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Robust Stabilization of Jump Polytopic Systems via Output Feedback

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Abstract— **The paper considers a class of systems composed of a finite set of linear plants with jumping transition between them determined by a homogeneous Markov chain. Each state of this chain corresponds to some mode of the system. Necessary and sufficient conditions for robust stabilization against polytopic uncertainty of modes and for simultaneous stabilization via output feedback are obtained. Some algorithms for computation of gain matrices of stabilizing controllers are given, along with an illustrative example. The algorithms effectively use LMI solvers.**

I. INTRODUCTION

In modern control engineering there exists a wide class of dynamic systems with random jumping changes of their structure or parameters such as aerospace systems, manufacturing systems, economic systems, etc., see, for example [2, 9, 13] and the references therein. These random changes may result from abrupt phenomena such as component and interconnection failures, parameters shifting, tracking and/or variation in the time frame of measurements. Systems of this type may be modeled as hybrid ones with many operating modes. Each mode corresponds to an individual deterministic or stochastic dynamic. Switching of the system modes is governed by a Markov process with a finite set of states $\mathbb{N} = \{1, 2, \ldots, \nu\}$ (Markov chain). When the mode $i \in \mathbb{N}$ is fixed the plant state evolves according to the corresponding individual dynamic. The state space of such systems is naturally hybrid: to the usual plant state in $\mathbb{R}^n$ we append a discrete variable taking values in the set $\mathbb{N}$.

The problem of stability and stabilizing control for jumping systems and their robustness has been intensively exploited over many years. Without any intention of being exhaustive here we quote [2, 9, 13] and the references therein.

One of the most important open questions in control theory is the output feedback problem including static output feedback [4, 6, 7, 12, 14, 17] despite the fact that the static output feedback represents the *simplest* closed loop control that can be realized in practice. There exist necessary and sufficient conditions for output feedback stabilization [4, 7, 12, 14] but these conditions are not readily implemented as numerical algorithms, except [4] in which an iterative LMI-based algorithm is proposed, and [18].

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The major difficulty arises from the non-convexity of the static output feedback solution set [7], which renders it a non-trivial computational task, analytical and computational alike. For the fixed-order dynamic output feedback the problem can be formulated exactly the same way as in the static output feedback case and has the same difficulties. However if we do not fix the controller order a priori then the set of solutions can be characterized by a convex set [7, 17].

The problem of output feedback becomes more difficult as one attempts to find robust stabilizing control. In the case of jumping systems we have especially weak development of the methods of solving this problem [13, 15, 16]. In view of these facts the goal of this paper is to obtain necessary and sufficient conditions for robust and simultaneous stabilization via output feedback for jump systems with polytopic uncertainties.

Parametrization of the set of stabilizing controllers is of great importance as it allows additional performance specifications to be incorporated into control design, with guaranteed closed-loop stability, by choosing the free parameters in the parametrization [7, 16]. The results of the parametrization obtained in [16] are effectively used in this paper.

The paper is organized as follows. In Section II we give a mathematical description of the considered system. In section III main notations and definitions are introduced and some parametrization results are formulated. In section IV necessary and sufficient conditions for robust and simultaneous stabilizations via output feedback are obtained. The results of section IV and the parametrization results allow us to formulate algorithms for computation of the gain matrices of robust stabilizing and simultaneously stabilizing controllers. These algorithms effectively uses LMI solvers and are presented in section V. An example is given in section VI.

II. SYSTEM DESCRIPTION

Consider a linear system described by the following equations:

$$\dot{x}_p(t) = A_p(r(t))x(t) + B_p(r(t))u_p(t), \tag{1}$$

$$y_p(t) = C_p(r(t))x_p(t), \quad t \geq 0, \tag{2}$$

$$x_p(\tau) = \Phi_{pij}x_p(\tau - 0), \tag{3}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state vector, $u_p \in \mathbb{R}^{m_p}$ is the control vector, $y_p \in \mathbb{R}^{k_p}$ is the output vector, $r(t)$ is a homogenous Markov chain whose state space is a set of integers $\mathbb{N} = \{1, 2, ..., \nu\}$ and transition matrix $P(\theta) = [P_{ij}(\theta)]_1^\nu = [\text{Prob}\{r(t+\theta) = j \mid r(t) = i\}]_1^\nu = \exp(\Pi\theta), \quad 0 \leq t \leq$

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$t+\theta,\ \Pi=[\pi_{ij}]_1^\nu$ with $\pi_{ij}\geq 0,\ j\neq i, \pi_{ii}=-\sum_{j\neq i}^{\nu}\pi_{ij}$, $\tau > t_0$ is the moment of discontinuous mode change, i.e. the moment of transition from $r(\tau-0)=i$ to $r(\tau)=j\neq i$, $\Phi_{pij},(i,j\in\mathbb{N})$ are $n_p\times n_p$ constant matrices, such that $\Phi_{pii}=I$. For each possible value of the process $r(t)\in\mathbb{N}$ we write $A_p(r(t))=A_{pi},\ \ B_p(r(t))=B_{pi},\ \ C_p(r(t))=C_{pi}$, when $r(t)=i$. These matrices have compatible dimensions and correspond to the different modes of the system.

