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Lyapunov exponent and joint spectral radius: some known and new results

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Abstract— **The logarithm of joint spectral radius of a set of matrices coincides with Lyapunov exponent of corresponding linear inclusions. Main results about Lyapunov exponents of discrete time and continuous time linear inclusions are presented. They include the existence of extremal norm; relations between Lyapunov indices of dual inclusions; maximum principle for linear inclusions; algebraic criteria for stability of linear inclusions; algorithm to find out the sign of Lyapunov exponents. The main result is extended to linear inclusions with delays. The Aizerman problem for three-ordered time-varying continuous time systems with one nonlinearity is solved. The Perron-Frobenius theorem is extended for three-ordered continuous time linear inclusions.**

I. Introduction

Assume $\mathcal{A}$ is a bounded set of $n \times n$ real matrices. Consider inclusions in continuous time

$$\frac{dx}{dt} \in \{ \, Ax \, : \, A \in \mathcal{A} \, \}, \tag{1}$$

and in discrete time

$$x_{k+1} \in \{ \, Ax_k \, : \, A \in \mathcal{A} \, \}. \tag{2}$$

The problem of stability of inclusions (1), (2) arises in the theory of absolute stability of feedback systems with time-varying nonlinearities satisfying sector conditions [1]. In [2], [1], [3] it has been shown that asymptotically stable inclusions possess Lyapunov functions. In [4], [5], [3] one can find some approximations to these Lyapunov functions. The variational approach (based on Pontryagin maximum principle) to the problem of absolute stability for continuous-time systems has been derived in [6], where the so-called dual system appeared for the first time. The discrete-time counterpart has been published in [2], but the original and the dual systems were presented in time with opposite directions (that is usual for Pontryagin maximum principle), which makes difficult the application of this result.

The problem of the existence of Lyapunov functions on the boundary of the set of absolute stability in the parameter space remained open until 1988. In [7], [8]. the Lyapunov exponents were introduced:

$$\rho(\mathcal{A}) = \sup_{x(0) \neq 0} \overline{\lim_{t \to \infty}} \frac{\ln \|x(t)\|}{t} \tag{3}$$

for inclusion (1), and

$$\rho(\mathcal{A}) = \sup_{x_0 \neq 0} \overline{\lim_{k \to \infty}} \frac{\ln \|x_k\|}{k} \tag{4}$$

for inclusions (2).

For the case of system (2) with finite set $\mathcal{A} = \{A_1, \ldots, A_q\}$ the Lyapunov exponent (4) coincides with logarithm of the joint spectral radius

$$\hat{\rho}(A_1, \ldots, A_q) = \lim_{m \to \infty} \max_{\sigma} \|A_{\sigma(1)} \ldots A_{\sigma(m)}\|^{1/m},$$

where σ runs the set of all substitutions σ: $\{1, \ldots, m\} \to \{1, \ldots, q\}$. This notion was introduced in [9] much earlier than there appeared papers on absolute stability mentioned above. Recently it attracts special attention [10], [11], [12] partly due to its close relation to the regularity exponent of Daubeshies wavelets.

In the sequel we assume that inclusions (1), (2) are irreducible, that is, the set of matrices $\mathcal{A}$ has no common proper invariant subspace. For the case of inclusions arising in the theory of absolute stability the property of irreducibility is equivalent to controllability and observability. Stability of any inclusion may be reduced to stability of a set of irreducible inclusions of less dimensions.

For any bounded set of matrices $\mathcal{A}$ we have $\rho(\mathcal{A}) = \rho(ex\ conv\ cl(\mathcal{A}))$ [7], [8], where $ex\ conv\ cl(\mathcal{A})$ is the set of extreme points of the closure of convex hull of $\mathcal{A}$. Therefore withou loss of generality in the sequel we assume that the set $\mathcal{A}$ is closed.

In the next two sections we list some of main results on Lyapunov exponents.

II. Discrete Time

In this section we use definition (4) for Lyapunov exponent $\rho(\mathcal{A})$. All the results are based on the following fundamental property of an irreducible set of matrices.

Theorem 2.1: [7] There exists a norm v in $\mathbb{R}^n$ such that $\max_{A \in \mathcal{A}} v(Ax) = e^{\rho(\mathcal{A})} v(x)$ for all $x \in \mathbb{R}$.

