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A noise–compensated estimation scheme for AR processes

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Abstract— **This paper deals with the problem of identifying autoregressive models in presence of additive measurement noise. A new approach, based on some theoretical results concerning the so-called dynamic Frisch scheme, is proposed. This method takes advantage of both low and high order Yule-Walker equations and allows to identify the AR parameters and the driving and output noise variances in a congruent way since the estimates assure the positive definiteness of the autocorrelation matrix of the AR process. Simulation results are reported to show the effectiveness of the proposed procedure and compare its performance with those of other identification methods.**

I. INTRODUCTION

Signals are frequently modelled as autoregressive (AR) processes in several engineering applications including, for example, spectral estimation, speech analysis, geoscience and time series forecasting. In many practical situations, however, the signals are corrupted by noise. In these cases, classical AR identification methods give misleading results; in fact it can be proved that the estimated AR poles are biased toward the center of the unit circle, leading thus to a smoothed spectrum [1].

Several approaches have been developed to recover the AR parameters from noisy measurements. Some of these approaches consist in standard ARMA parameter estimators, like prediction errors methods [2], [3]. It can be easily shown, in fact, that noisy AR processes admit an equivalent ARMA representation [4].

Another approach consists in solving the so–called high–order Yule–Walker (HOYW) equations [5], that constitute a set of linear equations. This method requires the knowledge of the autocorrelation function for higher lags that, however, are usually poorly estimated. To compensate the estimation errors, an overdetermined set of HOYW equations is often considered [6]. In [7]–[9] it has been recognized that better results can be obtained by relying on both low and high order YW equations. Starting from this set up, in [10] a new method has been proposed, related to signal/noise subspace techniques. This approach uses a modified set of low and high order YW equations leading to a quadratic eigenvalue problem whose solution gives an estimate of the AR parameters and of the noise variance.

Another class of identification methods is based on the bias–compensation principle [11]–[14]. These techniques are based on iterative least–squares procedures involving, at each step, the estimation of both AR parameters and output noise variance. A unified explanation of various bias–compensated least–squares (BCLS) schemes is reported in [13].

The new approach proposed in [15] allows to estimate the AR parameters, the driving noise variance and the variance of the additive noise. The method is based on some theoretical results concernig the dynamic Frisch scheme and makes use of the shift properties of time–invariant dynamic systems. As shown in [15], this approach yields better estimates than those obtained with BCLS techniques, especially in presence of low signal to noise ratios.

Starting from the same background, this paper introduces a more effective identification procedure that takes advantage of both low and high order Yule–Walker equations. This method, as the previous one, allows to identify the AR parameters, the driving noise variance and the output noise variance in a congruent way since the estimates assure the positive definiteness of the autocorrelation matrix of the AR process. The consistency of the new scheme is guaranteed provided that a minimum number of HOYW equations is considered. Numerical simulations show that the proposed algorithm significantly outperforms the approach introduced in [15] and gives better results than those obtained with other identification approaches.

The paper is organized as follows. Section II reports the formulation of the problem. Section III is focused on some important asymptotic properties of noisy AR models while Section IV describes the new identification algorithm. Section V shows how the proposed procedure can be used for the estimation of the model order. The effectiveness of the new approach is shown in Section VI where the results of Monte Carlo simulations and a comparison with other identification methods are reported. Section VII concludes the paper.

II. PROBLEM STATEMENT

Consider a noisy autoregressive model of order n described by the equations

$$x(t) = \alpha_1 x(t-1) + \cdots + \alpha_n x(t-n) + e(t), \quad (1)$$
$$y(t) = x(t) + w(t), \quad (2)$$

where $x(t)$ is the output of the noise–free AR model, driven by the input $e(t)$ while $y(t)$ is the available observation, affected by the noise process $w(t)$. The following assumptions will be made in the sequel.

A1. $e(t)$ and $w(t)$ are zero–mean white processes, mutually uncorrelated, with unknown variances σ_e^{2*} and σ_w^{2*}.
A2. $e(t)$, $x(t)$ and $w(t)$ are ergodic processes.
A3. The system order, n, is known.

