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Coupled Dynamic Systems: From Structure Towards State Agreement

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Abstract— **The state agreement problem is studied for nonlinear continuous-time systems. A general interconnection of nonlinear subsystems is treated, where the vector fields can switch within a finite family. Associated to each vector field is a directed graph based in a natural way on the interaction structure of the subsystems. With the assumption that the vector fields satisfy a certain sub-tangentiality condition, it is proved that asymptotic state agreement is achieved if and only if the dynamic interaction digraph has the property of being sufficiently connected over time. Applications of the main result are then made to the synchronization of coupled Kuramoto oscillators with time-varying interaction and to the analysis of a biochemical reaction network.**

I. Introduction

This paper studies the state agreement problem for coupled dynamic systems. State agreement means that the states of the subsystems are all equal. The problem arises naturally in biology, physics, engineering, ecology, and social science: e.g., synchronization [9], [20], consensus seeking [3], [5], [18], and rendezvous [2], [4], [10], [11]. Recent relevant work on this problem can be found in [7], [8], [12], [13], [16], [17].

Inspired by [17], our goal in this paper is to solve the state agreement problem for nonlinear continuous-time subsystems with time-varying interaction. Our setup is a general interconnection of nonlinear subsystems, where the vector fields can switch within a finite family. We associate to each vector field a directed graph based in a natural way on the interaction structure of the subsystems; this is called an *interaction digraph* in the present paper. Assuming that the vector fields satisfy a certain sub-tangentiality condition, we show that asymptotic state agreement is achieved if and only if the dynamic interaction digraph has the property of being sufficiently connected over time, in a certain technical sense.

As applications, we apply our main result to the synchronization of coupled Kuramoto oscillators with time-varying interaction and to the analysis of a biochemical reaction network.

All proofs are omitted due to pagelength requirements, which are available in [15].

II. Preliminaries

We first assemble some known and some novel concepts related to tangent cones and directed graphs.

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A. Tangent Cones

The convex hull of a finite set of points $x_1, \dots, x_n \in \mathbb{R}^m$ is a *polytope*, denoted $\mathrm{co}\{x_1, \dots, x_n\}$. Given a convex set $\mathcal{S} \subset \mathbb{R}^m$, its *relative interior*, denoted $\mathrm{ri}(\mathcal{S})$, is its interior in the smallest affine subspace containing $\mathcal{S}$ (which might be of dimension strictly less than m).

Fix any norm $\|\cdot\|$ in $\mathbb{R}^m$. For each nonempty set $\mathcal{S} \subset \mathbb{R}^m$ and each $y \in \mathbb{R}^m$, we denote the distance of y from $\mathcal{S}$ by $\|y\|_{\mathcal{S}} := \inf_{z \in \mathcal{S}} \|z - y\|$.

A nonempty set $\mathcal{K} \subset \mathbb{R}^m$ is called a *cone* if $\lambda y \in \mathcal{K}$ when $y \in \mathcal{K}$ and $\lambda > 0$. Let $\mathcal{S} \subset \mathbb{R}^m$ be a closed convex set and $y \in \mathcal{S}$. The *tangent cone* (often referred to as *contingent cone*) to $\mathcal{S}$ at y is the set

$$\mathcal{T}(y, \mathcal{S}) = \left\{ z \in \mathbb{R}^m : \liminf_{\lambda \to 0} \frac{\|y + \lambda z\|_{\mathcal{S}}}{\lambda} = 0 \right\}.$$

Note that if y is in the interior of $\mathcal{S}$, then $\mathcal{T}(y, \mathcal{S}) = \mathbb{R}^m$. Thus the set $\mathcal{T}(y, \mathcal{S})$ is non-trivial only on the boundary of $\mathcal{S}$. In particular, if $\mathcal{S}$ contains only one point, y, then $\mathcal{T}(y, \mathcal{S}) = \{0\}$. In geometric terms (see Fig. 1), the tangent

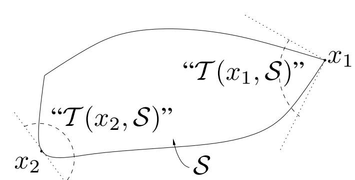

Fig. 1. Tangent cones $\mathcal{T}(x_1, \mathcal{S})$ and $\mathcal{T}(x_2, \mathcal{S})$ are obtained by translation of "$\mathcal{T}(x_1, \mathcal{S})$" and "$\mathcal{T}(x_2, \mathcal{S})$" to the origin.

cone for y in the boundary of $\mathcal{S}$ is a cone having center in the origin which contains all vectors whose directions point from y 'inside' (or they are 'tangent to') the set $\mathcal{S}$.

