of $\tilde{\Sigma}$. Hence, $\begin{bmatrix} \Sigma_{11} + Y_\gamma^z\Sigma_{21} & \Sigma_{12} + Y_\gamma^z\Sigma_{22} \end{bmatrix}$ is a basis for the stable invariant subspace of $\tilde{H}_\gamma(h)$, so that

$$\tilde{H}_\gamma(h) \in \text{dom}(\text{Ric}) \iff \det(\Sigma_{11} + Y_\gamma^z\Sigma_{21}) \neq 0$$

and $Y_\gamma(h) = (\Sigma_{11} + Y_\gamma^z\Sigma_{21})^{-1}(\Sigma_{12} + Y_\gamma^z\Sigma_{22})$ whenever exists.

On the other hand, it is a known result from the Riccati theory [6] that $\tilde{Y}_\gamma(h)$ as in the last formula satisfies $\tilde{Y}_\gamma(h) = Q_\gamma(h)$ whenever they exist. □

The next step is to establish some properties of the solution of (6), which are important in the analysis below.

Lemma 4: $Q_\gamma(t)$ is a monotonically non-decreasing function of t (in the sense that $Q_\gamma(t_1) \geqslant Q_\gamma(t_2)$ whenever $t_1 \geqslant t_2$) and $\lim_{t\to\infty} Q_\gamma(t) = \tilde{Y}_{\gamma,z_1}$.

Proof: See Appendix A. □

In particular, the result of Lemma 4 implies that $Q_\gamma(t)$ does not have finite escape points. Therefore, if $\gamma > \gamma_\infty$, then $\tilde{H}_\gamma(h) \in \mathrm{dom(Ric)}$ for all h.

Remark 2: It is worth emphasizing that unlike the uniform lag case, the DRE in (6) and, consequently, $\lim_{t\to\infty} Q_\gamma(t)$ depend on γ. This difference plays an important role in the analysis of both the limiting performance and the saturation (in the next section).

We are now at the position to state the main result of this section.

Theorem 2:

1) The performance limit as defined in (5) is $\gamma_{\min} = \max(\gamma_\infty, \gamma_1)$.
2) For any $\gamma > \gamma_{\min}$, $\mathbf{PSP_h}$ is solvable iff $Q_\gamma(h) > 0$, where $Q_\gamma(t)$ is defined by (6).

Proof: If $\gamma < \gamma_\infty$, then $H_{\gamma,z}$ has $j\omega$-axes eigenvalues, so that $H_{\gamma,z} \notin \mathrm{dom(Ric)}$. On the other hand for $\gamma > \gamma_\infty$, $\tilde{H}_\gamma(h) \in \mathrm{dom(Ric)}$ and therefore $Y_\gamma(h)$ is invertible whenever exists. This implies that $\mathbf{PSP_h}$ is solvable iff $\gamma > \gamma_\infty$ and $\tilde{Y}_\gamma(h) > 0$ (equivalently, $Q_\gamma(h) > 0$).

The proof is now completed now by noticing that for $\gamma > \gamma_\infty$ there exists a finite h such that $Q_\gamma(h) > 0$ iff $\lim_{t\to\infty} Q_\gamma(t) = \tilde{Y}_{\gamma,z_1} > 0$, which means $\gamma > \gamma_1$. □

This result has a partial intuitive interpretation: the smoothing performance with a finite partial lag can be neither better than that of the smoothing with an infinite uniform lag (γ_∞), nor than that of the filtering of the zero-lag channel (γ_1). A more intriguing fact, however, is that the performance limit is not greater or equal, but indeed *equal* to the maximum of γ_∞ and γ_1.

IV. Performance Saturation for a Finite Lag

The remarkable feature of H^∞ smoothing is that the achievable performance may saturate as a function of the smoothing lag. In other words, the performance limit may be achieved with a finite h and any further increase of the lag will not improve the performance.

Theorem (2) shows that $\gamma_{\min}$ depends both on γ_1 and γ_∞. It turns out that existence of the saturation depends on which of the mentioned factors is dominating and actually defines the performance limit.

First consider the case in which the limiting performance is defined by γ_∞.

Lemma 5: If $\gamma_{\min} = \gamma_\infty > \gamma_1$, than there exists a finite h_∞ such that $\gamma_o(h_\infty) = \gamma_{\min}$.

