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Dissipativity Theory for Singular Systems. Part I: Continuous-Time Case

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Abstract— **In this paper we develop dissipativity results for continual nonlinear and linear singular systems. To the best knowledge of author results are nonexistent. We generalize dissipativity theory to nonlinear continuous singular dynamical systems. Specifically, the classical concepts of system storage functions and supply rates are extended to singular dynamical systems providing a generalized system energy interpretation in terms of stored energy and dissipated energy over the continuous-time system dynamics. Furthermore, extended Kalman-Yakubovich-Popov conditions in terms of the singular system dynamics characterizing dissipativeness via system storage functions are derived. Finally, the framework is specialized to passive and nonexpansive singular systems to provide a generalization of the classical notions of passivity and nonexpansivity for nonlinear singular systems.**

I. Introduction

The key foundation in dissipativity theory of dynamical systems was presented by Willems in his seminal two-part paper [1], [2]. In particular, Willems [2] introduced definition of dissipativity for general dynamical systems in terms of an inequality involving a generalized system power input, or, supply rate, and a generalized energy function, or, storage function. Since Lyapunov functions can be viewed as generalization of energy functions for nonlinear dynamical systems, the notion of dissipativity, with appropriate storage functions and supply rates, can be used to construct Lyapunov functions for nonlinear feedback systems by appropriately combining storage functions for each subsystem. Even though the original work on dissipative dynamical systems was formulated in the state space setting describing the system dynamics in terms of continuous flows on appropriate manifolds, an input-output formulation for dissipative dynamical systems extending the notions of passivity [3], nonexpansivity [3], and conicity [3], [4] was presented in [5], [6], [7]. More recently, the notion of dissipativity theory was generalized in [8] to formalize the concepts of the nonlinear analog of strict positive realness and strict bounded realness. In particular, using exponentially weighted supply rates, the concept of exponential dissipativity was introduced in [8].

Dissipativity theory along with its connections to Lyapunov stability theory has been extensively developed for dynamical systems possessing continuous flows. Since singular systems [9], [10] more naturally describe the system and they are present in many applications including circuit systems theory, boundary problems, chemical and process industry, biological systems, to name just a few, it is important to develop dissipativity theory for this class of systems. To the best knowledge of the author results are nonexistent.

The contents of the paper are as follows. In Section II, we extend the notion of dissipative dynamical systems to develop the concept of dissipativity for singular dynamical systems. In Section III we develop Kalman-Yakubovich-Popov algebraic conditions in terms of the system dynamics for characterizing dissipativeness via system storage functions for singular systems. Furthermore, a generalized energy balance interpretation involving the system's stored or, accumulated energy and dissipated energy over the continuous-time dynamics is given. Specialization of these results to passive and nonexpansive singular systems is also provided. In Section IV we specialize the results of Section III to linear singular systems to obtain extended Kalman-Yakubovich-Popov equations for positive real and bounded real singular systems. Finally, we draw conclusion in Section V.

II. Dissipative Continuous Nonlinear Singular Systems: Input-Output and State Properties

We consider a continuous time nonlinear singular dynamical system $\mathcal{G}$ described by

$$E_{\rm c}\dot{x}(t) = f_{\rm c}(x(t)) + G_{\rm c}(x(t))u_{\rm c}(t), \quad x(0) = x_0 \quad (1)$$

$$y_{\rm c}(t) = h_{\rm c}(x(t)) + J_{\rm c}(x(t))u_{\rm c}(t). \quad (2)$$

where $t \geq 0$, $x(0) = x_0$, $x(t) \in \mathcal{D} \subset R^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u_{\rm c} \in \mathcal{U}_{\rm c} \subset R^{m_{\rm c}}$, input $u_{\rm c}(t), y_{\rm c}(t) \in R^{l_{\rm c}}$, $f_{\rm c} : \mathcal{D} \to R^n$ is Lipschitz continuous and satisfies $f_{\rm c}(0) = 0$, $\mathcal{G}_{\rm c} : \mathcal{D} \to n \times m_{\rm c}$, $h_{\rm c} : \mathcal{D} \to R^{l_{\rm c}}$ and satisfies $h_{\rm c}(0) = 0$, $J_{\rm c} : \mathcal{D} \to R^{l_{\rm c} \times m_{\rm c}}$. Matrix $E_{\rm c}$ is allowed to be a singular matrix. In case $E_{\rm c} = I$, (1)–(2) represent standard continuous time dynamical systems.

It is well known that Lyapunov functions can be viewed as generalizations of energy functions for a general nonlinear system. In this section we extend the notion of dissipative dynamical systems to develop the concept of dissipativity for nonlinear continuous singular systems. In particular, using concepts of nonlinear singular systems storage functions with appropriate supply rates, storage functions are developed as Lyapunov functions for nonlinear continuous singular systems.

The next result gives necessary and sufficient conditions for dissipativity, exponential disipativity, and losslessness.

Theorem 2.1: $\mathcal{G}$ is dissipative with respect to the supply rate $r_{\rm c}$ if and only if there exists a C^0 nonnegative-definite function $V_{\rm s} : R \times R^n \to R$, such that

$$V_{\rm s}(\hat{t}, E_{\rm c}x(\hat{t})) - V_{\rm s}(t, E_{\rm c}x(t)) \leq \int_t^{\hat{t}} r_{\rm c}(u_{\rm c}(s), y_{\rm c}(s)){\rm d}s. \quad (3)$$

Furthermore, $\mathcal{G}$ is exponentially dissipative with respect to the supply rate r_c if and only if there exists a C^0 nonnegative-definite function $V_s : R \times R^n \to R$, such that

$$e^{\epsilon\hat{t}}V_s(\hat{t}, E_c x(\hat{t})) - e^{\epsilon\hat{t}}V_s(t, E_c x(t)) \le \int_t^{\hat{t}} e^{\epsilon\hat{t}} r_c(u_c(s), y_c(s))\mathrm{d}s. \quad (4)$$

Finally, $\mathcal{G}$ is lossless with respect to the supply rate r_c if and only if there exists a C^0 nonnegative-definite function $V_s : R \times R^n \to R$ such that (3) is satisfied as equality.