Consider a fixed-order linear dynamic output feedback controller in the form of following equations

$$\dot{x}_c(t)=A_{ci}x_c(t)+B_{ci}y_p(t),\ \text{ if }\ r(t)=i, \tag{4}$$
$$u_p(t)=C_{ci}x_c(t)+D_{ci}y_p(t),\ \text{ if }\ r(t)=i, \tag{5}$$

where $x_c\in\mathbb{R}^{n_c}$ and matrices $A_{ci},\ B_{ci},\ C_{ci}$ and D_{ci} have compatible dimensions.

The system (1)–(5) can be written as

$$\dot{x}(t) = A(r(t))x(t)+B(r(t))u(t), \tag{6}$$
$$y(t) = C(r(t))x(t),\quad t\geq 0, \tag{7}$$
$$x(\tau) = \Phi_{ij}x(\tau-0), \tag{8}$$
$$u(t) = -G_i y(t),\ \text{if } r(t)=i, \tag{9}$$

where $x=[x_p^T\ \ x_c^T]^T\in\mathbb{R}^n,\ u\in\mathbb{R}^m,\ y\in\mathbb{R}^k,\ n=n_p+n_c,\ \ m=m_p+n_c,\ \ k=k_p+n_c\quad \Phi_{ij}=\text{diag}[\Phi_{pij}\ \ I_{n_c}]$ if $r(\tau-0)=i,\ r(\tau)=j$ and

$$A_i=\begin{bmatrix} A_{pi} & 0\\ 0 & 0\end{bmatrix}, B_i=\begin{bmatrix} B_{pi} & 0\\ 0 & I_{n_c}\end{bmatrix},$$

$$C_i=\begin{bmatrix} C_{pi} & 0\\ 0 & I_{n_c}\end{bmatrix}, G_i=-\begin{bmatrix} D_{ci} & C_{ci}\\ B_{ci} & A_{ci}\end{bmatrix}.$$

It is easy to see that this model gives a common description for systems with both static and fixed-order dynamic output feedback.

III. Preliminaries

A. Main notations and definitions

For every $i\in\mathbb{N}$ the plant state space of the system (6) can be presented in the form of the following partition

$$\mathbb{R}^n=\text{Im}(C_i^{\text{T}})\oplus\text{Ker}(C_i), \tag{10}$$

where $\text{Im}(C_i^{\text{T}})$ and $\text{Ker}(C_i)$ are orthogonal subspaces. For any $x\in\mathbb{R}^n$ we can write

$$x=x_{\text{I}}+x_{\text{K}},$$

where $x_{\text{I}}\in\text{Im}(C_i^{\text{T}})$ and $x_{\text{K}}\in\text{Ker}(C_i)$. Define the matrices

$$E_{\text{I}}(i)=C_i^{+}C_i,\ \ E_{\text{K}}(i)=I-E_{\text{I}}(i), \tag{11}$$

where C_i^+ is the Moore-Penrose inverse of C_i. According to the partition (10) the matrices (11) are projection matrices on $\text{Im}(C_i^{\text{T}})$ and on $\text{Ker}(C_i)$ correspondingly. These matrices are symmetric and unique. We use the notation $X^\perp$ for full rank matrix orthogonal to X. The matrix $X^\perp$ exists if and only if X has linearly dependent rows and for a given X the matrix $X^\perp$ is not unique.

Definition 1: The system (6) is said to be mean square stabilizable (MSS) via output feedback if there exists a control law in the form of (9) such that the system (6)–(9) is exponentially stable in the mean square, i.e. for any $x_0=x\in\mathbb{R}^n,\ \ r_0=i\in\mathbb{N},\ \ t\geq 0$ and for some $\alpha>0,\ \beta>0$ the following inequality holds [10]:

$$\mathcal{E}[\|\ x(t)\ \|^2|\ x_0=x,\ r_0=i]\leq$$
$$\beta\ \|\ x\ \|^2\exp(-\alpha(t-t_0)),\ t\geq t_0,$$

where $\mathcal{E}$ is the expectation operator.

An important role in the sequel along with the output feedback control (9) plays also the state feedback control

$$u(t)=-K(i)x(t),\ \text{ if }\ r(t)=i. \tag{12}$$

Definition 2: The control law (9) is said to be stabilizing control if it guarantees MSS of the closed loop system (6)–(9).

Definition 3: The control law (12) is said to be stabilizing control if it guarantees MSS of the closed loop system (6) – (8), (12).

B. Parametrization of stabilizing controllers with state feedback

Define the following set

$$\begin{aligned}\mathcal{L}_s\ =\ &\{H_i=H_i^T>0,\ \exists K_i\ \text{ such that }\\ &(A_i-B_iK_i)^TH_i+H_i(A_i-B_iK_i)+\\ &\sum_{j=1}^{\nu}\pi_{ij}\Phi_{ij}^TH_j\Phi_{ij}<0,\ i\in\mathbb{N}\}.\end{aligned}$$

The following theorem gives a characterization of the set of matrices of Lyapunov's functions $\mathcal{L}_s$ and a parametrization of stabilizing state feedback gains.

Theorem 1: Let the matrices H_i $(i\in\mathbb{N})$ are given. Then the following statements are equivalent.