Due to this property function v has been called the extremal norm. For irreducible sets of matrices $\mathcal{A}$ any inclusion (2) on the boundary of the domain of stability (i.e. when $\rho(\mathcal{A}) = 0$) has Lyapunov functions, and one of such functions is the extremal norm $v(\cdot)$.

According to theorem 2.1 for any initial value x_0 there exists a solution $\{x_k\}_{k=0}^{\infty}$ of inclusion (2) such that $v(x_k) = e^{\rho(\mathcal{A})} v(x_0)$ for all $k \geq 0$. Consider the sequence $y_k = e^{-\rho(\mathcal{A})} x_k$. Then $v(y_k) \equiv v(x_0)$. Denote by S the unit sphere with norm v: $S = \{x \, : \, v(x) = 1\}$. Denote by Ω the

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set of all $\omega-$limit points of all sequences $\{y_k\}$ with initial values $x(0) \in S$. Then $\Omega \subseteq S$. It is easy to see that the set $M = conv(\Omega)$ is closed and invariant with respect to inclusion $z_{k+1} \in \{e^{-\rho(\mathcal{A})}Az_k \ : \ A \in \mathcal{A}\}$, and moreover, $M = e^{-\rho(\mathcal{A})}conv(AM \ : \ A \in \mathcal{A})$. The set M does not belong to any hyperplane in $\mathbb{R}^n$ due to irreducibility of the set $\mathcal{A}$. Hence, M is a level set of a norm w in $\mathbb{R}^n$. This norm was introduced and investigated in [11]. It satisfies the property $w(Ax) \le e^{\rho(\mathcal{A})}w(x)$ for all $A \in \mathcal{A}$ and all $x \in \mathbb{R}^n$. As opposite to the norm v there exist points x such that the last inequality is strict for all $A \in \mathcal{A}$.

The problem of the existence of Lyapunov functions of particular type (piecewise-linear, quadratic, forms of even degree, polynomials) for asymptotically stable inclusions (2) has been studied for a long time (see for example [1]). The following statement shows that for asymptotically stable inclusions there always exists a Lyapunov function of the form of even degree.

Corollary 2.1: [7] If inclusion (2) is asymptotically stable (i.e. $\rho(\mathcal{A}) < 0$) then it has a Lyapunov function, which is a form of even degree.

Now consider dual inclusions. Denote $\mathcal{A}^* = \{A^* \ : \ A \in \mathcal{A}\ \}$. The inclusions (2) with the sets of matrices $\mathcal{A}$ and $\mathcal{A}^*$ are called dual. Their Lyapunov indices coincide:

Theorem 2.2: [7] $\rho(\mathcal{A}) = \rho(\mathcal{A}^*)$.

Denote $\partial v(x)$ the subdifferential of convex function v at point x. Assume $\det A \neq 0 \ \forall A \in \mathcal{A}$. Denote $v^*(l) = \max\{(l,x) \ : \ v(x) = 1\}$ the dual norm. The following result is a kind of Pontryagin maximum principle for linear inclusions (2).

Lemma 2.1: [7] For any $x \in \mathbb{R}^n$ there exist $A \in \mathcal{A}$ and $l \in \partial v(x)$ such that $v(Ax) = e^{\rho(\mathcal{A})}v(x)$ and $\max\{(l, A^{-1}Bx) \ : \ B \in \mathcal{A}\} = v(x)v^*(l)$.

In fact for all x there exists $l \in \partial v(x)$ such that the matrix A satisfying the last equality is unique. Function v being a convex function is differentiable almost everywhere. Moreover, if v is differentiable at x, $l = v'(x)$, and matrix $A = A(x)$ is defined as in lemma 2.1, then [7] function v is differentiable at Ax and its derivative is equal to $(A(x)^*)^{-1}l$. Hence, if for any vector x_0 we know that v is differentiable at x and we know $l = v'(x)$, then we can easily construct a sequence $\{x_k\}$ such that $v(x_k) = e^{\rho(\mathcal{A})k}v(x_0)$ for all $k \ge 0$.

An application of this result for the problem of absolute stability of systems with one time-varying nonlinearity is given below.

Define the set of matrices

$$\mathcal{A}_a = \{A + b\nu c^* \ : \ 0 \le \nu \le \mu\} \tag{5}$$

where A is $n \times n-$matrix, b, c are $n-$vectors, pair (A, b) is controllable, pair (A, c) is observable and μ is a positive number.