If in Theorem 2.1 $V_s(\cdot, x(\cdot))$ is C^1 a.e. on $[t_0, \infty)$, then an equivalent statement for dissipativeness of the nonlinear continuous singular system $\mathcal{G}$ with respect to the supply rate r_c is

$$\dot{V}_s(t, E_c x(t)) \le r_c(u_c(t), y_c(t)), \quad (5)$$

where $\dot{V}_s(\cdot,\cdot)$ denotes the total derivative of $V_s(t, E_c x(t))$ along the state trajectories $x(t)$ of the nonlinear continuous singular system (1), (2). Furthermore, an equivalent statement for exponential dissipativeness of the nonlinear continuous singular system $\mathcal{G}$ with respect to the supply rate r_c is given by

$$\dot{V}_s(t, E_c x(t)) + \epsilon V_s(t, E_c x(t)) \le r_c(u_c(t), y_c(t)). \quad (6)$$

The following theorem provides sufficient conditions for guaranteeing that all storage functions (resp., exponential storage functions) of a given dissipative (resp., exponentially dissipative) nonlinear continuous singular system are positive definite.

Theorem 2.2: Consider the nonlinear continuous singular system $\mathcal{G}$ given by (1), (2) and assume that $\mathcal{G}$ is completely reachable and zero-state observable. Furthermore, assume that $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate r_c and there exists function : $R^{l_c} \to R^{m_c}$ such that $r_c(\kappa_c(y_c), y_c) < 0$, $y_c \ne 0$. Then all the storage functions (resp., exponential storage functions) $V_s(t, x), (t, x) \in R \times R^n$, for $\mathcal{G}$ are positive definite, that is $V_s(\cdot, 0) = 0$ and $V_s > 0$, $(t, x) \in R \times R^n, x \ne 0$.

III. Kalman-Yakubovich-Popov Conditions for Continuous Singular Systems

Dissipativeness of a continuous singular system can be characterized in terms of the system functions $f_c(\cdot), G_c(\cdot), h_c(\cdot), J_c(\cdot)$. For the results in this section we consider the special case of dissipative continuous singular systems with quadratic supply rates and set $\mathcal{U}_c = R^{m_c}$. Specifically, let $Q_c \in S^{l_c}$, $S_c \in R^{l_c \times m_c}$, $R_c \in S^{m_c}$ be given and assume $r_c(u_c, y_c) = y_c^T Q_c y_c + 2y_c^T u_c + u_c^T R_c u_c$. In the reminder of the paper we assume that storage functions do not depend explicitly on time. Furthermore, we assume that there exist functions $\kappa_c : R^{l_c} \times R^{m_c}$ such that $r_c(\kappa_c(y_c), y_c) < 0, y_c \ne 0$, so that the storage function $V_s(E_c x)$, $x \in R^n$, is positive definite and we assume that $V_s(E_c x)$, $x \in R^n$, is continuously differentiable.

Theorem 3.1: Let $Q_c \in S^{l_c}$, $S_c \in R^{l_c \times m_c}$, $R_c \in S^{m_c}$. If there exist function $V_s : R^n \to R$, $L_c : R^n \to R^{p_c}, \mathcal{W}_c : R^n \to R^{p_c \times m_c}$, such that $V_s(\cdot)$ is C^1 and positive definite, $V_s(0) = 0$, and, for all $x \in R^n$,

$$0 = V_s'(E_c x) f_c(x) - h_c^T(x) Q_c h_c(x) + L_c^T(x) L_c(x), \quad (7)$$

$$0 = \frac{1}{2} V_s'(E_c x) G_c(x) - h_c^T(x)(Q_c J_c(x) + S_c) + L_c^T(x)\mathcal{W}_c(x), \quad (8)$$

$$0 = R_c + S_c^T J_c(x) + J_c^T(x) S_c + J_c^T(x) Q_c J_c(x) - \mathcal{W}_c^T(x)\mathcal{W}_c(x), \quad (9)$$

then the nonlinear continuous singular system $\mathcal{G}$ given by (1), (2) is dissipative with respect to the quadratic supply rate $r_c(u_c, y_c) = y_c^T Q_c y_c + 2y_c^T S_c u_c + u_c^T R_c u_c$. If, in addition,

$$\mathcal{N}_c(x) = R_c + S_c^T J_c(x) + J_c^T(x) S_c + J_c^T(x) Q_c J_c(x) > 0, \quad x \in R^n, \quad (10)$$

and there exists a C^1 function $V_s : R^n \to R$ such that $V_s(\cdot)$ is positive definite, $V_s(0) = 0$, and for all $x \in R^n$,

$$\begin{aligned} 0 \ge\ & V_s'(E_c x) f_c(x) - h_c^T(x) Q_c h_c(x) \\ & + [\tfrac{1}{2} V_s'(E_c x) G_c(x) - h_c^T(x)(Q_c J_c(x) + S_c)]\mathcal{N}_c^{-1}(x) \\ & \cdot [\tfrac{1}{2} V_s'(E_c x) G_c(x) - h_c^T(x)(Q_c J_c(x) + S_c)]^T \end{aligned} \quad (11)$$

then $\mathcal{G}$ is dissipative with respect to the quadratic supply rate $r_c(u_c, y_c) = y_c^T Q_c y_c + 2y_c^T S_c u_c + u_c^T R_c u_c$.

Proof: For any admissible input $u_c(\cdot)$, $t_0, t \in R$ it follows from (7)–(9) that

$$\begin{aligned} & V_s(E_c x(t)) - V_s(E_c x(t_0)) = \int_{t_0}^t \dot{V}_s(E_c x(s))\mathrm{d}s \\ & \le \int_{t_0}^t \dot{V}_s(E_c x(s)) + [L_c(x(s)) + \mathcal{W}_c(x(s))u_c(s)]^T \\ & \quad \cdot [L_c(x(s)) + \mathcal{W}_c(x(s))u_c(s)]\mathrm{d}s \\ & = \int_{t_0}^t V_s'(E_c x(s))(f_c(x(s)) + G_c(x(s))u_c(s)) \\ & \quad + L_c^T(x(s))L_c(x(s)) + 2L_c^T(x(s))\mathcal{W}_c(x(s))u_c(s) \\ & \quad + u_c^T(s)\mathcal{W}_c^T(x(s))\mathcal{W}_c(x(s))u_c(s)\mathrm{d}s \\ & = h_c^T(x)Q_c h_c(x) + 2h_c^T(x(s))(S_c + Q_c J_c(x(s)))u_c(s) \\ & \quad + u_c^T(s)(J_c^T(x(s))Q_c J_c(x(s)) + S_c^T J_c(x(s)) \\ & \quad + J_c^T(x(s))S_c + R_c)u_c(s)\mathrm{d}s \\ & = \int_{t_0}^t [y_c^T(s)Q_c y_c(s) + 2y_c^T(s)S_c u_c(s) + u_c^T(s)R_c u_c(s)]\mathrm{d}s \\ & = \int_{t_0}^t r_c(u_c(s), y_c(s))\mathrm{d}s, \end{aligned} \quad (12)$$

where $x(t)$, $t \in [t_0, t]$, satisfies (1), and $\dot{V}_s(\cdot)$ denotes the total derivative of the storage function along the trajectories $x(t)$, $t \in [t_0, t]$ of (1).