1) $H_i\in\mathcal{L}_s,\ i\in\mathbb{N}$;
2) $H_i,\ (i\in\mathbb{N})$ is a positive definite solution of the set of coupled matrix quadratic equations
$$A_i^TH_i+H_iA_i-H_iB_iR_i^{-1}B_i^TH_i+\sum_{j=1}^{\nu}\pi_{ij}\Phi_{ij}^TH_j\Phi_{ij}+Q_i=0$$
for some $Q_i>0$ and $R_i>0,\ (i\in\mathbb{N})$;
3) $H_i>0\ \ (i\in\mathbb{N})$ and these matrices satisfy the inequality
$$B_i^\perp(A_iH_i^{-1}+H_i^{-1}A_i^T+\sum_{j=1}^{\nu}\pi_{ij}H_i^{-1}\Phi_{ij}^TH_j\Phi_{ij}H_i^{-1}B_i^{\perp T}<0,$$
$$\text{if }\ B_iB_i^T\not>0,\ \ i\in\mathbb{N}.$$

All the stabilizing state feedback gains are given by

$$K_i=R_i^{-1}B_i^TH_i+R_i^{-\frac{1}{2}}\Lambda_iQ_i^{\frac{1}{2}},\ \ i\in\mathbb{N}, \tag{13}$$

where the matrices $H_i,\ Q_i,\ R_i$ are the ones in 2. and Λ_i is an arbitrary matrix such that $\|\ \Lambda_i\ \|<1$.

The proof is given in [16].

IV. Robust and simultaneous stabilization via output feedback

A. Robust stabilization of the systems with polytopic uncertainty of modes

Consider the case when in each fixed mode the matrices of the system are not exactly known. Assume that for each mode $i \in \mathbb{N}$

$$\Omega_i = [A_i \;\; B_i] \in \mathrm{Co}\{[A_{iq} \;\; B_{iq}], q = 1, ..., N\}, \tag{14}$$

or in other words $\Omega_i = \sum_{q=1}^{N} \lambda_{iq}\Omega_{iq}$ for some $0 \le \lambda_{iq} \le 1$. $\sum_{q=1}^{N} \lambda_q = 1$, where the N vertices of the polytope are described by

$$\Omega_{iq} = \{[A_{iq} \;\; B_{iq}], q = 1, ..., N\}, \; i \in \mathbb{N}.$$

Definition 4: The control law (9) is said to be robust stabilizing control for the system (6), (7) with the polytopic uncertainty (14) if it guarantees stochastic quadratic stability of this system i.e. there exist symmetric and positive definite matrices H_i, $(i \in \mathbb{N})$ such that for all $[A_i \;\; B_i]$ satisfying (14) the following inequalities hold:

$$(A_i - B_iG_iC_i)^T H_i + H_i(A_i - B_iG_iC_i) + \sum_{j=1}^{N} \pi_{ij}\Phi_{ij}^T H_j \Phi_{ij} < 0, \; i \in \mathbb{N}. \tag{15}$$

This control is said to be nonswitching robust control if $G_i = G$ $(i \in \mathbb{N})$.
It is well known [3] that these conditions hold if and only if there exist symmetric and positive definite matrices H_i, $(i \in \mathbb{N})$ such that

$$(A_{iq} - B_{iq}G_iC_i)^{\mathrm{T}} H_i + H_i(A_{iq} - B_{iq}G_iC_i) + \sum_{j=1}^{\nu} \pi_{ij}\Phi_{ij}^T H_j \Phi_{ij} < 0, \; i \in \mathbb{N}, \; q = 1, ..., N. \tag{16}$$

Lemma 1: Let $x, \; y, \; z \in \mathbb{R}^n, \; z = x + y, \; x^T y = 0, \; x \in \mathrm{Im}(B^T), \; y \in \mathrm{Ker}(B), \; x^T Ax \le x^T Bx, \; y^T By < 0$, where $A = A^T$ and B are some matrices of corresponding dimensions.Then

$$z^T Az \le z^T (B^T B + \beta I)z,$$

where $\beta = 2 \parallel A \parallel$.
The proof is given in [15].

Theorem 2: The system (6), (7) with the polytopic uncertainty of modes (14) is robust stabilizable via output feedback if and only if there exist matrices $M_{iq} = M_{iq}^T$, $R_{iq} = R_{iq}^T > 0$ and L_{iq} such that the system of matrix quadratic equations:

$$A_{iq}^T H_i + H_i A_{iq} - H_i B_{iq} R_{iq}^{-1} B_{iq} H_i + \sum_{j=1}^{\nu} \pi_{ij}\Phi_{ij}^T H_j \Phi_{ij} + M_{iq} = 0, \; i \in \mathbb{N}, \; q = 1, \ldots, N \tag{17}$$

has a positive definite solution $H_i = H_i^T$, satisfying the relations

$$(A_{iq} - B_{iq}K_{iq})^T H_i + H_i(A_{iq} - B_{iq}K_{iq}) + \sum_{j=1}^{\nu} \pi_{ij}\Phi_{ij}^T H_j \Phi_{ij} - S_{iq}^T K_{iq} - K_{iq}^T S_{iq} < 0, \; i \in \mathbb{N}, \; q = 1, \ldots, N, \tag{18}$$