Note that the Lyapunov indices of the sets of matrices $\mathcal{A}_a$ and $\{A,\ A + b\mu c^*\}$ coincide.

Corollary 2.2: [13] For all $y \in \mathbb{R}^n$ there exists a vector $l \in \partial v(y)$ such that for the solution of system

$$\begin{array}{c} x_{k+1} = Ax_k + b(c, x_k)u_k, \\ l_{k+1} = A^{-1\,*}l_k - A^{-1;*}c(A^{-1}b, l_k)u_k(1 + \mu(c, A^{-1}b)^{-1}, \\ u_k = \frac{\mu}{2}(1 + sign[(c, x_k)(A^{-1}b, l_k)]) \\ k = 0, 1, 2, \dots \end{array} \tag{6}$$

with initial value $x_0 = y$, $l_0 = l$, for some positive constants C_1, C_2 and for all $k = 0, 1, 2, \dots$ the following inequalities are true

$$\begin{array}{c} e^{-k\rho(\mathcal{A}_a)}\|x_k\|/\|y\| \in [C_1, C_2], \\ e^{k\rho(\mathcal{A}_a)}\|l_k\|/\|l\| \in [C_1, C_2] \end{array}.$$

It is important that both equations in (6) (for x and for l) are written in the same time direction, which is an essential improvement as compared with results of [2]. To find out an extremal solution (i.e. a solution with Lyapunov exponent $\rho(\mathcal{A}_a)$) it is sufficient to find for any nonzero vector $x(0)$ a vector $l(0)$ from the conditions of the corollary 2.2. But this problem is not solved up to now.

The following result is trivial but important.

Proposition 2.1: [7] The following statements are equivalent.

(i) Inclusion (2) is asymptotically stable.

(ii) Inclusion (2) is exponentially stable.

(iii) Lyapunov index $\rho(\mathcal{A})$ is negative.

There is a nice reformulation of negativeness of the Lyapunov exponent $\rho(\mathcal{A})$ in terms of pointwise properties of sequences in time and frequency domains.

Define the sequence $w_j = c^*A^{j-1}b$ if $j > 1$, $w_0 = -\mu^{-1}$.

Theorem 2.3: [13] The Lyapunov exponent $\rho(\mathcal{A}_a)$ is nonnegative (positive) if and only if there exist bounded (respectively, exponentially tending to zero at $\pm\infty$) nonzero sequences $\{f_k^1\}_{k=-\infty}^{\infty}$, $\{f_k^1\}_{k=-\infty}^{\infty}$ such that for all $k = 0, \pm 1, \pm 2, \dots$

$$f_k^1 = \sum_{j=0}^{\infty} w_j f_{k-j}^2, \quad f_k^1 \cdot f_k^2 = 0.$$

Note that $\{f_k^2\} \in l_2$ then $\{f_k^1\} \in l_2$ and the first equality means that the ratio of the Fourier transform of $\{f_k^1\}$ and the Fourier transform of $\{f_k^2\}$ is given function which is equal to the Fourier transform of $\{w_j\}$.

One of the main results of the section is given in the next theorem, which describes a procedure to compute the sign of the Lyapunov exponent (4) for the case of finite set $\mathcal{A} = \mathcal{A}_q = \{A_1, \dots, A_q\}$.

Theorem 2.4: [7] Assume Z_0 is a set of vertices of a rectangular in $\mathbb{R}^n$; zero is an interior point of $conv(Z_0)$. For any $k = 0, 1, 2, \dots$ denote $Z'_k = \{Ax \ : \ A \in \mathcal{A},\ x \in Z_k\}$, Z_{k+1} is the set of extremal points of a convex hull of the the finite set of points $Z_k \cup Z'_k$.

The Lyapunov exponent $\rho(\mathcal{A}_q)$ is positive if and only if $Z_k \cap Z_0 = \emptyset$ for some $k > 0$.

If Lyapunov exponent $\rho(\mathcal{A}_q)$ is negative, then $Z_k = Z_{k+1}$ for some $k > 0$. If $Z_k = Z_{k+1}$ for some $k > 0$ then the Lyapunov exponent $\rho(\mathcal{A}_q)$ is not positive.