Using (12) the result is immediate from Theorem 2.1.

To show (11) imply that $\mathcal{G}$ is dissipative with respect to quadratic supply rate r_c, note that (7)–(9) can be equivalently written as

$$\begin{bmatrix} \mathcal{A}_c(x) & \mathcal{B}_c(x) \\ \mathcal{B}_c^T(x) & \mathcal{C}_c(x) \end{bmatrix} = -\begin{bmatrix} L_c^T(x) \\ \mathcal{W}_c^T(x) \end{bmatrix} \begin{bmatrix} L_c(x) & \mathcal{W}_c(x) \end{bmatrix} \le 0, \quad x \in R^n \quad (13)$$

where $\mathcal{A}_c(x) = V_s'(E_c x) f_c(x) - h_c^T(x) Q_c$

Remark 3.1: Note that it follows from (6) that if the conditions in Theorem 3.1 are satisfied with (7) replaced by

$$0 = V_s'(E_c x) f_c(x) + \epsilon V_s(E_c x) - h_c^T(x) Q_c h_c(x) + L_c^T(x) L_c(x), \quad (14)$$

where $\epsilon > 0$, then the nonlinear continuous singular system $\mathcal{G}$ is exponentially dissipative. Similar remarks hold for Corollaries 3.1 and 3.2.

Using (7)–(9) it follows that, for $\hat{t} \geq t \geq 0$,

$$\begin{aligned}\int_t^{\hat{t}} r_c(u_c(s), y_c(s)) \mathrm{d}s = {} & V_s(E_c x(\hat{t})) - V_s(E_c x(t)) \\ & + \int_t^{\hat{t}} [L_c(x(s)) + \mathcal{W}_c(x(s)) u_c(s)]^T \\ & \cdot [L_c(x(s)) + \mathcal{W}_c(x(s)) u_c(s)] \mathrm{d}s \end{aligned} \quad (15)$$

which can be interpreted as a *generalized energy* balance equation where $V_s(E_c x(\hat{t})) - V_s(E_c x(t))$ is the stored or accumulated generalized energy of the nonlinear continuous singular system, the second path-dependent term on the right corresponds to the dissipated energy of the nonlinear continuous singular system over the continuous-time dynamics. Equivalently, it follows from Theorem 2.2 that (15) can be rewritten as

$$\begin{aligned}\dot{V}_s(E_c x(t)) = {} & r_c(u_c(t), y_c(t)) - [L_c(x(t)) + \mathcal{W}_c(x(t)) u_c(t)]^T \\ & \cdot [L_c(x(t)) + \mathcal{W}_c(x(t)) u_c(t)] \end{aligned} \quad (16)$$

which yields a set of generalized energy conservation equations. Specifically, (16) shows that the rate of change in generalized energy, or generalized power, is equal to the generalized system power minus the internal generalized system power dissipated.

Remark 3.2: Note that if $\mathcal{G}$ with $u_c(t) = 0$ and a C^1 positive definite, radially unbounded storage function is dissipative with respect to a quadratic supply rate where $Q_c \leq 0$, it follows that $\dot{V}_s(E_c x(t)) \leq y_c^T(t) Q_c y_c(t) \leq 0, t \geq 0$. Hence, the undisturbed $u_c(t) = 0$ nonlinear continuous singular system (1), (2) is Lyapunov stable. Alternatively, if $\mathcal{G}$ with $u_c(t) = 0$ and a C^1 positive-definite, radially unbounded storage function is exponentially dissipative and $Q_c \leq 0$, it follows that $\dot{V}_s(E_c x(t)) \leq -\epsilon V_s(E_c x(t)) + y_c^T(t) Q_c y_c(t) \leq -\epsilon V_s(E_c x(t))$, $t \geq 0$. Hence, the undisturbed nonlinear continuous singular system (1), (2) is asymptotically stable. If, in addition, there exist constants $\alpha, \beta > 0$ and $p \geq 1$ such that $\alpha \|x\|^p \leq V_s(E_c x) \leq \beta \|x\|^p$, $x \in R^n$, then the undisturbed nonlinear continuous singular system (1), (2) is exponentially stable.

Next, we provide necessary and sufficient conditions for the case where $\mathcal{G}$ given by (1), (2) is lossless with respect to a quadratic supply rate r_c.

Theorem 3.2: Let $Q_c \in S^{l_c}$, $S_c \in R^{l_c \times m_c}$, and $R_c \in S^{m_c}$. Then the nonlinear continuous singular system $\mathcal{G}$ given by (1), (2) is lossless with respect to the quadratic supply rate $r_c(u_c, y_c) = y_c^T Q_c y_c + 2 y_c^T S_c u_c + u_c^T R_c u_c$ if and only if there exists function $V_s(\cdot) : R^n \to R$ such that $V_s(\cdot)$ is C^1 and positive definite, $V_s(0) = 0$, and for all $x \in R^n$,

$$0 = V_s(E_c x) f_c(x) - h_c^T(x) Q_c h_c(x), \quad (17)$$

$$0 = \frac{1}{2} V_s'(E_c x) G_c(x) - h_c^T(x)(Q_c J_c(x) + S_c), \quad (18)$$

$$0 = R_c + S_c^T J_c(x) + J_c^T(x) S_c + J_c^T(x) Q_c J_c(x). \quad (19)$$

Proof: Sufficiency follows as in the proof of Theorem 3.1. To show necessity, suppose that the nonlinear continual singular system $\mathcal{G}$ is lossless with respect to the quadratic supply rate r_c. Then, it follows from Theorem 2.2 that

$$V_s(E_c x(\hat{t})) - V_s(E_c x(t)) = \int_t^{\hat{t}} r_c(u_c(s), y_c(s)) \mathrm{d}s. \quad (20)$$