$$R_{iq}^{-1}(B_{iq}^T H_i + L_{iq}) = R_{iq+1}^{-1}(B_{iq+1} H_i + L_{iq+1}), \quad i \in \mathbb{N}, \; q = 1, \ldots, N - 1, \tag{19}$$

where

$$S_{iq} = L_{iq}E_I(i) - B_{iq}^T H_i E_K(i), \tag{20}$$

$$K_{iq} = R_{iq}^{-1} B_{iq}^T H_i. \tag{21}$$

The corresponding robust output feedback control has the form (9) where the gain matrix is defined by the formula

$$G_i = R_{iq}^{-1}(B_{iq}^T H_i + L_{iq})C_i^+ \tag{22}$$

for arbitrary fixed $q \in \{1, \ldots, N\}$.

Proof: Necessity. Let the system (6), (7) with the polytopic uncertainty of modes (14) be robust stabilizable via output feedback. Then without a loss of generality we can suppose that there exists a matrix $\Theta_i = G_iC_iE_I(i) = \kappa_i E_I(i)$ and a positive definite matrix H_i $(i \in \mathbb{N})$ such that for all $x \in \mathbb{R}^n$

$$x^T[(A_{iq} - B_{iq}\Theta_i)^T H_i + H_i(A_{iq} - B_{iq}\Theta_i) + \sum_{j=1}^{\nu} \pi_{ij}\Phi_{ij}^T H_j \Phi_{ij}]x < 0, \quad i \in \mathbb{N}, \; q = 1, \ldots, N. \tag{23}$$

If $x \in \mathrm{Ker}\Theta_i$ then from (23) we have

$$x^T[A_{iq}^T H_i + H_i A_{iq} + \sum_{j=1}^{\nu} \pi_{ij}\Phi_{ij}^T H_j \Phi_{ij}]x < 0. \quad i \in \mathbb{N}, \; q = 1, \ldots, N. \tag{24}$$

Define for each $i \in \mathbb{N}$ the scalars α_i^* as follows

$$\alpha_i^* = \max_{x \in \mathrm{Im}\Theta_i^T, q=1,\ldots,N} \frac{x^T[A_{iq}^T H_i + H_i A_{iq}}{x^T \Theta_i^T \Theta_i x} + \frac{\sum_{j=1}^{\nu} \pi_{ij}\Phi_{ij}^T H_j \Phi_{ij}]x}{x^T \Theta_i^T \Theta_{iq} x}. \tag{25}$$

From the result of Lemma 1, (24) and (25) we obtain that

$$x^T[A_{iq}^T H_i + H_i A_{iq} + \sum_{j=1}^{\nu} \pi_{ij}\Phi_{ij}^T H_j \Phi_{ij}]x \le \alpha_i x^T E_I(i)\kappa_i^T \kappa_i E_I(i)x + \beta_{iq} x^T x, \; x \in \mathbb{R}^n, \; i \in \mathbb{N}, \; q = 1, \ldots, N,$$

where $\alpha_i > \max(0, \alpha_i^*)$ and β_{iq} is defined as in Lemma 1. Thus, for any symmetric matrix $R_{iq} > \alpha_i I$, we have

$$x^T[A_{iq}^T H_i + H_i A_{iq} + \sum_{j=1}^{\nu} \pi_{ij}\Phi_{ij}^T H_j \Phi_{ij}]x < x^T E_I(i)\kappa_i^T R_{iq} \kappa_i E_I(i)x + \beta_{iq} x^T x, \; x \in \mathbb{R}^n, \; i \in \mathbb{N}, \; q = 1, \ldots, N.$$

This inequality implies the existence of a symmetric matrix Q_{iq} $(i \in \mathbb{N})$, $q = 1, \ldots, N$, satisfying the system of the matrix equations

$$A_{iq}^T H_i + H_i A_{iq} - E_I(i) \kappa_i^T R_i \kappa_i E_I(i) + \sum_{j=1}^{\nu} \pi_{ij} \Phi_{ij}^T H_j \Phi_{ij} + Q_{iq} = 0, \; i \in \mathbb{N}, \; q = 1, \ldots, N. \quad (26)$$

Let us define $L_{iq} = R_{iq} \kappa_i - B_{iq}^T H_i$, then the equation (26) can be rearranged as

$$A_{iq}^T H_i + H_i A_{iq} - E_I(i)(H_i B_{iq} + L_{iq}^T) R_{iq}^{-1} (B_{iq}^T H_{iq} + L_{iq}) E_I(i) + \sum_{j=1}^{\nu} \pi_{ij}(\delta) \Phi_{ij}^T H_j \Phi_{ij} + Q_{iq} = 0, \; i \in \mathbb{N}, \; q = 1, \ldots, N, \quad (27)$$

moreover $\Theta_i = R_{iq}^{-1}(L_{iq} + B_{iq}^T H_i) E_I(i)$. Because Θ_i independent on q it follows from definition of L_{iq} that equations (19) hold and G_i is given by formula (22). Rewrite (27) in the following equivalent form

$$(A_{iq} - B_{iq} G_i C_i)^T H_i + H_i (A_{iq} - B_{iq} G_i C_i) + \sum_{j=1}^{\nu} \pi_{ij} \Phi_{ij}^T H_j \Phi_{ij} + H_i B_{iq} R_{iq}^{-1} B_{iq}^T H_i - S_{iq}^T R_i^{-1} S_{iq} + Q_{iq} = 0, \; i \in \mathbb{N}, \; q = 1, \ldots, N. \quad (28)$$