If $e^{\gamma}Z_k \subseteq Z_{k+1}$ for a number γ and some $k \ge 0$, then $\rho(\mathcal{A}_q) \le \gamma$.

If $\rho(\mathcal{A}_q) \neq 0$ then the procedure gives the $sign$ of $\rho(\mathcal{A}_q)$ in a finite number of steps. If $\rho(\mathcal{A}_q) = 0$ the values γ in the last statement may be taken arbitrary close to zero for sufficiently big k. For any given positive number ϵ in a finite number of steps we get the answer $\rho(\mathcal{A}_q) < \epsilon$.

The Lyapunov exponent of the set $\{e^{\beta}A \ : \ A \in \mathcal{A}_q\}$ is equal to $\rho(\mathcal{A}_q) + \beta$. Hence, we can get an estimation of the number $\rho(\mathcal{A}_q)$ more precisely by varying parameter β.

At the end of the section we present a couple of algebraic properties of Lyapunov exponent. The first one is an extension to discrete-time case and generalization of criterion published in [1]. The second one is based on a new idea.

Theorem 2.5: [7] The Lyapunov exponent $\rho(\mathcal{A})$ is negative if and only if there exist a number $m \geq n$ and a $n \times m$-matrix L of rank n such that for any matrix $A \in \mathcal{A}$ there exists a $m \times m$-matrix Γ which satisfies the following conditions:

(i) $A^*L = L\Gamma$;

(ii) all elements of matrix Γ are nonnegative and the sum of elements of each row is less than one.

The following result allows calculating the Lyapunov exponent of $\mathcal{A}_q$ using tenzor products. Denote by $A \otimes B$ the tenzor product of $n \times n$ and $m \times m$ matrices A and B: $(A \otimes B)_{(i-1)m+k,(j-1)m+l} = A_{i,j}B_{k,l}$ for all $1 \leq i, j \leq n$, $1 \leq k, l \leq m$. Note that the dimension of $A \otimes B$ is $nm \times nm$. The power $A^{\otimes k}$ is the tenzor product of matrix A by itself k times.

Theorem 2.6: For any integer p consider matrix

$$D = \sum_{i=1}^{q} A_i^{\otimes(2p)}.$$

If λ is the maximal eigenvalue of D, $\alpha = \lambda^{1/(2p)}$, $\kappa = q^{1/(2p)}$ then $\alpha/\kappa \leq e^{\rho(\mathcal{A}_q)} \leq \alpha$.

The value κ tends to one as $p \to \infty$. Hence, the last inequality provides an estimate to the Lyapunov exponent up to any given accuracy for sufficiently large p. Still, the dimension of matrix D increases so rapidly with increasing p that even for moderate values of $n = 3, 4$, $p = 5, 6$ the computational burden of calculation the spectral radius of D becomes too high.

III. Continuous Time

In this section we assume that Lyapunov exponent is defined as in (3). Introduce function [8]

$$v(y) = \sup(\overline{\lim}_{t\to\infty} e^{-\rho(\mathcal{A})t}\|x(t)\|),$$

where the *supremum* is taken over the set of solutions $x(t)$ of inclusion (1) with initial data $x(0) = y$.

The following result is a direct analogue of the theorem 2.1. It provides a basis for all the results of this section.

Theorem 3.1: [8] The following statements are true.

(i) v is a norm in $\mathbb{R}^n$.

(ii) For any solution $x(\cdot)$ of inclusion (1) and any $t \geq 0$ we have $v(x(t)) \leq e^{\rho(\mathcal{A})t}v(x(0))$.

(iii) For any $n-$vector y there exist a solution $x(\cdot)$ of inclusion (1) such that $x(0) = y$ and $v(x(t)) = e^{\rho(\mathcal{A})t}v(x(0))$ for all $t \geq 0$.

Function v is called extremal norm. It is a Lyapunov function for stable inclusions (1) *including* the boundary case $\rho(\mathcal{A}) = 0$.