By defining the vectors

$$\begin{aligned}\varphi_x(t) &= [x(t-n)\,\dots\,x(t-1)\;x(t)]^T, &(3)\\ \varphi_e(t) &= [\underbrace{0\,\dots\,0}_{n}\;e(t)]^T, &(4)\\ \varphi_y(t) &= [y(t-n)\,\dots\,y(t-1)\;y(t)]^T, &(5)\\ \varphi_w(t) &= [w(t-n)\,\dots\,w(t-1)\;w(t)]^T, &(6)\end{aligned}$$

and the parameter vector

$$\theta^* = \left[\,\alpha_n\;\cdots\;\alpha_1\;-1\,\right]^T, \qquad (7)$$

it is possible to represent model (1)–(2) in the form

$$\left(\varphi_x^T(t) - \varphi_e^T(t)\right)\,\theta^* = 0, \qquad (8)$$

$$\varphi_y(t) = \varphi_x(t) + \varphi_w(t), \qquad (9)$$

that will be used in the next section.

The problem to be solved can be stated as follows.

Problem 1: Estimate the AR parameters $\alpha_1, \dots, \alpha_n$ and the variances σ_e^{2*}, σ_w^{2*} on the basis of N noisy observations $y(1), y(2), \dots, y(N)$, generated by model (1)–(2) under assumptions A1–A3.

III. SOME ASYMPTOTIC PROPERTIES OF NOISY AR MODELS

Define now the covariance matrix

$$\begin{aligned}\bar{R}_x &= E\left[\varphi_x(t)\left(\varphi_x^T(t) - \varphi_e^T(t)\right)\right]\\ &= R_x - \text{diag}\;[0\;\cdots\;0\;\sigma_e^{2*}], \qquad (10)\end{aligned}$$

where $E[\cdot]$ denotes mathematical expectation. Starting from (8) it is possible to write the set of $n+1$ relations

$$\bar{R}_x\,\theta^* = 0, \qquad (11)$$

i.e.

$$\begin{bmatrix} r_x(0) & r_x(1) & \cdots & r_x(n) \\ r_x(1) & r_x(0) & \cdots & r_x(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ r_x(n) & r_x(n-1) & \cdots & r_x(0)-\sigma_e^{2*} \end{bmatrix} \theta^* = 0, \qquad (12)$$

where $r_x(k) = r_x(-k) = E\,[x(t)\,x(t-k)]$. The first n relations of (12) are the well–known Yule–Walker equations. In the present case, however, only the autocorrelations of $y(t)$ are available. From (9) and assumption A1 it follows that

$$R_y = E\,[\varphi_y(t)\,\varphi_y^T(t)] = R_x + \sigma_w^{2*}\,I_{n+1} = \bar{R}_x + \tilde{R}^*, \qquad (13)$$

where

$$\tilde{R}^* = \text{diag}\;[\sigma_w^{2*}\,I_n, \sigma_s^{2*}], \qquad (14)$$

and

$$\sigma_s^{2*} = \sigma_w^{2*} + \sigma_e^{2*}. \qquad (15)$$

Consider now the set of non-negative definite diagonal matrices $\tilde{R} = \text{diag}\;[\sigma_w^2\,I_n, \sigma_s^2]$ such that

$$R_y - \tilde{R} \geq 0, \qquad \det\,(R_y - \tilde{R}) = 0, \qquad (16)$$

described by the following theorem.

Theorem 1: The set of all matrices $\tilde{R}$ satisfying condition (16) defines the points $P = (\sigma_s^2, \sigma_w^2)$ of a convex curve $\mathcal{S}(R_y)$ belonging to the first quadrant of the noise plane $\mathcal{R}^2$ whose concavity faces the origin. Every point $P = (\sigma_s^2, \sigma_w^2)$ of $\mathcal{S}(R_y)$ satisfies the relation

$$\bar{R}_x(P) = R_y - \text{diag}\;[\sigma_w^2\,I_n, \sigma_s^2] \geq 0 \qquad (17)$$

and can be associated with a coefficient vector $\theta(P)$ satisfying the relation