Proof: See Appendix B. □

Roughly speaking, if $\gamma_\infty > \gamma_1$ the following inequality holds

$$\lim_{\gamma\to\gamma_{\min}} \tilde{Y}_{\gamma,z_1} = \tilde{Y}_{\gamma_\infty,z_1} > 0.$$

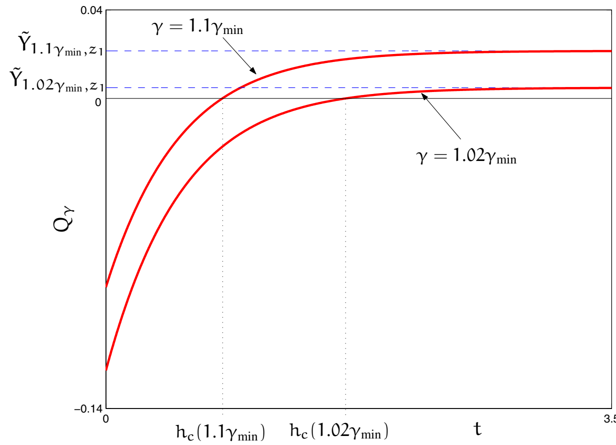

Fig. 2. $Q_\gamma(t)$ for Example 1

This situation is similar to that in the uniform lag case [11], and thus the saturation phenomenon exists.

The situation, however, is different when $\gamma_1 \geqslant \gamma_\infty$, as in this case one or more eigenvalues of $\tilde{Y}_{\gamma_{\min},z_1}$ can be zeros. This can be illustrated by the following academic example.

Example 1: Let

$$G_z = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{s-1} & 0 \end{bmatrix} \quad \text{and} \quad G_y = \begin{bmatrix} \frac{0.5}{s-1} & 1 \end{bmatrix},$$

for which $\gamma_{\min} = \gamma_1 = 2$ and $\det(\tilde{Y}_{\gamma_{\min},z_1}) = 0$.

Fig. 2 presents the graphs of $Q_\gamma(t)$ for $\gamma = 1.1\gamma_{\min}$ and $\gamma = 1.02\gamma_{\min}$. The dashed lines correspond to the values of $\tilde{Y}_{\gamma,z_1}$. We denote a value of t, for which $Q_\gamma(t)$ crosses the zero by $h_c(\gamma)$, so that the performance γ is achievable only if the smoothing lag is greater than $h_c(\gamma)$. Intuitively, by choosing the value of γ close enough to $\gamma_{\min}$, the value of $\tilde{Y}_{\gamma,z_1}$ may be placed arbitrary close to zero, which will make $h_c(\gamma)$ arbitrary large. Therefore there is no saturation in this example. (see Fig. 3) ♦

The saturation, however, may exist in some cases even

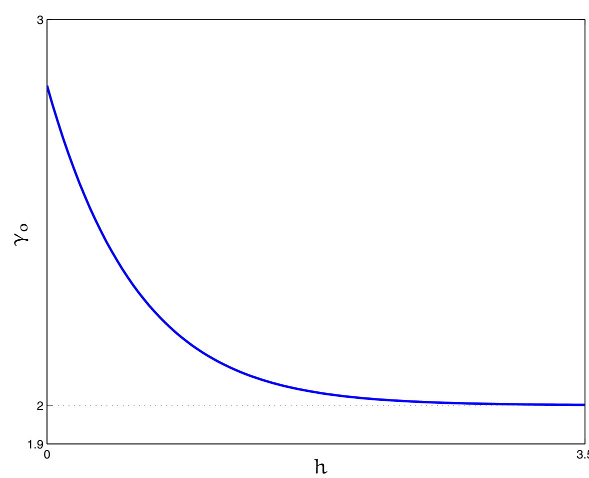

Fig. 3. $\gamma_o(h)$ for Example 1

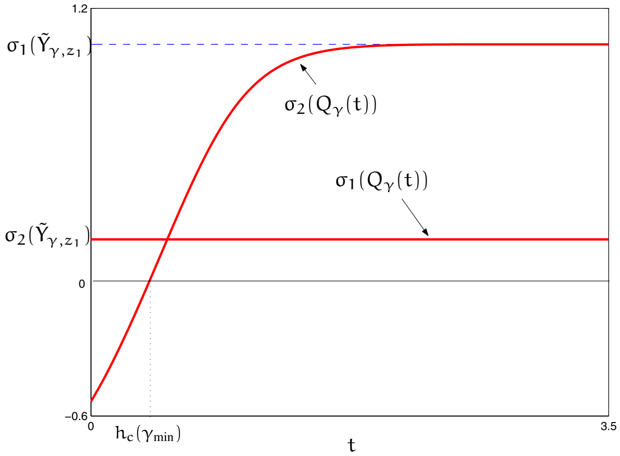

Fig. 4. Eigenvalues of $Q_\gamma(t)$ for Example 2

when the performance limit is defined by γ_1. To see this consider the following example.