Now, dividing (20) by $t - \hat{t}$ letting $\hat{t} \to t$, (20) is equivalent to

$$\begin{aligned}\dot{V}_s(E_c x(t)) &= V_s'(E_c x(t))[f_c(x(t)) + G_c(x(t)) u_c(t)] \\ &= r_c(u_c(t), y_c(t)). \end{aligned} \quad (21)$$

Next, with $t = 0$, it follows from (21) that

$$V_s'(E_c x_0)[f_c(x_0) + G_c(x_0) u_c(0)] = r_c(u_c(0), y_c(0)), \quad x_0 \in R^n, \quad u_c(0) \in R^{m_c}. \quad (22)$$

Since x_0 is arbitrary, it follows that

$$\begin{aligned} V_s'(E_c x)[f_c(x) + G_c(x) u_c] &= y_c^T Q_c y_c + 2 y_c^T S_c u_c + u_c^T R_c u_c \\ &= h_c(x) Q_c h_c(x) + 2 h_c^T(x)(Q_c J_c(x) + S_c) u_c \\ &\quad + u_c^T (R_c + S_c^T J_c(x) + J_c^T(x) S_c + J_c^T(x) Q_c J_c(x)) u_c, \\ &\quad x \in R^n, \quad u_c \in R^{m_c}. \end{aligned} \quad (23)$$

Now, equating coefficients of equal powers yields (17)-(19).

Next, we provide two definitions of nonlinear continuous singular system which are dissipative (resp., exponentially dissipative) with respect to the supply rates of a specific form.

Definition 3.1: A system $\mathcal{G}$ of the form (1), (2) with $m_c = l_c$ is *passive* (resp., *exponentially passive) if $\mathcal{G}$ is dissipative with respect to the supply rate* $r_c(u_c, y_c) = 2 u_c^T y_c$.

Definition 3.2: A system $\mathcal{G}$ of the form (1), (2) is *nonexpansive (resp.,* exponentially nonexpansive*) if $\mathcal{G}$ is dissipative with respect to the supply rate* $r_c(u_c, y_c) = (\gamma_c u_c^T u_c - y_c^T y_c)$, *where* $\gamma_c > 0$ *is given.*

The following results present the nonlinear versions of the Kalman-Yakubovich-Popov positive real lemma and the bounded real lemma for nonlinear continuous singular systems $\mathcal{G}$ of the form (1), (2).

Corollary 3.1: Consider the nonlinear continuous singular system $\mathcal{G}$ given by (1), (2). If there exist function $V_s : R^n \to R$, $L_c : R^n \to R^{p_c}, \mathcal{W}_c : R^n \to R^{p_c \times m_c}$, such that $V_s(\cdot)$ is C^1 and positive definite, $V_s(0) = 0$, and, for all $x \in R^n$,

$$0 = V_s'(E_c x) f_c(x) + L_c^T(x) L_c(x), \quad (24)$$

$$0 = \frac{1}{2} V_s'(E_c x) G_c(x) - h_c^T(x) + L_c^T(x) \mathcal{W}_c(x), \quad (25)$$

$$0 = J_c(x) + J_c^T(x) - \mathcal{W}_c^T(x) \mathcal{W}_c(x), \quad (26)$$

then the nonlinear continuous singular system $\mathcal{G}$ given by (1), (2) is passive. If, in addition, $J_c(x) + J_c^T(x) > 0, x \in R^n$, and there exists a C^1 function $V_s : R^n \to R$ such that $V_s(\cdot)$ is positive definite, $V_s(0) = 0$, and for all $x \in R^n$,

$$0 \geq V_{\mathrm{s}}'(E_{\mathrm{c}}x)f_{\mathrm{c}}(x) + [\frac{1}{2}V_{\mathrm{s}}'(E_{\mathrm{c}}x)G_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)] \cdot [J_{\mathrm{c}}(x) + J_{\mathrm{c}}^{\mathrm{T}}(x)]^{-1}[\frac{1}{2}V_{\mathrm{s}}'(E_{\mathrm{c}}x)G_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)], \quad (27)$$

then $\mathcal{G}$ is passive.

Proof: The result is a direct consequence of Theorem 3.1 with $l_{\mathrm{c}} = m_{\mathrm{c}}, Q_{\mathrm{c}} = 0, S_{\mathrm{c}} = I_{m_{\mathrm{c}}}, R_{\mathrm{c}} = 0$. Specifically, with $\kappa_{\mathrm{c}}(y_{\mathrm{c}}) = -y_{\mathrm{c}}$ it follows that $r_{\mathrm{c}}(\kappa_{\mathrm{c}}, y_{\mathrm{c}}) = -2y_{\mathrm{c}}^{\mathrm{T}}y_{\mathrm{c}}$, $y_{\mathrm{c}} \neq 0$, so that all of the assumptions of Theorem 3.1 are satisfied.

Corollary 3.2: Consider the nonlinear continuous singular system (1), (2). If there exist function $V_{\mathrm{s}} : R^n \to R$, $L_{\mathrm{c}} : R^n \to R^{p_{\mathrm{c}}}, \mathcal{W}_{\mathrm{c}} : R^n \to R^{p_{\mathrm{c}} \times m_{\mathrm{c}}}$, such that $V_{\mathrm{s}}(\cdot)$ is C^1 and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for all $x \in R^n$,

$$0 = V_{\mathrm{s}}'(E_{\mathrm{c}}x)f_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)h_{\mathrm{c}}(x) + L_{\mathrm{c}}^{\mathrm{T}}(x)L_{\mathrm{c}}(x), \quad (28)$$

$$0 = \frac{1}{2}V_{\mathrm{s}}'(E_{\mathrm{c}}x)G_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)J_{\mathrm{c}}(x) + L_{\mathrm{c}}^{\mathrm{T}}(x)\mathcal{W}_{\mathrm{c}}(x), \quad (29)$$

$$0 = \gamma_{\mathrm{c}}I_{m_{\mathrm{c}}} - J_{\mathrm{c}}(x)^{\mathrm{T}}J_{\mathrm{c}}(x) - \mathcal{W}_{\mathrm{c}}^{\mathrm{T}}(x)\mathcal{W}_{\mathrm{c}}(x), \quad (30)$$