$$\bar{R}_x(P)\,\theta(P) = 0. \qquad (18)$$

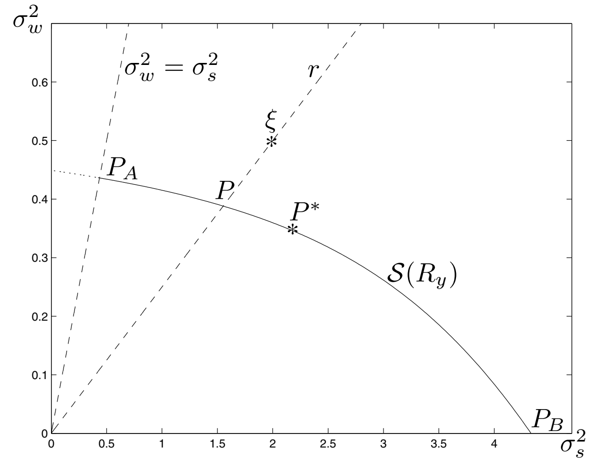

Fig. 1. Typical shape of $\mathcal{S}(R_y)$. All admissible solutions lie on the continuos line.

Theorem 1 can be proved in a way similar to that reported in [16] with reference to errors–in–variables models.

Relations (11) and (13) easily lead to the following important corollary.

Corollary 1: The point $P^* = (\sigma_s^{2*}, \sigma_w^{2*})$, associated with the true variances of $e(t)$ and $w(t)$, belongs to $\mathcal{S}(R_y)$ and the coefficient vector $\theta(P^*)$ is characterized (after a normalization of its $(n+1)$–th entry to -1) by the true AR parameters, i.e. $\theta(P^*) = \theta^*$.

Figure 1 shows a typical shape of $\mathcal{S}(R_y)$. Note that the points (σ_s^2, σ_w^2) of the curve with $\sigma_s^2 \leq \sigma_w^2$ (dotted line) are non admissible because they do not satisfy the condition $\sigma_e^2 = \sigma_s^2 - \sigma_w^2 > 0$. The set of admissible solutions (continuous line) is thus delimited by the straight lines $\sigma_w^2 = \sigma_s^2$ and $\sigma_w^2 = 0$.

Remark 1: The intersection of $\mathcal{S}(R_y)$ with the σ_s^2 axis is the point $P_B = (\sigma_{s\,\max}^2, 0)$ given by the least squares solution

$$\sigma_{s\,\max}^2 = \frac{\det\,(R_y)}{\det\,(R_y')}, \qquad (19)$$

where R_y' is obtained from R_y by deleting its $(n+1)$–th row and column. The intersection of $\mathcal{S}(R_y)$ with the straight line $\sigma_w^2 = \sigma_s^2$ is the point $P_A = (\sigma_{w\max}^2, \sigma_{w\max}^2)$, given by the eigenvector solution

$$\sigma_{w\max}^2 = \min\;\text{eig}\,(R_y). \qquad (20)$$

Since the point P_A corresponds to $\sigma_e^2 = 0$, it is not a solution of Problem 1.

The next theorem describes a parametrization of the singularity curve $\mathcal{S}(R_y)$ that allows to associate a solution of (16) with every straight line departing from the origin and lying in the first quadrant. This parameterization, introduced in [17] with reference to static errors–in–variables models, will play an important role in the implementation of the identification algorithm.

Theorem 2: Let $\xi = (\xi_1, \xi_2)$ be a generic point of the first quadrant of $\mathcal{R}^2$ and r the straight line from the origin through ξ (see Fig. 1). Its intersection with $\mathcal{S}(R_y)$ is the point $P = (\sigma_s^2, \sigma_w^2)$ given by

$$\sigma_s^2 = \frac{\xi_1}{\lambda_M}, \qquad \sigma_w^2 = \frac{\xi_2}{\lambda_M}, \tag{21}$$

where

$$\lambda_M = \max \mathrm{eig}\Big(R_y^{-1}\,\mathrm{diag}\big[\xi_2\, I_n, \xi_1\big]\Big)\,. \tag{22}$$