We can get an analogue of the norm w and invariant set M described in the previous section. Denote by S a unit sphere with norm v: $S = \{x \in \mathbb{R}^n \ : \ v(x) = 1\}$. For any vector $x(0) \in S$ consider solution $x(\cdot)$ of inclusion (1) such that $v(x(t)) = e^{\rho(\mathcal{A})t}v(x(0))$ for all $t \geq 0$. Denote $y(t) = e^{-\rho(\mathcal{A})t}x(t)$. Then $y(t) \in S$ for all $t \geq 0$. Denote by Ω the set of all $\omega-$limit points of all functions $y(\cdot)$ with $x(0) \in S$. The set $M = conv(\Omega)$ is closed, convex, invariant with respect to inclusion $dz/dt \in \{e^{-\rho(\mathcal{A})}A \ : \ A \in \mathcal{A}\}$ and does not belong to any hyperplane in $\mathbb{R}^n$. Moreover, $conv\{x(t) \ : \ x(0) \in M\} = e^{\rho(\mathcal{A})t}M$ for any $t \geq 0$. Hence, M is a level set of a norm w which is a direct analogue of the norm [11]. Norm w has the property $w(x(t)) \leq e^{\rho(\mathcal{A})t}w(x(0))$ for any $t \geq 0$ and any solution $x(\cdot)$ of inclusion (1). But it fails to have property (iii) of theorem 3.1.

The following proposition is trivial but important. The last statement shows that asymptotically stable inclusion (1) always has a Lyapunov function which is a form of (sufficiently high) even degree.

Proposition 3.1: [8] The following statements are equivalent.

(i) Inclusion (1) is asymptotically stable.

(ii) Inclusion (1) is exponentially stable.

(iii) Lyapunov index $\rho(\mathcal{A})$ is negative.

(iv) There exists a form of even degree, which is a Lyapunov function of inclusion (1).

Now consider the dual inclusion, that is, the inclusion (1) with the set of matrices $\mathcal{A}^* = \{A^* \ : \in \mathcal{A}\}$. Denote by $\rho(\mathcal{A}^*)$ its Lyapunov exponent.

Theorem 3.2: [8] Lyapunov exponents $\rho(\mathcal{A})$ and $\rho(\mathcal{A}^*)$ coincide.

(compare with theorem 2 in [14]). The next theorem is an analogue of Pontryagin maximum principle for continuous time linear inclusions (1).

Theorem 3.3: [8] For any nonzero $x \in \mathbb{R}^n$ there exists $l \in \partial v(x)$ such that $\max\{(l, (A - \rho(\mathcal{A}))x), \ : \ A \in \mathcal{A} \ \} = 0$. For any $l \in \partial v(x)$, $A \in \mathcal{A}$ we have $(l, (A - \rho(\mathcal{A}))x) \leq 0$.

Note that for almost all x the subdifferential $\partial v(x)$ consists of a single vector $l = v'(x)$.

Now consider some algebraic criteria for stability of inclusions (1). We need

Definition 3.1: An $m \times m$-matrix Γ satisfies condition $U(\lambda)$ if all off-diagonal elements of matrix Γ are nonnegative, and the sum of all elements of each row of matrix Γ is less than λ.

The following statement is a generalization of criterion [1].

Theorem 3.4: [8] For any positive ϵ there exists an integer number $m \geq n$ and an $n \times m-$matrix L such that the linear span of rows of L is $\mathbb{R}^n$, and for any matrix $A \in \mathcal{A}$

there exists a $m \times m-$matrix Γ such that matrix Γ satisfies condition $U(\rho(\mathcal{A})+\epsilon)$ and $A^*\Gamma = L\Gamma$.

In particular, this criterion may be reformulated as follows.

Theorem 3.5: The Lyapunov exponent $\rho(\mathcal{A})$ is negative if and only if there exist an integer $m \geq 1$, a number $\lambda < 0$, a positive definite diagonal matrix H and $n \times m-$matrix G such that for any matrix $A \in \mathcal{A}$ there exist a $m \times n-$matrix A' and $m \times m-$matrix A'' such that matrix

$$\Gamma(A) = \begin{pmatrix} H^{-1}AH + GA' & -H^{-1}A^*HG + GA'' \\ A' & A'' \end{pmatrix} \tag{7}$$

satisfies condition $U(\lambda)$.

If we denote $\tilde{\Gamma}(A) = diag\{H, I\}\Gamma(A)$, $\tilde{G} = HG$, then criterion of theorem 3.5 amounts to n bilinear and m linear scalar inequalities with respect to unknown entries of matrices $\tilde{G}$, H, A', A''. To check these inequalities we can use suitable numerical methods.