then the nonlinear continuous singular system $\mathcal{G}$ given by (1), (2) is nonexpansive. If, in addition, $\gamma_{\mathrm{c}}I_{m_{\mathrm{c}}} - J_{\mathrm{c}}^{\mathrm{T}}(x)J_{\mathrm{c}}(x) > 0, x \in R^n$, and there exists a C^1 function $V_{\mathrm{s}} : R^n \to R$ such that $V_{\mathrm{s}}(\cdot)$ is positive definite, $V_{\mathrm{s}}(0) = 0$, and for all $x \in R^n$,

$$0 \geq V_{\mathrm{s}}'(E_{\mathrm{c}}x)f_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)h_{\mathrm{c}}(x) + [\frac{1}{2}V_{\mathrm{s}}'(E_{\mathrm{c}}x)G_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)J_{\mathrm{c}}(x)][\gamma_{\mathrm{c}}I_{m_{\mathrm{c}}} - J_{\mathrm{c}}^{\mathrm{T}}(x)J_{\mathrm{c}}(x)]^{-1} \cdot [\frac{1}{2}V_{\mathrm{s}}'(E_{\mathrm{c}}x)G_{\mathrm{c}}(x) + h_{\mathrm{c}}^{\mathrm{T}}(x)J_{\mathrm{c}}(x)]^{\mathrm{T}} \quad (31)$$

then $\mathcal{G}$ is nonexpansive.

Proof: The result is a direct consequence of Theorem 3.1 with $Q_{\mathrm{c}} = -l_{\mathrm{c}}, S_{\mathrm{c}} = 0, R_{\mathrm{c}} = \gamma_{\mathrm{c}}I_{m_{\mathrm{c}}}$. Specifically, with $\kappa_{\mathrm{c}}(y_{\mathrm{c}}) = -\frac{1}{2\gamma_{\mathrm{c}}}y_{\mathrm{c}}$ it follows that $r_{\mathrm{c}}(\kappa_{\mathrm{c}}, y_{\mathrm{c}}) = -\frac{3}{4}y_{\mathrm{c}}^{\mathrm{T}}y_{\mathrm{c}}$, $y_{\mathrm{c}} \neq 0$, so that all of the assumptions of Theorem 3.1 are satisfied.

Next, we provide necessary and sufficient conditions for dissipativity of a nonlinear continuous singular system $\mathcal{G}$, of the form (1), (2).

Theorem 3.3: Let $Q_{\mathrm{c}} \in S^{l_{\mathrm{c}}}$, $S_{\mathrm{c}} \in R^{l_{\mathrm{c}} \times m_{\mathrm{c}}}$, $R_{\mathrm{c}} \in S^{m_{\mathrm{c}}}$. Then the nonlinear continuous singular system $\mathcal{G}$ given by (1), (2) is dissipative with respect to the supply rate $r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) = y_{\mathrm{c}}^{\mathrm{T}}Q_{\mathrm{c}}y_{\mathrm{c}} + 2y_{\mathrm{c}}^{\mathrm{T}}S_{\mathrm{c}}u_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}}R_{\mathrm{c}}u_{\mathrm{c}}$ if and only if there exist function $V_{\mathrm{s}} : R^n \to R$, $L_{\mathrm{c}} : R^n \to R^{p_{\mathrm{c}}}, \mathcal{W}_{\mathrm{c}} : R^n \to R^{p_{\mathrm{c}} \times m_{\mathrm{c}}}$, such that $V_{\mathrm{s}}(\cdot)$ is C^1 and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for all $x \in R^n$,

$$0 = V_{\mathrm{s}}'(E_{\mathrm{c}}x)f_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)Q_{\mathrm{c}}h_{\mathrm{c}}(x) + L_{\mathrm{c}}^{\mathrm{T}}(x)L_{\mathrm{c}}(x), \quad (32)$$

$$0 = \frac{1}{2}V_{\mathrm{s}}'(E_{\mathrm{c}}x)G_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)(Q_{\mathrm{c}}J_{\mathrm{c}}(x) + S_{\mathrm{c}}) + L_{\mathrm{c}}^{\mathrm{T}}(x)\mathcal{W}_{\mathrm{c}}(x), \quad (33)$$

$$0 = R_{\mathrm{c}} + S_{\mathrm{c}}^{\mathrm{T}}J_{\mathrm{c}}(x) + J_{\mathrm{c}}^{\mathrm{T}}(x)S_{\mathrm{c}} + J_{\mathrm{c}}^{\mathrm{T}}(x)Q_{\mathrm{c}}J_{\mathrm{c}}(x) - \mathcal{W}_{\mathrm{c}}^{\mathrm{T}}(x)\mathcal{W}_{\mathrm{c}}(x). \quad (34)$$

Proof: Sufficiency follows from Theorem 3.1. Necessity follows from Theorem 2.2 using a similar construction as in the proof of Theorem 3.2.

Finally, we present two key results on linearization of nonlinear continuous singular systems. For these results, we assume there exists function $\kappa_{\mathrm{c}} : R^{l_{\mathrm{c}}} \to R^{m_{\mathrm{c}}}$ such that $r_{\mathrm{c}}(\kappa_{\mathrm{c}}, y_{\mathrm{c}}) < 0$, $y_{\mathrm{c}} \neq 0$, and the available storage $V_{\mathrm{a}}(x)$, $x \in R^n$, is a C^3 function.