B. Directed Graphs

For a directed graph (digraph for short) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \dots, v_n\}$ is the set of nodes and $\mathcal{E}$ is the set of arcs, if there is a path in $\mathcal{G}$ from one node v_i to another node v_j, then v_j is said to be *reachable* from v_i, written $v_i \to v_j$. Note that every node of a digraph is reachable from itself.

A digraph is said to be *quasi strongly connected* (QSC) (called *arbitrated* in [6]) if for every two nodes v_i and v_j there is a node v from which v_i and v_j are reachable.

III. Definitions and Main Results

To formalize the notion of a switched interconnected system, suppose that we are given a family of systems represented by the equations

$$\begin{aligned}\dot{x}_1 &= f_p^1(x_1, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_p^n(x_1, \ldots, x_n),\end{aligned}$$

where $x_i \in \mathbb{R}^m$ is the state of subsystem i and where the index p lives in a finite set $\mathcal{P}$. Notice that the subsystems share a common state space, $\mathbb{R}^m$.

Introducing the *aggregate state* $x \in \mathbb{R}^{mn}$, we have the concise form

$$\dot{x} = f_p(x), \quad p \in \mathcal{P}, \tag{1}$$

where for each $p \in \mathcal{P}$, $f_p : \mathbb{R}^{mn} \to \mathbb{R}^{mn}$.

We now associate to each vector field f_p an interaction digraph $\mathcal{G}_p$ capturing the interaction structure of the n subsystems (agents).

Definition 1: An *interaction digraph* $\mathcal{G}_p$ consists of

- a finite set $\mathcal{V}$ of n nodes, each node i modeling agent i;
- an arc set $\mathcal{E}_p$ representing the links between agents. An arc from node j to node i indicates that agent j is a neighbor of agent i in the sense that f_p^i depends on x_j, i.e., there exist $x_j^1, x_j^2 \in \mathbb{R}^m$ such that

$$f_p^i(x_1, \ldots, x_j^1, \ldots, x_n) \neq f_p^i(x_1, \ldots, x_j^2, \ldots, x_n).$$

The set of neighbors of agent i is denoted $\mathcal{N}_i(p)$.

Let $\mathcal{C}_p^i = \mathrm{co}\{x_i, x_j : j \in \mathcal{N}_i(p)\}$ denote the polytope in $\mathbb{R}^m$ formed by the states of agent i and its neighbors. Also, it's convenient to introduce a subset $\mathcal{S} \subset \mathbb{R}^m$ of the common state space that plays the role of a region of focus. In our state agreement problem, initial states of the agents will be in $\mathcal{S}$ and agreement will occur in $\mathcal{S}$. Let $\mathcal{I}_0$ denote the index set $\{1, \ldots, n\}$ and assume that, for each $i \in \mathcal{I}_0$ and each $p \in \mathcal{P}$, the vector fields $f_p^i : \mathbb{R}^{mn} \to \mathbb{R}^m$ satisfy the following two assumptions:

A1: f_p^i is locally Lipschitz on $\mathcal{S}^n$;

A2: For all $x \in \mathcal{S}^n$, $f_p^i(x) \in \mathrm{ri}\left(\mathcal{T}(x_i, \mathcal{C}_p^i)\right)$.

Assumption A2 is sometimes referred to as a *strict sub-tangentiality condition*. Fig. 2 illustrates two example situations of A2. In the left-hand example, agent 1 has only one

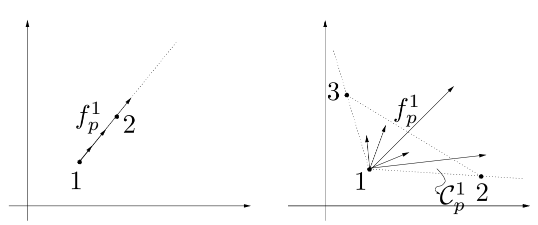

Fig. 2. Some examples of vector fields f_p^i satisfying assumption A2.

neighbor, agent 2; the convex hull $\mathcal{C}_p^1$ is the line segment joining x_1 and x_2; the tangent cone $\mathcal{T}(x_1, \mathcal{C}_p^1)$ is the closed ray $\{\lambda(x_2 - x_1) : \lambda \geq 0\}$ (in the picture it's shown translated to x_1); the relative interior $\mathrm{ri}\left(\mathcal{T}(x_1, \mathcal{C}_p^1)\right)$ is the open ray $\{\lambda(x_2 - x_1) : \lambda > 0\}$; and A2 means that f_p^1 is nonzero and points in the direction of $x_2 - x_1$. In the right-hand example, agent 1 has two neighbors, agents 2 and 3; the convex hull $\mathcal{C}_p^1$ is the triangle with vertices x_1, x_2, x_3; the tangent cone $\mathcal{T}(x_1, \mathcal{C}_p^1)$ is

$$\{\lambda_1(x_2 - x_1) + \lambda_2(x_3 - x_1) : \lambda_1, \lambda_2 \geq 0\}$$