Example 2: Let now

$$G_z = \begin{bmatrix} \frac{1}{s-1} & 0 & 0 & 0 \\ 0 & \frac{1}{s-1} & 0 & 0 \end{bmatrix} \quad \text{and} \quad G_y = \begin{bmatrix} \frac{s-5}{(s-1)^2} & 0 & 1 & 0 \\ 0 & \frac{s-3}{(s-1)^2} & 0 & 1 \end{bmatrix}$$

for which $\gamma_{\min} = \gamma_1 = 0.42$.

Two relevant eigenvalues of $Q_\gamma(t)$ are presented as functions of t in Fig. 4 for some $\gamma > \gamma_{\min}$. It can be seen that one of these eigenvalues coincides with the minimal eigenvalue of $\tilde{Y}_{\gamma,z_1}$ for all $t > 0$ (this is actually true for all $\gamma > \gamma_{\min}$). Therefore the approach of zero by the minimal eigenvalue of $\tilde{Y}_{\gamma,z_1}$ does not yield unbounded growth of $h_c(\gamma)$ and the saturation exists, seen Fig. 5. ♦

Summarizing the discussion above, the achievable performance generically saturates if $\gamma_\infty > \gamma_1$, while if $\gamma_\infty \leqslant \gamma_1$ it might either saturate or not. The quantification of this case is the subject of the current research.

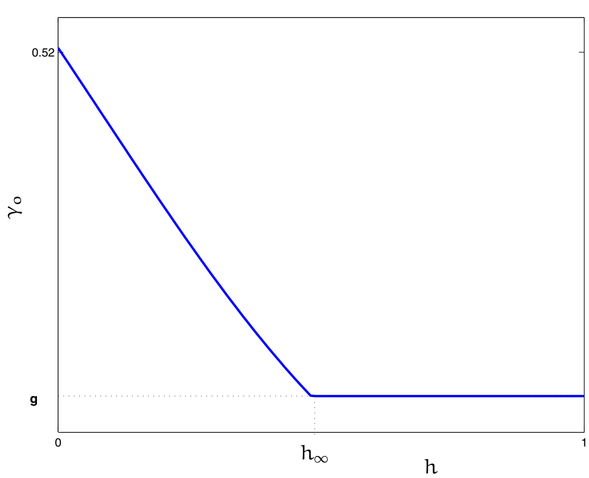

Fig. 5. $\gamma_o(h)$ for Example 2

V. Concluding Remarks

In this paper the achievable H^∞ performance in the continuous-time fixed-lag smoothing with a partial fixed lag has been studied. In particular, the limit of the performance that can be achieved with a partial lag has been established. It turns out to be equal to the maximum of the achievable performances of the following two estimation problems:

- the H^∞ filtering of the zero-lag estimation channel;
- the H^∞ fixed interval smoothing of both channels (of the entire z signal).

The saturation of the achievable performance as a function of the smoothing lag has been studied as well. It has been shown, that the saturation exists if the dominating constrain, i.e., the one which determines the performance limit, is the performance of the H^∞ fixed interval smoothing.

The necessary and sufficient conditions for the existence of the saturation, as well as the extension of the approach adopted in this paper to more general multi-channel problems, are the subject of the current research.

Appendix

A. Proof of Lemma 4

Differentiating (6) by t we get:

$$\ddot{Q}_\gamma = -(A + BB'Q_\gamma)'\dot{Q}_\gamma - \dot{Q}_\gamma(A + BB'Q_\gamma).$$

Denoting by $\Phi_Q(t,0)$ the state transition matrix associated with $-(A + BB'Q_\gamma)$ and noting that $\dot{Q}_\gamma(0) = \frac{1}{\gamma^2}C_2'C_2$ we obtain:

$$\dot{Q}_\gamma(t) = \tfrac{1}{\gamma^2}\Phi_Q'(t,0)C_2'C_2\Phi_Q(t,0) \geqslant 0,$$

which proves the monotonicity statement of the claim.

Now, define the monotonically non-increasing function $Q_\Delta(t) \doteq \tilde{Y}_{\gamma,z_1} - Q_\gamma(t)$. It is readily verified that $Q_\Delta(t)$ satisfies

$$\dot{Q}_\Delta = A_{\gamma,z_1}'Q_\Delta + Q_\Delta A_{\gamma,z_1} + Q_\Delta BB'Q_\Delta, \qquad Q_\Delta(0) = \tilde{Y}_{\gamma,z_1} - \tilde{Y}_{\gamma,z} \geqslant 0, \quad (7)$$

where $A_{\gamma,z_1} \doteq -(A + BB'\tilde{Y}_{\gamma,z_1})$ is Hurwitz.