Proof: Let $\tilde{R}(P) = \mathrm{diag}\,[\sigma_w^2\, I_n, \sigma_s^2]$. Since both ξ and P belong to r it follows that $\xi = \lambda\, P$. Moreover, the coordinates (σ_s^2, σ_w^2) of P must satisfy the condition

$$\det\, (R_y - \tilde{R}(P)) = 0,$$

with

$$R_y - \tilde{R}(P) \geq 0,$$

so that

$$\det\, R_y^{-1}\left(I_{n+1} - \frac{1}{\lambda} R_y^{-1} \tilde{R}_\xi\right) = 0, \tag{23}$$

where $\tilde{R}_\xi = \mathrm{diag}\,[\xi_2\, I_n, \xi_1]$. The scalar λ that solves (23) is given by

$$\lambda = \max \mathrm{eig}(R_y^{-1} \tilde{R}_\xi),$$

which leads to (21) and (22). ∎

IV. AR IDENTIFICATION

As shown in the previous section, the determination of P^* on $\mathcal{S}(R_y)$ leads to the solution of Problem 1. The search procedure for P^* proposed in [15] is based on the shift properties of dynamic systems. Here we introduce a more effective procedure that relies on the use of high–order Yule Walker equations.

Define the $q \times 1$ vector

$$\varphi_x^h(t) = [\,x(t{-}n{-}q)\;\ldots\;x(t{-}n{-}2)\;x(t{-}n{-}1)\,]^T, \quad q \geq n, \tag{24}$$

and the $q \times (n+1)$ matrix

$$\begin{aligned} R_x^h &= E\,[\varphi_x^h(t)\,\varphi_x^T(t)] \qquad (25)\\ &= \begin{bmatrix} r_x(q) & r_x(q+1) & \cdots & r_x(q+n) \\ r_x(q-1) & r_x(q) & \cdots & r_x(q+n-1) \\ \vdots & \vdots & \ddots & \vdots \\ r_x(1) & r_x(2) & \cdots & r_x(n+1) \end{bmatrix}. \end{aligned}$$

Since

$$r_y(k) = r_x(k) \qquad \text{for } k \neq 0, \tag{26}$$

we have

$$R_y^h = E\,[\varphi_y^h(t)\,\varphi_y^T(t)] = R_x^h, \tag{27}$$

where

$$\varphi_y^h(t) = [\,y(t-n-q)\;\ldots\;y(t-n-2)\;y(t-n-1)\,]^T. \tag{28}$$

It is thus possible to write a set of q Yule–Walker equations

$$R_y^h\, \theta^* = 0, \tag{29}$$

that does not involve the output noise variance σ_w^{2*}.

Because of this property and of Corollary 1, the search for P^* along $\mathcal{S}(R_y)$ can be performed on the basis of the following cost function

$$J(P) = \|R_y^h\, \theta(P)\|_2^2 = \theta^T(P)(R_y^h)^T\, R_y^h\, \theta(P). \tag{30}$$

In fact, by taking $q \geq n$, $J(P)$ exhibits the following properties

i) $J(P) \geq 0$
ii) $J(P) = 0 \Leftrightarrow P = P^*$,

i.e. the high–order Yule–Walker equations are satisfied only by the true AR parameters θ^*. Note that $q < n$ would not guarantee property ii). In fact, in this case, two distinct AR models $x_1(t), x_2(t)$ of order n characterized by

$$\begin{aligned} r_{x_1}(0) &\neq r_{x_2}(0), \\ r_{x_1}(k) &= r_{x_2}(k), \quad k = 1, 2, \ldots, n-1, \end{aligned}$$

may exist [18].

All previous considerations allow to formulate the following identification algorithm.

Algorithm 1.