(again, it's shown translated to x_1); the relative interior $\mathrm{ri}\left(\mathcal{T}(x_1, \mathcal{C}_p^1)\right)$ is

$$\{\lambda_1(x_2 - x_1) + \lambda_2(x_3 - x_1) : \lambda_1, \lambda_2 > 0\};$$

and A2 means that f_p^1 points into this open cone. In general, A2 requires that $f_p^i(x)$ have the form

$$\sum_{j \in \mathcal{N}_i(p)} \alpha_j(x)(x_j - x_i),$$

where $\alpha_j(x)$ are non-negative scalar functions, and that $f_p^i(x)$, now viewed as a vector applied at the vertex x_i, not be tangent to the relative boundary of the convex set $\mathcal{C}_p^i$.

When the index p in (1) is replaced by a piecewise constant function $\sigma : [0, \ \infty) \to \mathcal{P}$, we obtain a *switched interconnected system*

$$\dot{x}(t) = f_{\sigma(t)}(x(t)). \tag{2}$$

The function σ is called a *switching signal*. The case of infinitely fast switching (chattering), which would call for a concept of generalized solution, is not considered here. As a matter of fact, we shall show in the next section by means of a counterexample that even piecewise constant switching signals $\sigma(t)$ do not have sufficient regularity for asymptotic agreement of the switched interconnected system (2). Let $\mathcal{S}_{dwell}$ denote the class of piecewise constant switching signals such that any consecutive discontinuities are separated by no less than some fixed positive constant τ_D, the *dwell time*. We make the following assumption:

A3: $\sigma(t) \in \mathcal{S}_{dwell}$.

Having replaced p by a switching signal $\sigma(t)$, we similarly replace the interaction digraph $\mathcal{G}_p$ by a dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$.

Definition 2: Given a switching signal $\sigma(t)$, $\sigma : [0, \infty) \to \mathcal{P}$, the *dynamic interaction digraph* $\mathcal{G}_{\sigma(t)}$ is the pair $\left(\mathcal{V}, \mathcal{E}_{\sigma(t)}\right)$. Given two real numbers $t_1 \leq t_2$, *the union digraph* $\mathcal{G}\left([t_1, t_2]\right)$ is the digraph whose arcs are obtained from the union of the arcs in $\mathcal{G}_{\sigma(t)}$ over the time interval $[t_1, t_2]$, that is, $\mathcal{G}\left([t_1, t_2]\right) = \left(\mathcal{V}, \bigcup_{t \in [t_1, t_2]} \mathcal{E}_{\sigma(t)}\right)$.

Definition 3: A dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$ is *uniformly quasi strongly connected (UQSC)* if there exists $T > 0$ such that for all $t \geq 0$, the union digraph $\mathcal{G}([t, t + T])$ is QSC.

We show in the following that if, and only if, the dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$ is UQSC, then the switched interconnected system achieves asymptotic state agreement on $\mathcal{S}$.

Now comes the precise meaning of state agreement.

Definition 4: The switched interconnected system (2) has the property of

1) *state agreement (SA) on* $\mathcal{S}$ if $\forall \zeta \in \mathcal{S}$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall t_0 \geq 0$

$$(\forall i) \ \ (\|x_i(t_0) - \zeta\| \leq \delta) \wedge (x_i(t_0) \in \mathcal{S})$$

$$\implies (\forall t \geq t_0)(\forall i)\ \|x_i(t) - \zeta\| \leq \varepsilon;$$

2) *asymptotic state agreement (ASA) on $\mathcal{S}$* if it has the property of state agreement on $\mathcal{S}$ and in addition $\forall \varepsilon > 0,\ \forall c > 0,\ \exists T > 0$ such that $\forall t_0 \geq 0$

$$(\forall i)\ (\|x_i(t_0)\| \leq c) \wedge (x_i(t_0) \in \mathcal{S})$$

$$\implies (\exists \zeta \in \mathcal{S})(\forall t \geq t_0 + T)(\forall i)\ \|x_i(t) - \zeta\| \leq \varepsilon;$$

3) *global asymptotic state agreement (GASA)* if it has the property of ASA on $\mathbb{R}^m$.