Assume that there exists a $t_f > 0$ such that $Q_\Delta(t_f) \not\geqslant 0$. Then there must exist a vector η such that $\eta'Q_\Delta(t_f)\eta < 0$. Consider the following scalar analytic function:

$$f(t) \doteq \eta'Q_\Delta(t)\eta.$$

It is readily seen that $f(0) \geqslant 0$ and $f(t_f) < 0$. Therefore there should exist $t_c \in [0, t_f)$ such that $f(t_c) = 0$. By the inspection of (7) it can be shown that in this case all derivatives of $f(t)$ at $t = t_c$ vanish as well. This, together with the analyticity of $f(t)$, implies that $f(t) \equiv 0$, which is a contradiction. Therefore, $Q_\Delta(t) \geqslant 0$ for all $t > 0$.

So far we proved that Q_Δ is monotonically non-increasing and bounded. Hence, its limit as $t \to \infty$ exists. Finally, since A_{γ,z_1} is Hurwitz, the only equilibrium of Q_Δ in the positive semi-definite region is $Q_\Delta = 0$.

B. Proof of Lemma 5

It is known [12] that as $\gamma_\infty > 0$, the following limit exists and we may define

$$\tilde{Y}_m \doteq \lim_{\gamma \searrow \gamma_\infty} \tilde{Y}_{\gamma,z}.$$

To rule out the trivial situation when any performance level $\gamma > \gamma_\infty$ is achievable with $h = 0$, we assume that at least one eigenvalue of $\tilde{Y}_m$ is negative, i.e., that smoothing over-performs filtering.

Let $Q_m(t)$ be a solution of DRE

$$\begin{aligned} -\dot{Q}_m(t) &= A'Q_m(t) + Q_m(t)A + \gamma_\infty^{-2}C_1'C_1 \\ &\quad - C_y'C_y + Q_m(t)BB'Q_m(t), \quad Q_m(0) = \tilde{Y}_m. \end{aligned} \tag{8}$$

This, in a sense, is the extension of the definition of $Q_\gamma(t)$ for $\gamma = \gamma_\infty$ and $Q_m(t) = \lim_{\gamma \searrow \gamma_\infty} Q_\gamma(t)$ for any $t > 0$. The following technical result shows that $Q_m(t)$ has properties similar to those of $Q_\gamma(t)$.

Claim 1: $Q_m(t)$ is monotonically non-decreasing function of t and $\lim_{t\to\infty} Q_m(t) = \tilde{Y}_{\gamma_\infty,z_1}$.

Proof: This result is similar to Lemma 4, and can be proved using the same arguments. □

Note that $\gamma_\infty > \gamma_1$, so that $\tilde{Y}_{\gamma_\infty,z_1}$ exists and is positive definite. As a result we may define h_m as a maximal value of t, for which $Q_m(t)$ has a zero eigenvalue. It is readily seen that for any $t > h_m$, $Q_m(t) > 0$. The following result plays an important role in the sequel.

Claim 2: For any $\gamma > \gamma_\infty$ and for any $t > 0$ the following inequality holds

$$\mathcal{Q}_\gamma(t) \doteq Q_\gamma(t) - Q_m(t) \geqslant 0$$

Proof: Subtracting (8) from (6) and performing some algebraic manipulations yield that $\mathcal{Q}_\gamma(t)$ satisfies the following time varying DRE

$$\begin{aligned} \dot{\mathcal{Q}}_\gamma(t) &= -\mathcal{A}'(t)\mathcal{Q}_\gamma(t) - \mathcal{Q}_\gamma(t)\mathcal{A}(t) + (\gamma_\infty^{-2} - \gamma^{-2})C_1'C_1 \\ &\quad - \mathcal{Q}_\gamma(t)BB'\mathcal{Q}_\gamma(t), \quad \mathcal{Q}_\gamma(0) = \tilde{Y}_{\gamma,z} - \tilde{Y}_m \geqslant 0 \end{aligned} \tag{9}$$

where

$$\mathcal{A}(t) \doteq A + BB'Q_m(t).$$

Assume that there exists a $t_f > 0$ such that $\mathcal{Q}_\gamma(t_f) \ngeqslant 0$. Then there must exist a vector η such that $\eta'\mathcal{Q}_\gamma(t_f)\eta < 0$. Consider the scalar analytic function

$$f(t) \doteq \eta'\mathcal{Q}_\gamma(t)\eta$$

It is readily seen that $f(0) \geqslant 0$ and $f(t_f) < 0$. Therefore there should exist $t_c \in [0, t_f)$ such that $f(t_c) = 0$. On the other hand, it follows from (9) that if $\eta'\mathcal{Q}_\gamma(t_c)\eta = 0$, then

$$\dot{f}(t_c) = (\gamma_\infty^{-2} - \gamma^{-2})\|C_1\eta\|^2 \geqslant 0.$$

There are two different situations:

1) If $\eta \in \ker C_1$, then $\dot{f}(t_c) = 0$ and it can also be shown that all other derivatives of f are zero as well. Then the proof follows that in Appendix A.
2) If $\eta \notin \ker C_1$, then $\dot{f}(t_c) > 0$ and we have that whenever $f(t) = 0$, its derivative is positive. It is then clear that $f(t)$ can never become negative.

Thus, in both cases we have a contradiction and therefore $\mathcal{Q}_\gamma(t) \geqslant 0$ for all $t > 0$. □

The proof is then completed now by noticing that for any $h_\infty > h_m$ the following statement holds

$$Q_\gamma(h_\infty) \geqslant Q_m(h_\infty) > 0, \quad \forall \gamma > \gamma_\infty.$$

Therefore any performance level greater then $\gamma_{\min} = \gamma_\infty$ is achievable with a smoothing lag h_∞, and hence $\gamma_o(h_\infty) = \gamma_{\min}$.

References

[1] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*. Englewood Cliffs, NJ: Prentice-Hall, 1979.

[2] S. Crisafulli, J. D. Mills, and R. R. Bitmead, "Kalman filtering techniques in speech coding," in *Proc. IEEE ICASSP-92*, vol. 1, 1992, pp. 77–80.

[3] P. Gahinet, "On the game Riccati equations arising in H^∞ control problems," *SIAM J. Control Optim.*, vol. 32, no. 3, pp. 635–647, 1994.

[4] A. Kojima and S. Ishijima, "Formulas on preview and delayed H^∞ control," in *Proc. 42nd IEEE Conf. Decision and Control*, Maui, HW, 2003, pp. 6532–6538.

[5] M. Kristalny and L. Mirkin, "On the multi-channel H^∞ fixed-lag smoothing," in *Proc. 5th IFAC Workshop on Time Delay Systems*, K.U. Leuven, Belgium, 2004.

[6] V. Kučera, "Riccati equations and their solutions," in *The Control Handbook*, W. S. Levine, Ed. Boca Raton: CRC Press, 1996, pp. 595–606.

[7] A. K. Mahalanabis, B. Zhou, and N. K. Bose, "Improved multi-target tracking in clutter by PDA smoothing," *IEEE Trans. Aerospace and Electronic Systems*, vol. 26, no. 1, pp. 113–121, 1990.

[8] D. Martinus, B. Soenarko, and Y. Y. Nazaruddin, "Optimal control design with preview for semi-active suspension on a half-vehicle model," in *Proc. 35th IEEE Conf. Decision and Control*, vol. 3, Kobe, Japan, 1996, pp. 2798–2803.

[9] G. Meinsma and L. Mirkin, "H^∞ control of systems with multiple I/O delays via decomposition to adobe problems," *IEEE Trans. Automat. Control*, vol. 50, no. 2, pp. 199–211, 2005.

[10] L. Mirkin, "On the H^∞ fixed-lag smoothing: How to exploit the information preview," *Automatica*, vol. 39, no. 8, pp. 1495–1504, 2003.

[11] L. Mirkin and G. Meinsma, "When does the H^∞ fixed-lag smoothing performance saturate for a finite smoothing lag?" *IEEE Trans. Automat. Control*, vol. 49, no. 1, pp. 131–134, 2004.

[12] C. Scherer, "H^∞-control by state-feedback and fast algorithms for the computation of optimal H^∞-norms," *IEEE Trans. Automat. Control*, vol. 35, no. 10, pp. 1090–1099, 1990.

[13] G. Tadmor and L. Mirkin, "H^∞ control and estimation with preview—Part I: Matrix ARE solutions in continuous time," *IEEE Trans. Automat. Control*, vol. 50, no. 1, pp. 19–28, 2005.

[14] N. Wiener, *Extrapolation, Interpolation and Smoothing of Stationary Time Series*. Cambridge, MA: The MIT Press, 1949.

[15] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[16] Q. Zou and S. Devasia, "Preview-based optimal inversion for output tracking: application to scanning tunneling microscopy," *IEEE Trans. Control Syst. Technology*, vol. 12, no. 3, pp. 375–386, 2004.