1) Compute, on the basis of the observed sequence $y(1), \ldots, y(N)$, an estimate of R_y and R_y^h, e.g.

$$\begin{aligned} \hat{R}_y &= \frac{1}{N-n} \sum_{t=n+1}^{t=N} \varphi_y(t)\, \varphi_y^T(t), \\ \hat{R}_y^h &= \frac{1}{N-n-q} \sum_{t=n+q+1}^{t=N} \varphi_y^h(t)\, \varphi_y^T(t), \quad q \geq n. \end{aligned}$$

2) Start from a generic direction r belonging to the region of the first quadrant of $\mathcal{R}^2$ delimited by the straight lines $\sigma_w^2 = \sigma_s^2$ and $\sigma_w^2 = 0$.
3) Compute, by means of (21)–(22), the intersection $P = (\sigma_s^2, \sigma_w^2)$ between r and $\mathcal{S}(\hat{R}_y)$.
4) Compute $\bar{R}_x(P)$ and $\theta(P)$ by means of the relations

$$\begin{aligned} \bar{R}_x(P) &= \hat{R}_y - \mathrm{diag}\,[\sigma_w^2\, I_n,\, \sigma_s^2], \\ \bar{R}_x(P)\, \theta(P) &= 0, \end{aligned}$$

and normalize the $(n+1)$–th entry of $\theta(P)$ to -1.
5) Compute the cost function $J(P)$ (30).
6) Move to a new direction $r \pm \Delta r$ corresponding to a decrease of $J(P)$.
7) Repeat steps 3–6 until the point $P^\circ = (\sigma_s^{2\circ}, \sigma_w^{2\circ})$ associated with the minimum of $J(P)$ is found.
8) The estimates of the AR parameters and noise variance are thus given by $\theta(P^\circ)$ and $\sigma_w^{2\circ}$. The estimate of the driving noise variance is then

$$\sigma_e^{2\circ} = \sigma_s^{2\circ} - \sigma_w^{2\circ}.$$

Remark 2: The minimum of cost function (30) along $\mathcal{S}(\hat{R}_y)$ can be found by means of any standard search algorithm since this function exhibits a single absolute minimum.

TABLE I

TRUE AND ESTIMATED VALUES OF PARAMETERS AND VARIANCES FOR ALGORITHM 1, SR METHOD, PEM AND SUBSPACE METHOD. A MONTE CARLO SIMULATION OF 100 RUNS HAS BEEN PERFORMED WITH $N = 1000$ AND SNR ≈ 10 DB.

	α_1	α_2	α_3	α_4	σ_e^{2*}	σ_w^{2*}
true	2.4	-3.03	1.986	-0.6586	1	4
Alg. 1 ($q = 4$)	2.3701 ± 0.1182	-2.9708 ± 0.1909	1.9279 ± 0.1559	-0.6408 ± 0.0427	1.1107 ± 0.3542	3.9578 ± 0.2329
SR	2.3219 ± 0.4636	-2.8867 ± 0.9119	1.8539 ± 0.8530	-0.6160 ± 0.3259	1.6114 ± 1.9891	3.8633 ± 0.4464
PEM	2.3832 ± 0.2662	-3.0269 ± 0.5281	1.9901 ± 0.4794	-0.6779 ± 0.1953	–	–
SS ($q = 4$)	1.1858 ± 0.0527	-0.8367 ± 0.0959	0.0516 ± 0.0882	0.0281 ± 0.0353	–	3.0098 ± 0.1761

The results reported in Section VI have been obtained by using the routine *fminsearch* of MATLAB. Also a crude search procedure would converge in a few steps.

V. ESTIMATION OF THE MODEL ORDER

This section describes how Algorithm 1 can be profitably used for the estimation of the order n of the autoregressive model (1).

By using the spectral factorization, the noisy observation $y(t)$ can be uniquely represented as an ARMA model [1]

$$y(t) = \frac{C(z^{-1})}{A(z^{-1})}\varepsilon(t), \tag{31}$$

where $A(z^{-1}), C(z^{-1})$ are stable polynomials in the unit delay operator z^{-1}

$$A(z^{-1}) = 1 + \alpha_1\, z^{-1} + \cdots + \alpha_n\, z^{-n} \tag{32}$$

$$C(z^{-1}) = 1 + \gamma_1\, z^{-1} + \cdots + \gamma_n\, z^{-n}, \tag{33}$$

while $\varepsilon(t)$ is a zero mean white noise with variance σ_ε^2. In particular, $C(z^{-1})$ and σ_ε^2 are given by the relation

$$\sigma_\varepsilon^2\, C(z^{-1})\, C(z) = \sigma_w^{2*}\, A(z^{-1})\, A(z) + \sigma_e^{2*}. \tag{34}$$