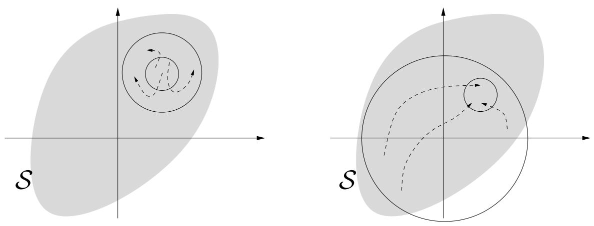

Fig. 3. Asymptotic state agreement on $\mathcal{S}$.

These definitions are illustrated in Fig. 3 and can be said roughly speaking as follows. State agreement (the left-hand figure) means, for every point ζ in $\mathcal{S}$, the agents stay arbitrarily close to ζ if they start sufficiently close to ζ, uniformly with respect to the starting time. Asymptotic state agreement (the two figures together) means, in addition, the agents converge to a common location in $\mathcal{S}$.

These state agreement definitions are related to stability with respect to a set. Let Ω denote the set of aggregate states such that the subsystem states are all equal and in $\mathcal{S}$, i.e.,

$$\Omega = \{x \in \mathbb{R}^{nm} : x_1 = \cdots = x_n \in \mathcal{S}\}.$$

Then state agreement is equivalent to uniform stability with respect to Ω.

Finally, a new definition of positive invariance specially for interconnected systems:

Definition 5: A set $\mathcal{A} \subset \mathbb{R}^m$ is said to be *positively invariant* for the switched interconnected system (2) if

$$(\forall t_0 \geq 0)(\forall i)\ x_i(t_0) \in \mathcal{A} \implies (\forall t \geq t_0)(\forall i)\ x_i(t) \in \mathcal{A}.$$

Our first result establishes the positive invariance property of any compact convex set in $\mathcal{S}$ without needing any property of the interaction digraph. This result can perhaps be understood intuitively as follows. For $m = 2$, all agents move in the plane. Let $\mathcal{A}$ be a compact convex set in $\mathcal{S}$ and assume all agents start in $\mathcal{A}$. Let $\mathcal{C}(t)$ denote the convex hull of the agents' locations at time t. Because $\mathcal{A}$ is convex, clearly $\mathcal{C}(0) \subset \mathcal{A}$. Now invoke assumption A2. An agent that is initially in the interior of $\mathcal{C}(0)$ can head off in any direction at $t = 0$, but an agent that is initially on the boundary of $\mathcal{C}(0)$ is constrained to head into its interior. In this way, $\mathcal{C}(t)$ is non-increasing (if $t_2 > t_1$, then $\mathcal{C}(t_2) \subset \mathcal{C}(t_1)$), and $\mathcal{A}$ is therefore positively invariant for the switched interconnected system (2).

Theorem 1: Let $\mathcal{A} \subset \mathcal{S}$ be a compact convex set. Then $\mathcal{A}$ is positively invariant for the switched interconnected system (2).

The second result establishes state agreement of the system, again without needing any property of the interaction digraph.

Theorem 2: Suppose $\mathcal{S}$ is closed and convex. The switched interconnected system (2) has the property of state agreement on $\mathcal{S}$.

Now comes our main result.

Theorem 3: Suppose $\mathcal{S}$ is closed and convex. The switched interconnected system (2) has the property of asymptotic state agreement on $\mathcal{S}$ if and only if the dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$ is UQSC.

Remark. When $\mathcal{S} = \mathbb{R}^m$ in assumptions A1 and A2, the switched interconnected system (2) has the global asymptotic state agreement property if and only if $\mathcal{G}_{\sigma(t)}$ is UQSC.

In the special case when $\sigma(t)$ is a constant signal, that is, $\sigma(t) \equiv p$ for some $p \in \mathcal{P}$, then the switched interconnected system becomes time-invariant and $\mathcal{G}_{\sigma(t)}$ is just a fixed interaction digraph $\mathcal{G}_p$. In this case, the property of UQSC is equivalent to QSC. Thus, we arrive at the following special result.

Corollary 4: Suppose $\sigma(t) = p$ and $\mathcal{S} = \mathbb{R}^m$. Then, the interconnected system (2) has the globally asymptotic state agreement property if and only if $\mathcal{G}_p$ is QSC.