Theorem 3.4: Let $Q_{\mathrm{c}} \in S^{l_{\mathrm{c}}}$, $S_{\mathrm{c}} \in R^{l_{\mathrm{c}} \times m_{\mathrm{c}}}$, $R_{\mathrm{c}} \in S^{m_{\mathrm{c}}}$, and suppose that the nonlinear continuous singular system $\mathcal{G}$ given by (1), (2) is dissipative with respect to the quadratic supply rate $r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) = y_{\mathrm{c}}^{\mathrm{T}}Q_{\mathrm{c}}y_{\mathrm{c}} + 2y_{\mathrm{c}}^{\mathrm{T}}S_{\mathrm{c}}u_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}}R_{\mathrm{c}}u_{\mathrm{c}}$. Then there exists matrices $P \in R^{n \times n}$, $L_{\mathrm{c}} : R^n \to R^{p_{\mathrm{c}}}, W_{\mathrm{c}} : R^n \to R^{p_{\mathrm{c}} \times m_{\mathrm{c}}}$, with P nonnegative definite, such that

$$0 = A_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{c}} - C_{\mathrm{c}}^{\mathrm{T}}Q_{\mathrm{c}}C_{\mathrm{c}} + L_{\mathrm{c}}^{\mathrm{T}}L_{\mathrm{c}}, \quad (35)$$

$$0 = PB_{\mathrm{c}} - C_{\mathrm{c}}^{\mathrm{T}}(Q_{\mathrm{c}}D_{\mathrm{c}} + S_{\mathrm{c}}) + L_{\mathrm{c}}^{\mathrm{T}}W_{\mathrm{c}}, \quad (36)$$

$$0 = R_{\mathrm{c}} + S_{\mathrm{c}}^{\mathrm{T}}D_{\mathrm{c}} + D_{\mathrm{c}}^{\mathrm{T}}S_{\mathrm{c}} + D_{\mathrm{c}}^{\mathrm{T}}Q_{\mathrm{c}}D_{\mathrm{c}}(x) - W_{\mathrm{c}}^{\mathrm{T}}W_{\mathrm{c}}, \quad (37)$$

where

$$A_{\mathrm{c}} = \frac{\partial f_{\mathrm{c}}}{\partial x}|_{x=0},\ B_{\mathrm{c}} = G_{\mathrm{c}}(0),\ C_{\mathrm{c}} = \frac{\partial h_{\mathrm{c}}}{\partial x}|_{x=0},\ D_{\mathrm{c}} = J_{\mathrm{c}}(0). \quad (38)$$

If, in addition, $(A_{\mathrm{c}}, C_{\mathrm{c}})$ is observable, then $P > 0$.

Proof: First note that since $\mathcal{G}$ is dissipative with respect to the supply rate r_{c} it follows from Theorem 2.2 that there exists a storage function $V_{\mathrm{s}} : R^n \to R$ such that

$$V_{\mathrm{s}}(E_{\mathrm{c}}x(\hat{t})) - V_{\mathrm{s}}(E_{\mathrm{c}}x(t)) \leq \int_t^{\hat{t}} r_{\mathrm{c}}(u_{\mathrm{c}}(s), y_{\mathrm{c}}(s))\mathrm{d}s. \quad (39)$$

Now, dividing (39) by $\hat{t} - t$, (39) is equivalent to

$$\dot{V}_{\mathrm{s}}(E_{\mathrm{c}}x(t)) = V_{\mathrm{s}}(E_{\mathrm{c}}x(t))[f_{\mathrm{c}}(x(t)) + G_{\mathrm{c}}(x(t))u_{\mathrm{c}}(t)] \leq r_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)). \quad (40)$$

Next, with $t = 0$, it follows that

$$V_{\mathrm{s}}'(E_{\mathrm{c}}x_0)[f_{\mathrm{c}}(x_0) + G_{\mathrm{c}}(x_0)u_{\mathrm{c}}(0)] \leq r_{\mathrm{c}}(u_{\mathrm{c}}(0), y_{\mathrm{c}}(0)), \quad x_0 \in R^n, \quad u_{\mathrm{c}}(0) \in R^{m_{\mathrm{c}}}. \quad (41)$$

Since $x_0 \in R^n$ is arbitrary, it follows that

$$V_{\mathrm{s}}'(E_{\mathrm{c}}x)[f_{\mathrm{c}}(x) + G_{\mathrm{c}}(x)u_{\mathrm{c}}] \leq r_{\mathrm{c}}(u_{\mathrm{c}}, h_{\mathrm{c}}(x) + J_{\mathrm{c}}(x)u_{\mathrm{c}}), \quad x \in R^n, \quad u_{\mathrm{c}} \in R^{m_{\mathrm{c}}}. \quad (42)$$

Next, it follows from (42) that there exists smooth function $d_{\mathrm{c}} : R^n \times R^{m_{\mathrm{c}}} \to R$ such that $d_{\mathrm{c}}(x, u_{\mathrm{c}}) \geq 0$, $d_{\mathrm{c}}(0, 0) = 0$ and

$$0 = V_{\mathrm{s}}'(E_{\mathrm{c}}x)[f_{\mathrm{c}}(x) + G_{\mathrm{c}}(x)u_{\mathrm{c}}] - r_{\mathrm{c}}(u_{\mathrm{c}}, h_{\mathrm{c}}(x) + J_{\mathrm{c}}(x)u_{\mathrm{c}}) + d_{\mathrm{c}}(x, u_{\mathrm{c}}), \quad x \in R^n, \quad u_{\mathrm{c}} \in R^{m_{\mathrm{c}}}. \quad (43)$$

Now, expanding $V_{\mathrm{s}}(\cdot)$, $d_{\mathrm{c}}(\cdot, \cdot)$ via a Taylor series expansion about $x = 0, u_{\mathrm{c}} = 0$ and using the fact that $V_{\mathrm{s}}(\cdot), d_{\mathrm{c}}(\cdot, \cdot)$ are nonnegative definite and $V_{\mathrm{s}}(0) = 0$, $d_{\mathrm{c}}(0, 0) = 0$, it follows there exist matrices $P \in R^{n \times n}$, $L_{\mathrm{c}} : R^n \to R^{p_{\mathrm{c}}}, W_{\mathrm{c}} : R^n \to R^{p_{\mathrm{c}} \times m_{\mathrm{c}}}$, with P nonnegative definite, such that

$$V_{\mathrm{s}}(E_{\mathrm{c}}x) = x^{\mathrm{T}}E_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}}x + V_{\mathrm{r}}(E_{\mathrm{c}}x), \quad (44)$$

$$d_{\mathrm{c}}(x, u_{\mathrm{c}}) = (L_{\mathrm{c}}x + W_{\mathrm{c}}u_{\mathrm{c}})^{\mathrm{T}}(L_{\mathrm{c}}x + W_{\mathrm{c}}u_{\mathrm{c}}) + d_{\mathrm{cr}}(x, u_{\mathrm{c}}), \quad (45)$$

where $V_{\mathrm{r}} : R^n \to R$, $d_{\mathrm{cr}} : R^n \times R^{m_{\mathrm{c}}} \to R$ contain the higher-order terms of $V_{\mathrm{s}}(\cdot)$ and $d_{\mathrm{c}}(\cdot, \cdot)$, respectively. Next, let $f_{\mathrm{c}}(x) = A_{\mathrm{c}}x + f_{\mathrm{cr}}(x), h_{\mathrm{c}}(x) = C_{\mathrm{c}}x + h_{\mathrm{cr}}(x)$, where $f_{\mathrm{cr}}(\cdot), h_{\mathrm{cr}}(\cdot)$ contain the nonlinear terms of $f_{\mathrm{c}}(x), h_{\mathrm{c}}(x)$, respectively, and let $G_{\mathrm{c}}(x) = B_{\mathrm{c}} + G_{\mathrm{cr}}$,