Taking into account (23) it is easy to see from (28) that

$$H_i B_{iq} R_{iq}^{-1} B_{iq}^T H_i - S_{iq}^T R_{iq}^{-1} S_{iq} + Q_{iq} > 0, \quad i \in \mathbb{N}, \; q = 1, \ldots, N. \quad (29)$$

On the other hand

$$(A_{iq} - B_{iq} K_{iq})^T H_i + H_i (A_{iq} - B_{iq} K_{iq}) + \sum_{j=1}^{\nu} \pi_{ij} \Phi_{ij}^T H_j \Phi_{ij} - S_{iq}^T K_{iq} - K_{iq}^T S_{iq} = -(H_i B_{iq} R_{iq}^{-1} B_{iq}^T H_{iq} - S_{iq}^T R_{iq}^{-1} S_{iq} + Q_{iq}). \quad (30)$$

It follows from (29), (30) that the inequalities (18) hold. The equations (27) are equivalent to (17) with $M_{iq} = Q_{iq} - S_{iq}^T K_{iq} - K_{iq}^T S_{iq}$.

Sufficiency. Let (17) – (19) are valid and $G(i)$ is given by (22). Then substituting (20), (21) in (18), and taking into account (11), (22) we obtain

$$0 > (A_{iq} - B_{iq} K_{iq})^T H_i + H_i (A_{iq} - B_{iq} K_{iq}) + \sum_{j=1}^{\nu} \pi_{ij} \Phi_{ij}^T H_j \Phi_{ij} - S_{iq}^T K_{iq} - K_{iq}^T S_{iq} + (A_{iq} - B_{iq} G_i C_i)^T H_i + H_i (A_{iq} - B_{iq} G_i C_i) + \sum_{j=1}^{\nu} \pi_{ij} \Phi_{ij}^T H_j \Phi_{ij}, \; i \in \mathbb{N}, \; q = 1, \ldots, N.$$

It follows from these inequalities and (16) that the control law (9) with the matrix G_i given by (22) guarantees stochastic quadratic stability of the system (6), (7) with polytopic uncertainty of modes (14) and thus it is robust stabilizing control. ∎

Corollary 1: The system (6), (7) with $C_i = C$, $(i \in \mathbb{N})$ is robust stabilizable via nonswitching output feedback if and only if for some symmetric matrices M_{iq}, and $R_{iq} > 0$ there exist a positive definite solution $H_i = H_i^T$ of the system of the coupled Riccati equations (17) and matrices L_{iq} of compatible dimensions, satisfying for all $i \in \mathbb{N}$, $q = 1, \ldots, N-1$ the system of inequalities (18) and the following system of equations

$$R_{iq}^{-1}(B_{iq}^T H_i + L_{iq}) = R_{i+1q+1}^{-1}(B_{i+1q+1} H_i + 1 + L_{i+1q+1}), i = 1, \ldots, \nu - 1 \in \mathbb{N}, \; q = 1, \ldots, N-1.$$

The robust stabilizing control has the form of (9) where the gain matrix is given by (22) for an arbitrary fixed $i \in \mathbb{N}$, $q = 1, \ldots, N$. If, in addition, $\Phi_{ij} = I$, $(i, j \in \mathbb{N})$ and there exists a a common solution of (17), (18) $H_i = H > 0$ $(i \in \mathbb{N})$, then this control stabilizes the system (6), (7) independently of the mode change process.

B. Simultaneous stabilization

Consider a set of single linear systems described by the following equations

$$\dot{x}(t) = A_i x(t) + B_i u(t), \quad (31)$$

$$y(t) = C x(t), \quad t \geq 0. \quad (32)$$

We seek an output feedback control law

$$u(t) = -G y(t), \quad (33)$$

that will simultaneously stabilize all the systems (31), (32). The necessary and sufficient conditions for the control law (33) to be stabilizing one are the existence of symmetric and positive definite matrices H_i, $(i \in \mathbb{N})$ such that

$$(A_i - B_i G C_i)^T H_i + H_i (A_i - B_i G C_i) < 0, \; i \in \mathbb{N}. \quad (34)$$

Based on the inequalities (34) and on the outline of the proof of the previous theorem we obtain the following

Theorem 3: The set of systems (31), (32) is simultaneously stabilizable via output feedback if and only if there exist matrices $M_i = M_i^T$, $R_i = R_i^T > 0$ and L_i such that the system of matrix quadratic equations:

$$A_i^T H_i + H_i A_i - H_i B_i R_i^{-1} B_i H + M_i = 0, \; i \in \mathbb{N},$$

has a positive definite solution $H_i = H_i^T$ satisfying the relations

$$(A_i - B_i K_i)^T H_i + H_i (A_i - B_i K_i) - S_i^T K_i - K_i^T S_i < 0, \; i \in \mathbb{N},$$

$$R_i^{-1}(B_i^T H_i + L_i) = R_{i+1}^{-1}(B_{i+1} H_{i+1} + L_{i+1}), \quad i = 1, \ldots, \nu - 1,$$

where

$$S_i = L_i E_I(i) - B_i^T H_i E_K(i), \quad K_i = R_i^{-1} B_i^T H_i.$$

The corresponding output stabilizing control has the form (33) where the gain matrix is defined by the formula

$$G = R_i^{-1}(B_i^T H_i + L_i) C_i^+ \quad (35)$$

for arbitrary fixed $i \in \mathbb{N}$.