Remark. For this special case we can actually relax the assumptions on the vector fields $f_p^i : \mathbb{R}^{mn} \to \mathbb{R}^m$ as follows:
A1$'$: f_p^i is continuous on $\mathbb{R}^{mn}$;
A2$'$: For all $x \in \mathbb{R}^{mn}$, $f_p^i(x) \in \mathcal{T}\left(x_i, \mathcal{C}_p^i\right)$, but $f_p^i(x) \neq 0$ if $\mathcal{C}_p^i$ is not a singleton and x_i is its vertex.

The sketch of the proof can be found in [14]. Unlike the proof of Theorem 3, the proof in [14] relies on LaSalle's invariance principle. As shown in the next section by means of a counterexample, when the interaction digraph is dynamic assumption A1$'$ is too weak for sufficiency in Theorem 3 to hold.

IV. Some Examples and Further Remarks

In this section we present some examples to better illustrate the nature of our assumptions.

A. Concerning Assumption A1

We now present an example showing that Theorem 3 may fail to hold when the vector fields are just continuous instead of locally Lipschitz.

Example 4.1. Consider three agents, 1, 2, and 3, with state space $\mathbb{R}$. There are three possible vector fields:

$$p = 1: \left\{\begin{array}{l} \dot{x}_1 = g(x_3 - x_1) \\ \dot{x}_2 = 0 \\ \dot{x}_3 = 0 \end{array}\right\}, \quad p = 2: \left\{\begin{array}{l} \dot{x}_1 = g(x_2 - x_1) \\ \dot{x}_2 = 0 \\ \dot{x}_3 = 0 \end{array}\right\},$$

$$p = 3: \left\{\begin{array}{l} \dot{x}_1 = 0 \\ \dot{x}_2 = g(x_1 - x_2) \\ \dot{x}_3 = 0 \end{array}\right\},$$

where $g(y) := \mathrm{sign}(y) \cdot |y|^{\frac{1}{2}}$, $y \in \mathbb{R}$. The function g has the property that each solution of the differential equation $\dot{y} = g(y)$ reaches the origin (asymptotically stable equilibrium) in finite time.

For each $p \in \mathcal{P} = \{1, 2, 3\}$, the associated interaction digraphs are depicted in Fig. 4. Let $\mathcal{S} = \mathbb{R}$. Obviously, the function $g(\cdot)$ is only continuous (not locally Lipschitz on $\mathbb{R}$), so assumption A1 does not hold, but it can be easily checked that A2 holds. Let us set a switching signal $\sigma(t)$ to

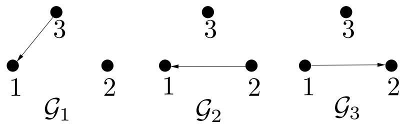

Fig. 4. The interaction digraphs $\mathcal{G}_p$, $p = 1, 2, 3$.

be periodic with period of 12 seconds, that is,

$$\sigma(t) = \begin{cases} 1, & t \in [12k, 12k+4), \\ 2, & t \in [12k+4, 12k+8), \\ 3, & t \in [12k+8, 12k+12), \end{cases} \qquad k = 0, 1, \ldots.$$

Thus, assumption A3 holds.

For the switched interconnected system corresponding to the switching signal above, the dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$ is UQSC. To see that, simply let $T = 12$ and notice that for any $t > 0$, $\mathcal{G}([t, t+T]) = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ is QSC. However, this switched interconnected system does not have the property of asymptotic state agreement on $\mathcal{S}$ as shown by a simulation in Fig. 5. Intuitively, for the period of

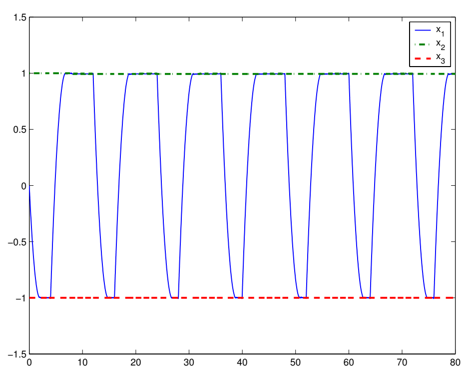

Fig. 5. Time evolution of three coordinates not tending to a common value.

$\sigma(t) = 1$, agent 1 moves toward agent 3 and the others remain stationary, whereas for the period of $\sigma(t) = 2$, agent 1 moves toward agent 2 and the others remain stationary. However, agent 1 reaches the location of agent 2 and stays there during this period. Then, when the system switches to $p = 3$, agent 2 starts to move toward agent 1, but since agents 1 and 2 are already collocated, agent 2 keeps stationary. Hence, only agent 1 moves forward and backward between the locations of agent 2 and 3 while the others are stationary.