$J_c(x) = D_c + J_c$, where G_{cr}, J_{cr} contain the non-constant terms of $G_c(x), J_c(x)$, respectively. Using the above expressions, (43) can be written as

$$0 = 2x^T P(E_c^T A_c x + B_c u_c) - (x^T C_c^T Q_c C_c x + 2x^T C_c^T Q_c D_c u_c + u_c^T D_c^T Q_c D_c u_c + 2x^T C_c^T S_c u_c + 2u_c^T D_c^T S_c u_c + u_c^T R_c u_c) + (L_c x + W_c u_c)^T \cdot (L_c x + W_c u_c) + (x, u_c) \quad (46)$$

where (x, u_c) is such that

$$\lim_{\|x\|^2+\|u_c\|^2 \to 0} \frac{|(x, u_c)|}{\|x\|^2 + \|u_c\|^2} = 0. \quad (47)$$

Now, viewing (46) as the Taylor series expansion of (43) about $x = 0$ and $u_c = 0$ it follows that

$$0 = x^T(A_c^T P E_c + E_c^T P A_c - C_c^T Q_c C_c + L_c^T L_c)x + 2x^T(PB_c - C_c^T S_c - C_c^T Q_c D_c + L_c^T W_c)u_c + u_c^T(W_c^T W_c - D_c^T Q_c D_c - D_c^T S_c - S_c^T D_c - R_c)u_c, \quad x \in R^n, \quad u_c \in R^{m_c}. \quad (48)$$

Next, equating coefficients of equal powers in (48) yields (35)–(37).

Finally, to show that $P > 0$ in the case where (A_c, C_c) is observable, note that it follows from Theorem 3.1 and (35)–(37) that the linearized system $\mathcal{G}$ with storage function $V_s(x) = x^T E_c^T P E_c x$ is dissipative with respect to the quadratic supply rate $r_c(u_c, y_c)$. Now, the positive definiteness of P follows from Theorem 2.2.

IV. Specialization to Continuous Linear Singular Systems

In this section we specialize the results of Section III to the case of linear continuous singular systems. Specifically, setting $f_c(x) = A_c x, G_c(x) = B_c, h_c(x) = C_c x, J_c(x) = D_c$, the nonlinear continuous singular system (1), (2) specializes to

$$E_c \dot{x}(t) = A_c x(t) + B_c u_c(t), \qquad x(0) = x_0, \quad (49)$$

$$y_c(t) = C_c x(t) + D_c u_c(t), \quad (50)$$

where $A_c \in R^{n\times n}$, $B_c \in R^{n\times m_c}$, $C_c \in R^{l_c \times n}$.

Theorem 4.1: Let $Q_c \in S^{l_c}$, $S_c \in R^{l_c \times m_c}$, $R_c \in S^{m_c}$, consider the nonlinear continuous singular system $\mathcal{G}$ given by (49), (50), and assume $\mathcal{G}$ is minimal. Then the following statements are equivalent:

i) $\mathcal{G}$ is dissipative with respect to the quadratic supply rate $r_c(u_c, y_c) = y_c^T Q_c y_c + 2y_c^T S_c u_c + u_c^T R_c u_c$.

ii) There exist matrices $P \in R^{n\times n}$, $L_c \in R^{p_c \times n}, W_c \in R^{p_c \times m_c}$, with P positive definite, such that (35)–(37) are satisfied.

If, in addition, $R_c + S_c^T D_c + D_c^T S_c + D_c^T Q_c D_c > 0$, where P satisfies (35)–(37), then $\mathcal{G}$ is dissipative with respect to the quadratic supply rate $r_c(u_c, y_c) = y_c^T Q_c y_c + 2y_c^T S_c u_c + u_c^T R_c u_c$ if and only if there exists an $n \times n$ positive-definite matrix P such that

$$0 \geq A_c^T P E_c + E_c^T P A_c - C_c^T Q_c C_c + [PB_c - C_c^T(Q_c D_c + S_c)][R_c + S_c^T D_c + D_c^T S_c + D_c^T Q_c D_c]^{-1} [PB_c - C_c^T(Q_c D_c + S_c)]^T \quad (51)$$

Proof: The fact that ii) implies i) follows from Theorem 3.1 with $f_c(x) = A_c x$, $G_c(x) = B_c$, $h_c(x) = C_c x$, $J_c(x) = D_c$, $V_s = x^T E_c^T P E_c x$, $L_c(x) = L_c x$, $\mathcal{W}_c(x) = W_c$. To show that i) implies ii), note that if the linear continuous singular system given by (49), (50) is dissipative, then it follows from Theorem 3.4 with $f_c(x) = A_c x$, $G_c(x) = B_c$, $h_c(x) = C_c x$, $J_c(x) = D_c$ that there exists matrices $P \in R^{n\times n}$, $L_c \in R^{p_c \times n}, W_c \in R^{p_c\times m_c}$, with P positive definite, such that (35)–(37) are satisfied. Finally, (51) follow from (11) and Theorem 3.4 with the linearization given above.

Remark 4.1: Note that the proof of Theorem 4.1 relies on Theorem 3.4 which *a priori* assumes that the storage function $V_s(x), x \in R^n$ is C^3. Unlike linear, time-invariant dissipative dynamical systems with continuous flows [2], there does not always exists a smooth (i.e. C^∞) storage function $V(E_c x)$, $x \in R^n$, for linear dissipative singular dynamical systems.