V. The algorithms for computation of stabilizing gains

A. Algorithm based on direct solution of coupled Riccati equations (CRE)

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Step 1. Solve the system of CRE (17) by the LMI optimization method [1,5] and find the matrices $H_i = H_i^T > 0$ and $K_{iq},\ i \in \mathbb{N},\ q = 1, \ldots, N$.

Step 2. If the LMI/LME problem (18)-(19) is not feasible then correct the LQR parameters (weight matrices) and go to step 1, else if this LMI/LME problem is feasible find the matrices L_{iq} and calculate the matrices G_i according to the formula (22).

B. Algorithm based on parametrization of stabilizing solutions of CRE

The parametrization results (Theorem 1) along with Theorem 2 (Theorem 3) allow us to formulate an algorithm for synthesis of robust stabilizing control (9) (simultaneously stabilizing control (33)). Let us consider in detail the case of robust stabilizing control. According to Theorem 1 all the state feedback matrices are given by the formula

$$K_{iq} = R_{iq}^{-1} B_{iq}^T P_i + R_{iq}^{-\frac{1}{2}} \Lambda_{iq} Q_{iq}^{\frac{1}{2}}, \quad i \in \mathbb{N}, \qquad (36)$$

where $\| \Lambda_{iq} \| < 1$ and $P_i = P_i^T > 0$ $(i \in \mathbb{N})$ satisfy the matrix quadratic equations

$$A_{iq}^T P_i + P_i A_{iq} - P_i B_{iq} R_{iq}^{-1} B_{iq}^T P_i + \sum_{j=1}^{\nu} \pi_{ij} \Phi_{ij}^T P_j \Phi_{ij} + Q_{iq} = 0 \qquad (37)$$

for given $Q_{iq} = Q_{iq}^T > 0$ and $R_{iq} = R_{iq}^T > 0$. On the other hand

$$K_{iq} = R_{iq}^{-1} B_{iq}^T H_i,$$

where H_i $(i \in \mathbb{N})$ satisfy the equation (17).

Let matrices $H_i > 0$ $(i \in \mathbb{N})$ be a solution of LME

$$R_{iq}^{-1} B_{iq}^T H_i = R_{iq}^{-1} B_{iq}^T P_i + R_{iq}^{-\frac{1}{2}} \Lambda_{iq} Q_{iq}^{\frac{1}{2}},$$

where $P_i > 0$ and Λ_{iq} $(i \in \mathbb{N},\ q = 1, \ldots, N)$ are the same as in (36), then it is easy to see that this solution satisfies (17) for some $M_{iq} = M_{iq}^T$. Taking into account these results we can formulate the algorithm as follows

Step 1. Solve LMI's with respect to the variables Y_i:

$$\begin{bmatrix} \Gamma_{11}(i,q) & \Gamma_{12}(i,q) \\ \Gamma_{12}^T(i,q) & \Gamma_{22}(i,q) \end{bmatrix} < 0, \qquad (38)$$

$$\Gamma_{11}(i,q) = B_{iq}^{\perp} (A_{iq} Y_i + Y_i A_{iq}^T + \pi_{ii} Y_i) B_{iq}^{\perp T},$$
$$\Gamma_{22}(i,q) = \mathrm{diag}[-Y_1 \ \ldots - Y_{i-1} \ \ - Y_{i+1} \ldots - Y_\nu],$$
$$\Gamma_{12}(i) = \Big[B_{iq}^{\perp} \pi_{i1}^{\frac{1}{2}} Y_i \Phi_{i1}^T \ldots B_{iq}^{\perp} \pi_{ii-1}^{\frac{1}{2}} Y_i \Phi_{ii-1}^T \ \ B_{iq}^{\perp} \pi_{ii+1}^{\frac{1}{2}} Y_i \Phi_{ii+1}^T \ldots B_{iq}^{\perp} \pi_{i\nu}^{\frac{1}{2}} Y_i \Phi_{i\nu}^T \Big].$$

Step 2. Find $P_i = Y_i^{-1}$ and then the matrices $Q_{iq} > 0$, $R_{iq} > 0$ from the equations (37).

Step 3. Put $H_i = P_i$ $(i \in \mathbb{N})$, $\varepsilon = \varepsilon_0 < 1$.

Step 4. Find $K_{iq} = R_{iq}^{-1} B_{iq}^T H_i$.