For a particular set of matrices $\mathcal{A}_a$ (5) we have the following equivalent statements from different areas of mathematics.

Theorem 3.6: [19] The following statements are equivalent.

(i) The Lyapunov exponent $\rho(\mathcal{A}_a)$ is negative.

(ii) There is only one bounded on $(0, \infty)$ and continuous solution $y(\cdot)$ to the equality

$$y(t)\{-\frac{1}{\mu}y(t) + \int_0^\infty (c, e^{A\tau}b)y(t+\tau)d\tau\} = 0 \quad \forall t \geq 0,$$

and this solution is trivial: $y \equiv 0$.

(iii) Let ξ be a stationary stochastic process, $P\{\xi(t) = 0\} \equiv P\{\xi(t) = 1\} \equiv 1/2$. There exists an invariant with respect to system

$$dx/dt = (A + b\xi c^*)x + \xi b$$

stochastic measure with bounded support.

(iv) For all solutions of system

$$\begin{gathered} dx/dt = Ax + b(c,x)u, \\ dl/dt = -A^*l - c(b,l)u, \\ u = \mu\{1 + sign[(c,x)(b,l)]\}/2 \end{gathered}$$

it follows $x(t) \to 0$ as $t \to \infty$.

Finally we point out the relation between Lyapunov exponent of a continuous time inclusion (1) and Lyapunov exponent of the discrete time first approximation. This result coupled with theorem 2.4 may be used to prove stability of continuous time inclusion (1).

For any positive number τ denote by $\rho_\tau(\mathcal{A})$ the discrete time Lyapunov exponent (4) of inclusion (2) with the set of matrices $\{e^{A\tau} \ : \ A \in \mathcal{A}\}$.

Theorem 3.7: [15] There exists a constant C such that $\rho_\tau(\mathcal{A}) \leq \tau\rho_(\mathcal{A}) \leq \rho_\tau(\mathcal{A}) + C\tau^2$ for all $\tau \in (0, 1)$.

IV. INCLUSIONS WITH DELAYS

Given a positive number τ and a bounded closed set of pairs of real $n \times n-$matrices $\mathcal{A}$ consider the following linear inclusion:

$$\frac{dx}{dt}(t) \in \{A_1x(t) + A_2x(t-\tau) \ : \ (A_1, A_2) \in \mathcal{A}\}. \tag{8}$$

The state of the inclusion is a pair $(x_t, x(t))$, where $x_t = \{x(t+\theta) -\tau \leq \theta < 0\} \in L_2(\tau, 0)$ and $x(t) \in \mathbb{R}^n$. The initial data for this inclusion is a pair $(x_0, x(0)) \in L_2(-\tau, 0) \times \mathbb{R}^n$.

Introduce the Lyapunov exponent [16]

$$\rho(\mathcal{A}) = sup_{(x_0,x(0))\neq 0}\overline{\lim_{t\to\infty}}\frac{\ln(\|x_t\|_{L_2(-\tau,0)} + \|x(0)\|)}{t}.$$

where *supremum* is taken over all solutions of inclusion (8) with nonzero initial data $(x_0, x(0))$. Due to the boundedness of the set $\mathcal{A}$ the number $\rho(\mathcal{A})$ is finite.

We assume as above, that the set of pairs of matrices $\mathcal{A}$ is irreducible, that is, there is no proper subspace in $\mathbb{R}^n$ which is invariant with respect to inclusion (8) (i.e. with respect to any matrix of any pair $(A_1, A_2) \in \mathcal{A}$).

The following theorem generalizes the results of theorem 3.1 to the case of inclusions with delay (8).

Theorem 4.1: [16] There exists a norm w in $L_2(-\tau, 0) \times \mathbb{R}^n$ such that

(i) for any solution $(x_t, x(t))$ of inclusion (8) we have $w(x_t, x(t)) \leq e^{\rho(\mathcal{A})t}w(x_0, x(0))$ for all $t \geq 0$;

(ii) for any initial data (y_0, y) there exists a solution solution $(x_t, x(t))$ of inclusion (8) such that $y_0 = x_0$, $y = x(0)$ and $w(x_t, x(t)) = e^{\rho(\mathcal{A})t}w(x_0, x(0))$ for all $t \geq 0$.