Once the AR parameters $\alpha_1, \ldots, \alpha_n$ and the variances $\sigma_w^{2*}, \sigma_e^{2*}$ have been estimated by means of Algorithm 1, it is possible to compute the estimates $\hat{C}(z^{-1}), \hat{\sigma}_\varepsilon^2$ of $C(z^{-1}), \sigma_\varepsilon^2$ by using (34). A possible way for estimating n consists thus in applying Algorithm 1 for an increasing sequence of possible orders $k = 1, 2, \ldots$ and in observing the stabilization of the corresponding estimated variances $\hat{\sigma}_\varepsilon^2$. An alternative more robust procedure consists in the reconstruction of the innovation $\varepsilon(t)$ by means of the whitening filter

$$\varepsilon(t) = \frac{A(z^{-1})}{C(z^{-1})}y(t), \tag{35}$$

and in the subsequent application of a whiteness test.

On the basis of the previous observations the following algorithm can be devised for estimating the model order n.

Algorithm 2.

1) Start from the order $k = 1$.
2) Compute, by means of Algorithm 1, the estimates $\hat{\theta}^k, \sigma_w^{2k}, \sigma_e^{2k}$ and construct the polynomial $\hat{A}^k(z^{-1})$ (32).
3) Compute the estimates $\hat{C}^k(z^{-1})$ and σ_ε^{2k} with (34).
4) Compute the innovation sequence
$$\hat{\varepsilon}^k(1), \hat{\varepsilon}^k(2), \ldots, \hat{\varepsilon}^k(N) \tag{36}$$
by means of the whitening filter
$$\hat{\varepsilon}^k(t) = \frac{\hat{A}^k(z^{-1})}{\hat{C}^k(z^{-1})}y(t).$$
5) Apply a χ^2 test to check the whiteness of the sequence (36) [19].
6) If the test has been superated accept k as model order, otherwise set $k = k + 1$ and return to step 2.

VI. NUMERICAL RESULTS

In this section, the effectiveness of the proposed identification algorithm is tested by means of numerical simulations. The results are then compared with those of other approaches. For this purpose, let us consider the 4–th order AR process

$$\begin{aligned} x(t) \;\; = \;\; & 2.4\, x(t-1) - 3.03\, x(t-2) + 1.986\, x(t-3) \\ & - 0.6586\, x(t-4) + e(t), \end{aligned}$$

already used in [13], [15], where $e(t)$ is a white noise with variance $\sigma_e^{2*} = 1$. The variance of the observation noise is $\sigma_w^{2*} = 4$, corresponding to a signal to noise ratio (SNR) of about 10 dB, i.e.

$$\text{SNR} = 20\,\log_{10}\sqrt{\frac{E[x^2(t)]}{E[w^2(t)]}} = 10\,\log_{10}\,\frac{r_x(0)}{\sigma_w^{2*}} \approx 10\ \text{dB}.$$

A Monte Carlo simulation of 100 runs has been performed by using, in every run, $N = 1000$ samples of the noisy output $y(t)$. Several methods have been considered:

- Algorithm 1, with the minimum number $q = 4$ of HOYW equations;
- the algorithm proposed in [15] which will be denoted as shifted–relation (SR) method;
- the standard prediction error method (PEM) (routine *pem* of MATLAB);
- the subspace approach (SS) proposed in [10] with the minimum number $q = 4$ of HOYW equations.

The results are summarized in Table I that reports the true values of parameters and variances, the means of their estimates and the associated standard deviations.

It is possible to note that Algortihm 1 significantly outperforms the approach proposed in [15] and gives parameter

TABLE II

TRUE AND ESTIMATED VALUES OF PARAMETERS AND VARIANCES FOR ALGORITHM 1 AND SS METHOD. A MONTE CARLO SIMULATION OF 100 RUNS HAS BEEN PERFORMED WITH $N = 1000$ AND SNR ≈ 10 DB.