Step 5. If LMI/LME's (18)-(19) with respect to the variables L_{iq} are feasible then calculate the gain matrix using the formula (22), stop; else find matrices $H_i = H_i^T > 0$ as a solution of the following LMI optimization problem

$$\mathrm{trace} \sum_{i=1}^{\nu} H_i \to \max,$$
$$H_i > 0,$$
$$R_{iq}^{-1} B_{iq}^T H_i = R_{iq}^{-1} B_{iq}^T P_i + R_{iq}^{-\frac{1}{2}} \Lambda_{iq} Q_{iq}^{\frac{1}{2}},$$
$$\begin{bmatrix} \varepsilon I & \Lambda_{iq} \\ \Lambda_{iq}^T & I \end{bmatrix} > 0, i \in \mathbb{N},\ q = 1, \ldots, N.$$

Step 6. Put $\varepsilon = \varepsilon + \Delta\varepsilon$.

Step 7. If $\varepsilon > 1$, then stop, else go to step 4.

The algorithm for computation of the simultaneously stabilizing gain follows from the algorithm above as a special case.

VI. An example of application of the proposed method

In flight control practice it is very important to obtain an output feedback controller with a constant gain to stabilize the aircraft in all the possible flight modes. In this section we briefly demonstrate the application of the proposed method in the design of a control system for the linearized model of the angular longitudinal aircraft motion. This model is given by the following equations

$$\begin{aligned} \dot{\vartheta} &= \omega_z, \\ \dot{\omega}_z &= -a_{mz}^{\alpha} \vartheta - a_{mz}^{\omega z} \omega_z + a_{mz}^{\alpha} \Theta + a_{mz}^{\delta} \delta, \\ \dot{\Theta} &= -a_y^{\alpha} \vartheta + a_y^{\alpha} \Theta, \end{aligned} \qquad (39)$$

where ϑ is the pitch angle, ω_z is the angular velocity, $\Theta = \vartheta - \alpha$, α is the angle of attack, δ is the elevator angle. In this case the state and control vectors of the system (6) are

$$x(t) = [\vartheta \ \omega_z \ \Theta]^T, \ \ u(t) = \delta,$$

Usually only ϑ and ω_z are available for direct measurement and we have

$$y(t) = [\vartheta \ \omega_z]^T.$$

The considered aircraft has nine typical flight modes with uncertainty of each mode given by

$$\begin{aligned} A_{ilk} &= A_{ilk}^0 + \Delta A_{ilk}^0 \\ B_{ik} &= B_{ik}^0 + \lambda B_{ik}^0,\ i = 1, \ldots, 9,\ l = 2, 3,\ k = 1, 2, 3, \\ \underline{\Delta} &\le \Delta \le \overline{\Delta},\ \underline{\lambda} \le \lambda \le \overline{\lambda}. \end{aligned}$$

The numerical values of the parameters are the following [11]:

$$A_1^0 = \begin{bmatrix} 0 & 1 & 0 \\ -4.2 & -1.5 & 4.2 \\ 0.77 & 0 & -0.77 \end{bmatrix},\ B_1^0 = \begin{bmatrix} 0 \\ -7.4 \\ 0 \end{bmatrix},$$

$$A_2^0 = \begin{bmatrix} 0 & 1 & 0 \\ -7.1 & -1.9 & 7.1 \\ 1 & 0 & -1 \end{bmatrix},\ B_2^0 = \begin{bmatrix} 0 \\ -12.7 \\ 0 \end{bmatrix},$$

$$A_3^0 = \begin{bmatrix} 0 & 1 & 0 \\ -78 & -4.1 & 78 \\ 2.8 & 0 & -2.8 \end{bmatrix},\ B_3^0 = \begin{bmatrix} 0 \\ -57 \\ 0 \end{bmatrix},$$

$$A_4^0 = \begin{bmatrix} 0 & 1 & 0 \\ -4 & -1.4 & 4 \\ 0.62 & 0 & -0.62 \end{bmatrix},\ B_4^0 = \begin{bmatrix} 0 \\ -7.5 \\ 0 \end{bmatrix},$$

$$A_5^0 = \begin{bmatrix} 0 & 1 & 0 \\ -116 & -2.36 & 116 \\ 2.3 & 0 & -2.3 \end{bmatrix},\ B_5^0 = \begin{bmatrix} 0 \\ -42 \\ 0 \end{bmatrix},$$

$$A_6^0 = \begin{bmatrix} 0 & 1 & 0 \\ -7.9 & -1.1 & 7.9 \\ 0.56 & 0 & -0.56 \end{bmatrix},\ B_6^0 = \begin{bmatrix} 0 \\ -13.8 \\ 0 \end{bmatrix},$$

$$A_7^0 = \begin{bmatrix} 0 & 1 & 0 \\ -55 & -0.66 & 55 \\ 0.84 & 0 & -0.84 \end{bmatrix},\ B_7^0 = \begin{bmatrix} 0 \\ -22.5 \\ 0 \end{bmatrix},$$

$$A_8^0 = \begin{bmatrix} 0 & 1 & 0 \\ -14.5 & -0.43 & 14.5 \\ 0.33 & 0 & -0.33 \end{bmatrix},\ B_8^0 = \begin{bmatrix} 0 \\ -8.6 \\ 0 \end{bmatrix},$$

$$A_9^0 = \begin{bmatrix} 0 & 1 & 0 \\ -18 & -0.31 & 18 \\ 0.34 & 0 & -0.34 \end{bmatrix},\ B_9^0 = \begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix},$$

$$C_i^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, i = 1, \ldots, 9,\ \underline{\Delta} = -0.2,\ \overline{\Delta} = 0.2,$$

$$\underline{\lambda} = -0.25\ \overline{\lambda} = 0.25.$$

The problem is to stabilize the system (39) independently on the mode change process for given uncertain parameters of each mode by means of constant static output feedback control (33).