B. Concerning Assumption A2

Our next example is concerned with the necessity of the strictness in assumption A2. This cannot be relaxed to just $f_p^i(x) \in \mathcal{T}(x_i, \mathcal{C}_p^i)$, as shown next.

Example 4.2. Consider two agents, 1 and 2, with state space $\mathbb{R}$. There is only one vector field:

$$p = 1 : \left\{ \begin{array}{l} \dot{x}_1 = f_1^1(x_1, x_2) = 0 \\ \dot{x}_2 = f_1^2(x_1, x_2) = g(x_1 - x_2) \end{array} \right\}$$

where the smooth function $g : \mathbb{R} \to \mathbb{R}$ is given in Fig. 6.

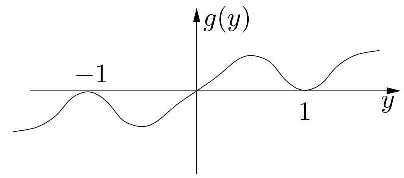

Fig. 6. A smooth function $g(y)$.

The interconnected system above has fixed coupling structure, that is, $\sigma(t) \equiv 1$. So assumption A3 is trivially satisfied. Let $\mathcal{S} = \mathbb{R}$. Assumption A1 holds, but A2 does not hold since $f_1^2(x_1, x_2) = g(x_1 - x_2) = 0 \notin \text{ri}\left(\mathcal{T}(x_2, \mathcal{C}_1^2)\right)$ when $x_1 = x_2 + 1$ by noticing that $\mathcal{C}_1^2 = \text{co}\{x_1, x_2\}$ is the line segment joining x_1 and x_2. However, $f_1^1(x_1, x_2)$ and $f_1^2(x_1, x_2)$ are in $\mathcal{T}(x_1, \mathcal{C}_1^1)$ and $\mathcal{T}(x_2, \mathcal{C}_1^2)$ respectively for all $(x_1, x_2) \in \mathcal{S} \times \mathcal{S}$.

In the associated interaction digraph of the unique vector field ($p = 1$), there is an arc from node 1 to 2. So it is QSC. Recalling that the property of UQSC is equivalent to QSC for fixed digraph, the dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$ is UQSC. But this interconnected system fails to achieve asymptotic state agreement on $\mathcal{S} = \mathbb{R}$ when, for example, initially $x_1(0) = x_2(0) + 1$.

However, if we choose $\mathcal{S} = [a, b]$, where a, b are real numbers such that $b - a < 1$, then assumptions A1, A2, and A3 hold. Thus, it follows that this interconnected system achieves asymptotic state agreement on $\mathcal{S}$ since the dynamic interaction digraph $\mathcal{G}_{\sigma(t)}$ is UQSC as shown before.

C. Concerning Assumption A3

Although the switched interconnected system (2) has the property of state agreement under piecewise constant switching signals, additional regularity conditions on the switching signal $\sigma(\cdot)$ are needed in order to guarantee asymptotic state agreement. This is illustrated by the following very simple linear example.

Example 4.3. Consider just two agents, 1 and 2, with state space $\mathbb{R}$. There are two possible vector fields:

$$p = 1 : \left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2 - x_1 \\ \dot{x}_2 & = & 0 \end{array} \right\}, \quad p = 2 : \left\{ \begin{array}{lcl} \dot{x}_1 & = & 0 \\ \dot{x}_2 & = & 0 \end{array} \right\}$$

Thus agent 2 has no neighbor and never moves. For $p = 1$ agent 1 moves toward agent 2, whereas for $p = 2$ agent 1 has no neighbor and therefore doesn't move. Assumptions A1 and A2 hold for $\mathcal{S} = \mathbb{R}$. Let us define switching times τ_k by setting $\tau_0 = 0$ and defining the intervals $\delta_k = \tau_{k+1} - \tau_k$ as follows:

k	0	1	2	3	4	5	6	$\cdots$
δ_k	1	1	1/2	1	$1/2^2$	1	$1/2^3$	$\cdots$

Then we define $\sigma(t)$ to be the alternating sequence $1, 2, 1, 2, \ldots$ over the time intervals, respectively,

$$[\tau_0, \tau_1), [\tau_1, \tau_2), [\tau_2, \tau_3), [\tau_3, \tau_4), \ldots$$

This switching signal is piecewise constant and the dynamic interaction digraph is UQSC. However, if $x_1(0) \neq x_2(0)$, $x_1(t)$ does not converge to $x_2(t)$—asymptotic state agreement does not occur.