	α_1	α_2	α_3	α_4	σ_e^{2*}	σ_w^{2*}
true	2.4	-3.03	1.986	-0.6586	1	4
Alg. 1 ($q = 7$)	2.3976 ± 0.0994	-3.0307 ± 0.1477	1.9871 ± 0.1118	-0.6666 ± 0.0254	1.0169 ± 0.2679	3.9631 ± 0.2295
SS ($q = 350$)	2.3362 ± 0.0210	-3.0165 ± 0.0390	2.0105 ± 0.0356	-0.7068 ± 0.0154	–	4.5520 ± 0.2524

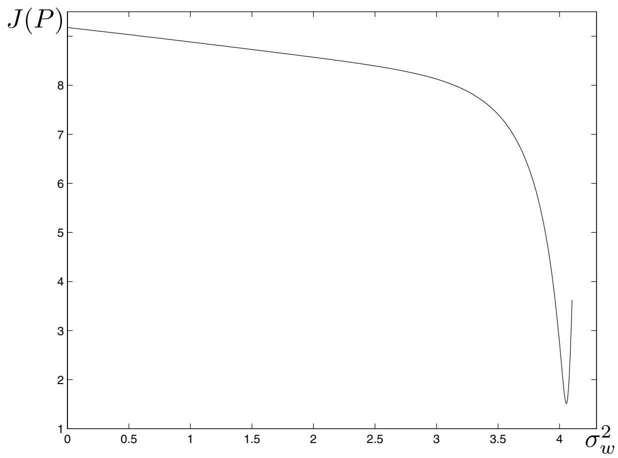

Fig. 2. Values of $J(P)$ along $\mathcal{S}(\hat{R}_y)$ in a typical run of the Monte Carlo simulation, $q = 7$, $N = 1000$, SNR ≈ 10 dB.

estimates better than those of PEM, which is also more computational demanding. It must also be pointed out that the PEM performance can be remarkably worse when only poor initial parameter estimates are available. Note also that very poor results have been obtained with the SS approach.

The number of HOYW equations has then been varied in order to test the improvement obtainable by means of Algorithm 1 and the SS method. For Algorithm 1 the best performance has been obtained with $q = 7$ while the SS approach gives the best results with $q = 350$. These results are summarized in Table II.

It is worth noting that the SS approach fails to give good estimates when the number of HOYW equations is low while satisfactory results can be obtained with a large number of equations. This can be a serious drawback when short sequences of data are available since a large number of high–order autocorrelation lags must be estimated. On the contrary, a low number of HOYW equations is sufficient to obtain good results with the proposed method. Algorithm 1 provides also very good estimates of the driving and measurement noise variances $\sigma_e^{2*}, \sigma_w^{2*}$. Note that an estimate of σ_e^{2*} cannot be directly obtained with the PEM and SS approaches.

Fig. 2 reports the values of $J(P)$ versus the noise variance σ_w^2 along $\mathcal{S}(\hat{R}_y)$ in a typical run of the Monte Carlo simulation with $q = 7$ and shows the good selectivity of cost function (30).

Finally, Algorithm 2 has been tested for the estimation of the model order n on sequences of $N = 1000$ samples for SNR ranging from 5 dB to 25 dB. For every SNR a 100 runs Monte Carlo simulation has been performed. A test $\chi^2_{0.01}(8)$ has been used in step 5.

TABLE III

ESTIMATION OF THE MODEL ORDER BY MEANS OF ALGORITHM 2.

SNR (dB)	5	8	10	12	15	20	25
$k = 4$	73	78	92	93	100	100	100

Table III reports the numbers of runs associated with the detection of the correct order $k = 4$. It can be observed that very good results have been obtained for SNR greater than 8 dB.

VII. CONCLUSIONS

Identification of autoregressive models in presence of noise is an important issue in many engineering applications. Many classical approaches are based on the solution of an overdetermined set of high order Yule–Walker equations. Nevertheless, it has been proved that the use of both low and high order YW equations leads to significant estimation improvements.

In this paper, an identification approach that relies on the dynamic Frisch scheme properties and takes advantage of both low and high order YW equations has been proposed. This scheme is consistent when a minimum number of HOYW equations is used.

The Monte Carlo simulations that have been carried out show the good performance of the new algorithm with respect to other estimation approaches and also the good estimate of the driving and output noise variances.

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