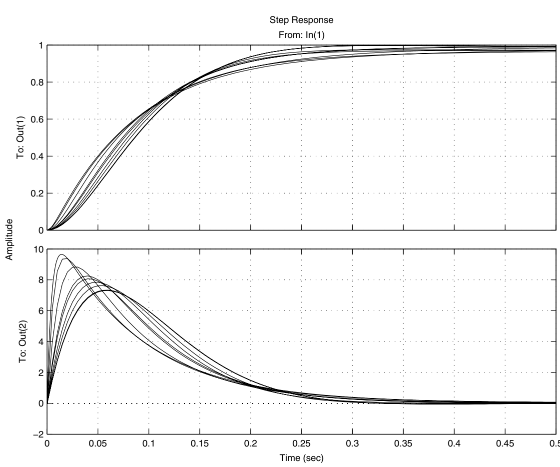

Fig. 1. The step responses in different modes.

To find the gain matrix (35) of this control we used the proposed method and the algorithm based on direct solving of CRE.

For all the modes it is supposed that the control weight matrices $R(i) = I$ and the state weight matrices M_{iq} were computed by a special routine based on the Jonson's method [8].

Figure 1 shows the step responses of the closed-loop system in all the nine modes with the computed gain matrix $G = [-57.76\ -5.26]$.

All LMI/LME programming was done within the framework of the YALMIP interface to the SeDuMi solver for MATLAB.

REFERENCES

[1] M. Ait Rami and L. El Ghaoui, "LMI optimization for nonstandard Riccati equation arising in stochastic control", IEEE Trans. Automat. Control, vol.41, pp. 1666-1671, 1996.

[2] E. K. Boukas and Z. Liu, *Deterministic and stochastic time-delay systems*, Birkhauser, Boston, 2001.

[3] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear matrix inequalities in control and system theory*, SIAM, Philadelphia, 1994.

[4] Y. Y. Cao, J. Lam and Y. X. Sun, "Static output feedback stabilization: an ILMI approach", Automatica, vol. 34, pp. 1641-1645, 1998.

[5] L. El Ghaoui and M. Ait Rami, "Robust state feedback stabilization of jump linear systems via LMI's" Int. J. Robust and Nonlinear Control", vol. 6, pp. 1015-1022, 1996.

[6] G. Garcia, B. Pradin, S. Tarbouriech and F. Zeng, "Robust stabilization and guaranteed cost control for discrete-time linear systems by static output feedback", Automatica, vol.39, pp. 1635-1641, 2003.

[7] T. Iwasaki and R. E. Skelton, "Parametrization of all stabilizing controllers via quadratic Lyapunov functions", *J. Optimization Theory and Appl.*, vol. 85, pp. 291-307, 1995.

[8] C.D. Johnson, "The 'unreachable poles' defect in LQR theory: analysis and remedy", *Int. J. Control*, vol. 47, pp. 697-709, 1988.

[9] I. Ya. Kats, *Lyapunov function method in problems of stability and stabilization of random structure systems*, UGAPS, Ekaterinburg, 1998, (in Russian).

[10] R. Z. Khas'minskii. *Stochastic stability of differential equations*. Sijthoff & Nordhoff, (1980).

[11] A.A. Krasovskii *Systems of the authomatic flight control and their analytical design*. Nauka, Moscow, 1973.

[12] V. Kučera and C. E. De Souza, "A necessary and sufficient conditions for output feedback stabilizability", *Automatica*, vol.31, pp. 1357-1359, 1995.

[13] M. Mariton, *Jump linear systems in automatic control*, Marcel Dekker, New York, 1990.

[14] A. Trofino-Neto and V. Kučera, "Stabilization via output feedback", *IEEE Trans. Automatic Control*, vol.38, pp. 764-765, 1993.

[15] P. V. Pakshin and D. M. Retinsky, "Robust stabilization of jumping systems via static output feedback", *Proceedings of the European Control Conference*, EUCA, Cambridge, UK, 2003, pp. 1-6. (CD ROM).

[16] P. V. Pakshin , I. N. Mitrofanov " Parametrization of stabilizing controllers and robust stabilization via static output feedback for jump linear systems." *Proceedings of the 6th IFAC Symposium Nonlinear Control Systems*, Stuttgart, Germany, September 1-3, 2004. Vol.3. pp. 1151 - 1156.

[17] V. L. Syrmos, C. T. Abdallah, P. Dorato and K. Grigoriadis, "Static output feedback - A survey", *Automatica*, vol.33, pp. 125-137, 1997.

[18] J. Yu "A convergent algorithm for computing stabilizing static output feedback gains" *IEEE Trans. Automat. Control*, vol.49, pp.2271-2275, 2004.