The example suggests that in order to obtain asymptotic state agreement, one needs to impose some restrictions on the admissible switching signals. One way to address this problem is to make sure that the switching signal has a dwell time, that is, there exists $\tau_D > 0$ such that

$$(\forall k)\ (\tau_{k+1} - \tau_k) \geq \tau_D.$$

This is precisely the assumption A3, and is ubiquitous in the switching control literature.

V. Some Applications

In this section we discuss some applications of our main results.

A. Synchronization of Coupled Oscillators

The Kuramoto model describes the dynamics of a set of n phase oscillators θ_i with natural frequencies ω_i. More details can be found in [9], [20]. The time evolution of the i-th oscillator is given by

$$\dot{\theta}_i = \omega_i + k_i \sum_{j \in \mathcal{N}_i(t)} \sin(\theta_j - \theta_i),$$

where $k_i > 0$ is the coupling strength and $\mathcal{N}_i(t)$ is the set of neighbors of oscillator i at time t. The interaction structure can be general so far, that is, $\mathcal{N}_i(t)$ can be an arbitrary set of other nodes and can be dynamic.

The neighbor sets $\mathcal{N}_i(t)$ define $\mathcal{G}_{\sigma(t)}$ and the switched interconnected system

$$\dot{\theta} = f_{\sigma(t)}(\theta),$$

where $\theta = [\theta_1 \ \cdots \ \theta_n]^T$ and $\sigma(t)$ is a suitable switching signal. For identical coupled oscillators (i.e., $\omega_i = \omega, \forall i$), the transformation $x_i = \theta_i - \omega t$ yields

$$\dot{x}_i = k_i \sum_{j \in \mathcal{N}_i(t)} \sin(x_j - x_i), \quad i = 1, \ldots, n. \qquad (3)$$

Let a, b be any real numbers such that $0 \leq b - a < \pi$, and define $\mathcal{S} = [a, b]$. It is easily seen that A1 and A2 are satisfied. Suppose $\sigma(t)$ here is regular enough satisfying A3. Then from Theorem 3 it follows that if, and only if, $\mathcal{G}_{\sigma(t)}$ is UQSC, the switched interconnected system (3) has the property of asymptotic state agreement on $\mathcal{S}$. This implies that there exists $\bar{x} \in \mathbb{R}$ such that

$$\theta_i(t) \to \bar{x} + \omega t, \quad \dot{\theta}_i(t) \to \omega,$$

and the oscillators synchronize. This is an extension of Theorem 1 in [9], which assumes the interaction graph is undirected and static and the initial state $\theta_i(0) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for all i.

As an example, three Kuramoto oscillators with dynamic interaction structure are simulated. The initial conditions are $\theta_1 = 0$, $\theta_2 = 1$, $\theta_3 = -1$. The natural frequency ω_i equals 1, and the coupling strength k_i is set to 1 for all i. The interaction structure switches among three possible interaction structures periodically, shown in Fig. 7. It can

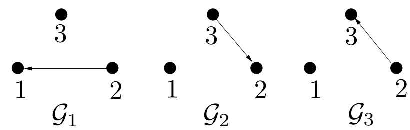

Fig. 7. Three interaction digraphs $\mathcal{G}_p,\ p = 1, 2, 3$.

be checked that $\mathcal{G}_{\sigma(t)}$ is UQSC. So these three oscillators achieve asymptotical synchronization as we conclude by our main theorem. Fig. 8 shows the plots of $\sin(\theta_i),\ i = 1, 2, 3$ and of the switching signal $\sigma(t)$. Synchronization is evident.

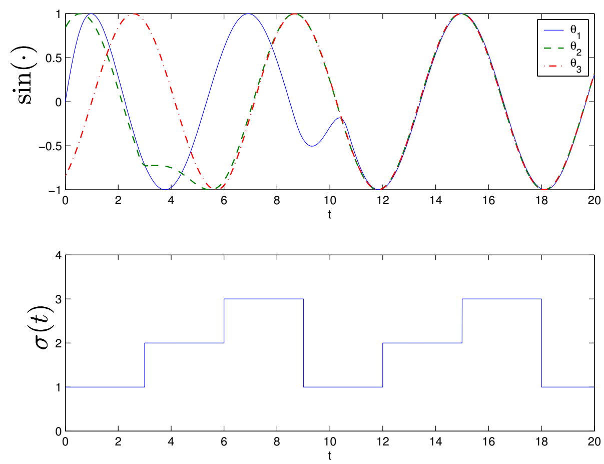

Fig. 8. Synchronization of three interacting oscillators.