Remark 4.2: Note that (35–(37) are equivalent to

$$\begin{bmatrix} \mathcal{A}_c & \mathcal{B}_c \\ \mathcal{B}_c^T & \mathcal{D}_c \end{bmatrix} = \begin{bmatrix} L_c^T \\ W_c^T \end{bmatrix} \begin{bmatrix} L_c & W_c \end{bmatrix} \geq 0, \quad (52)$$

where $\mathcal{A}_c = -A_c^T P E_c - E_c^T P A_c + C_c^T Q_c C_c$, $\mathcal{B}_c = -PB_c + C_c^T(Q_c D_c + S_c)$, $\mathcal{D}_c = R_c + S_c^T D_c + D_c^T S_c + D_c^T Q_c D_c$. Hence dissipativity of linear continuous singular system with respect to quadratic supply rates can be characterized via Linear Matrix Inequalities (LMI's) [11]. Similar remarks hold for the passivity and nonexpansivity results given in Corollaries 4.1 and 4.2, respectively.

The following results present generalizations of the positive real lemma and the bounded real lemma for linear continuous singular systems, respectively.

Corollary 4.1: Consider the linear continuous singular system $\mathcal{G}$ given by (49), (50) with $m_c = l_c$ and assume $\mathcal{G}$ is minimal. Then the following statements are equivalent:

i) $\mathcal{G}$ is passive

ii) There exists matrices $P \in R^{n\times n}$, $L_c \in R^{p_c\times n}, W_c \in R^{p_c\times m_c}$, with P positive definite, such that

$$0 = A_c^T P E_c + E_c^T P A_c + L_c^T L_c, \quad (53)$$

$$0 = PB_c - C_c^T + L_c^T W_c, \quad (54)$$

$$0 = D_c + D_c^T - W_c^T W_c. \quad (55)$$

If, in addition $D_c + D_c^T > 0$ where P satisfies (53)–(55), then $\mathcal{G}$ is passive if and only if there exists an $n \times n$ positive-definite matrix P such that

$$0 \geq A_c^T P E_c + E_c^T P A_c + (PB_c - C_c^T)(D_c + D_c^T)^{-1} \cdot (PB_c - C_c^T)^T. \quad (56)$$

Proof: The result is a direct consequence of Theorem 4.1 with $m_c = l_c$, $Q_c = 0$, $S_c = I_{m_c}$, and $R_c = 0$.

Remark 4.3: Equations (53)–(55) are generalization of the equations appearing in the continuous time positive real lemma [12] used to characterize positive realness for continuous linear systems in the state space to singular systems. Similar remark hold for Corollary 4.2.

Corollary 4.2: Consider the linear continuous singular system $\mathcal{G}$ given by (49), (50) and assume $\mathcal{G}$ is minimal. Then the following statements are equivalent:

i) $\mathcal{G}$ is nonexpansive.

ii) There exists matrices $P \in R^{n\times n}$, $L_{\rm c} \in R^{p_{\rm c}\times n}, W_{\rm c} \in R^{p_{\rm c}\times m_{\rm c}}$, with P positive definite, such that

$$0 = A_{\rm c}^{\rm T} P E_{\rm c} + E_{\rm c}^{\rm T} P A_{\rm c} + C_{\rm c}^{\rm T} C_{\rm c} + L_{\rm c}^{\rm T} L_{\rm c}, \quad (57)$$

$$0 = P B_{\rm c} - C_{\rm c}^{\rm T} D_{\rm c} + L_{\rm c}^{\rm T} W_{\rm c}, \quad (58)$$

$$0 = \gamma_{\rm c} I_{m_{\rm c}} - D_{\rm c}^{\rm T} D_{\rm c} - W_{\rm c}^{\rm T} W_{\rm c}. \quad (59)$$

If, in addition $\gamma_{\rm c}^2 I_{m_{\rm c}} - D_{\rm c}^{\rm T} D_{\rm c} > 0$ where P satisfies (57)–(59), then $\mathcal{G}$ is nonexpansive if and only if there exists an $n \times n$ positive-definite matrix P such that

$$0 \geq A_{\rm c}^{\rm T} P E_{\rm c} + E_{\rm c}^{\rm T} P A_{\rm c} + (P B_{\rm c} + C_{\rm c}^{\rm T} D_{\rm c})(\gamma_{\rm c}^2 I_{m_{\rm c}} + D_{\rm c}^{\rm T} D_{\rm c})^{-1}(P B_{\rm c} - C_{\rm c}^{\rm T} D_{\rm c})^{\rm T} + C_{\rm c}^{\rm T} C_{\rm c}. \quad (60)$$

Proof: The result is direct consequence of Theorem 4.1 with $Q_{\rm c} = I_{l_{\rm c}}$, $S_{\rm c} = 0$, and $R_{\rm c} = \gamma_{\rm c}^2 I_{m_{\rm c}}$.

Remark 4.4: It follows from Remark 4.4 that if (53) and (57) are replaced, respectively, by

$$0 = A_{\rm c}^{\rm T} P E_{\rm c} + E_{\rm c}^{\rm T} P A_{\rm c} + \epsilon P + L_{\rm c}^{\rm T} L_{\rm c}, \quad (61)$$

$$0 = A_{\rm c}^{\rm T} P E_{\rm c} + E_{\rm c}^{\rm T} P A_{\rm c} + C_{\rm c}^{\rm T} C_{\rm c} + \epsilon P + L_{\rm c}^{\rm T} L_{\rm c}, \quad (62)$$

where $\epsilon > 0$, then (61), (54), (55) provide necessary and sufficient conditions for exponential passivity, while (62), (58), (59) provide necessary and sufficient conditions for exponential nonexpansivity. These conditions present generalizations of the strict positive real lemma and the strict bounded real lemma for linear continuous singular systems, respectively.

V. Conclusion

In this paper we have extended the classical notions of dissipativity theory to nonlinear singular dynamical systems. Specifically, the concept of storage functions and supply rates are extended to singular dynamical systems providing a generalized system energy interpretation in terms of stored energy and dissipated energy over the continuous-time dynamics. Furthermore, extended Kalman-Yakubovich-Popov algebraic conditions in terms of the singular system dynamics for characterizing dissipativeness via system storage functions are derived. In the case of quadratic supply rates involving net system power and input-output energy, these results provide generalizations of the classical notions of passivity and nonexpansivity. In addition, for linear singular systems, the proposed results provide a generalization of the positive real lemma and the bounded real lemma.

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