B. Biochemical Reaction Network Analysis

A biochemical reaction network is a finite set of reactions among a finite set of species. Consider, for example, two reversible reactions among three compounds C_1, C_2, and C_3, in which C_1 is transformed into C_2, C_2 is transformed into C_3, and vice versa:

$$C_1 \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C_2 \underset{k_4}{\overset{k_3}{\rightleftharpoons}} C_3$$

The constants $k_1 > 0$, $k_2 > 0$ are the forward and reverse rate constants of the reaction $C_1 \rightleftharpoons C_2$; similarly for $k_3 > 0$, $k_4 > 0$. Denote the concentrations of C_1, C_2, and C_3, respectively, by x_1, x_2, and x_3. Only nonnegative concentrations are physically possible. Such a reaction network gives rise to a dynamical system, which describes how the state of the network changes over time.

Suppose the dynamics of both reactions are dictated by the mass action principle. This leads to the model

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1^\alpha + k_2 x_2^\alpha, \\ \dot{x}_2 &= k_1 x_1^\alpha - k_2 x_2^\alpha - k_3 x_2^\alpha + k_4 x_3^\alpha, \\ \dot{x}_3 &= k_3 x_2^\alpha - k_4 x_3^\alpha, \end{aligned} \qquad (4)$$

where $\alpha \geq 1$ is an integer. For more on modeling and analysis of biochemical reaction networks, we refer to [1], [21].

The linear transformation

$$y_1 = \left(\frac{k_1}{k_2}\right)^{\frac{1}{\alpha}} x_1, \quad y_2 = x_2, \quad y_3 = \left(\frac{k_4}{k_3}\right)^{\frac{1}{\alpha}} x_3,$$

leads to

$$\begin{aligned} \dot{y}_1 &= h_1(y_1, y_2)(y_2 - y_1), \\ \dot{y}_2 &= h_2(y_1, y_2)(y_1 - y_2) + h_3(y_2, y_3)(y_3 - y_2), \\ \dot{y}_3 &= h_4(y_2, y_3)(y_2 - y_3), \end{aligned} \tag{5}$$

where $h_1(y_1, y_2)$, $h_2(y_1, y_2)$, $h_3(y_2, y_3)$, and $h_4(y_2, y_3)$ are suitable terms; for example

$$h_1(y_1, y_2) = \frac{k_1^{1/\alpha} k_2}{k_2^{1/\alpha}} \frac{y_2^\alpha - y_1^\alpha}{y_2 - y_1}.$$

It can be easily verified that $h_1(y_1, y_2) \geq 0$ and $h_1(y_1, y_2) = 0$ if and only if $y_1 = y_2 = 0$. The same observations hold for $h_2(y_1, y_2)$, $h_3(y_2, y_3)$, and $h_4(y_2, y_3)$. It thus follows that each point in the set $\Omega = \{y : y_1 = y_2 = y_3 \geq 0\}$ is an equilibrium. Physically, when $y \in \Omega$, the reaction network is at a chemical equilibrium.

Consider now the interaction digraph associated with (5). Physically, each node represents a compound and each arc connecting two nodes represents a reaction between two compounds. This digraph is QSC (actually, it is strongly connected). Since there is no switching in the system (i.e., $\sigma(t)$ is constant), assumption A3 is obviously satisfied and the dynamic interaction digraph is UQSC. In addition, it can be easily checked that, for $\mathcal{S} = [0, \infty)$, the vector field in the above system satisfies assumptions A1 and A2. Hence, Theorem 3 can be applied to conclude that system (5) has the property of asymptotic state agreement on $\mathcal{S}$. This result coincides with the analysis using Theorem 5.2 in [1]. Our analysis can be extended to more complicated biochemical reaction networks containing a set of compounds and a set of reversible reactions. Their asymptotic state agreement property is captured by the interaction digraph.

VI. Conclusions

In this paper, we have studied the state agreement problem for a class of switched interconnected large-scale systems with a family of admissible vector fields. Necessary and sufficient conditions, in terms of the interaction graph, are obtained to assure that the system achieves asymptotic state agreement. On the other hand, our results can be understood as connective stability, as in the framework of [19]. Achieving asymptotic state agreement of a large-scale interconnected system is robust with respect to either the coupling structure or parameter values.

The notion of state agreement in this paper is that the states of the subsystems are all equal and constant. This notion can potentially be generalized in the following two directions. First, state agreement could mean equality of all the trajectories of the subsystems. This would be of interest in formation control of multi-agent systems. Second, state agreement could mean equality of all the states after suitable state transformations. An example is the biochemical reaction network studied in this paper.

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