V. AIZERMAN PROBLEM FOR ONE TIME-VARYING NONLINEARITY

Consider a feedback system with one nonlinearity

$$\begin{gathered} \tfrac{dx}{dt} = Ax + b\xi, \quad \sigma = c^*x, \\ \xi(t) = \varphi(t, \sigma), \end{gathered} \tag{9}$$

where A, b, c are constant matrices of dimension $n \times n$, $n \times 1$ and $n \times 1$ respectively, and φ is a function from the set

$$\mathcal{M} = \{\varphi \ : \ 0 \leq \frac{\varphi(t,s)}{s} \leq \mu \quad \forall s \neq 0, t \geq 0\}.$$

where μ is given positive number.

Aizerman problem consists of finding necessary and sufficient conditions in the space of parameters (A, b, c, μ) for absolute stability of system (9) in class $\mathcal{M}$ (i.e. global asymptotic stability of system (9) with all nonlinearities $\varphi \in \mathcal{M}$).

It is easy to see that system (9) is absolute stable in class $\mathcal{M}$ if and only if inclusion (1) with the set of matrices $\mathcal{A} = \mathcal{A}_a = \{A + b\nu c^* \ : \ 0 \leq \nu \leq \mu\}$ is asymptotically stable, which is equivalent in turn to the inequality $\rho(\mathcal{A}_a) < 0$.

The first solution to this problem for two-dimensional systems (i.e. if $n = 2$) was presented in [17] (see also a more general result in [18]). It may be formulated in terms of parameters A, b, c, μ or as follows.

Theorem 5.1: [19] For two-ordered systems ($n=2$) the Lyapunov exponent $\rho(\mathcal{A}_a)$ is negative if and only if the following system is stable

$$dx/dt = Ax + b(c,x)\frac{\mu}{2}\{1 + sign[(c,x)(b,l)]\}.$$

This theorem provides the extremal solutions for system (9). Namely, these solutions correspond to nonlinearity $\xi(t) = \sigma(t)$ if $sign[(c,x)(b,l)] \geq 0$ and $\xi(t) = 0$ otherwise.

For three-ordered systems it is much more difficult to find out such extremal solutions.

Denote by I the identity matrix. Introduce shorthand $\rho = \rho(\mathcal{A}_a)$. Consider the following system

$$\begin{array}{c} dx/dt = (A-\rho I)x + b(c,x)u, \\ dl/dt = -(A-\rho I)^* l - c(b,l)u, \\ u = \frac{\mu}{2}\{1 + sign[(c,x)(b,l)]\}. \end{array} \tag{10}$$

Denote by S the unit sphere of extremal norm v: $S = \{x : v(x) = 1\}$. Then for any solution of equation (10) we have $v(x(t)) \leq v(x(0))$ and for any $x(0) \in S$ there exists a solution $x(\cdot)$ of equation (10) such that $x(t) \in S$ for all $t \geq 0$. These solution are called extremal. The values $x(t+\tau)$ of extremal solutions $x(\cdot)$ with $t > 0$ do not intersect and depend continuously on initial values $x(\tau)$ if $\tau > 0$. The set S is two-dimensional, and hence it is reasonable to suggest that each extremal solution tends to a constant point or (orbitarily) to a periodic extremal solution.

The proof of this statement (and the following theorem) is rather long, technical [19] and uses auxiliary seminorms and careful study of the geometry of the set S. The final result occurs to be simple. Denote $B = A + b\mu c^*$.

Theorem 5.2: [19] Assume $n=3$ and matrices $A - \rho I$ and $B - \rho I$ are Hurwitz (i.e. all their eigenvalues have negative real parts). Then there exists a solution $(x^0(\cdot), l^0(\cdot))$ of system (10) and a positive number T such that

(i) $(x^0(t), l^0(t)) = (-x^0(t+T), -l^0(t+T))$ for all $t \geq 0$; functions $(c, x^0(\cdot))$ and $(b, l^0(\cdot))$ have only one (and simple) root on the interval $[0,T)$;

(ii) for any extremal solution of equation (10) with $x(0) \in S$ we have $x(t) \to \{x^0(s) : 0 \leq s < 2T\}$ as $t \to \infty$.

The convergence in (ii) is in Hausdorf metrics. This theorem states that provided matrices $A - \rho I$ and $B - \rho I$ are Hurwitz all the extremal solutions of system (10) tend to an antiperiodic solution, which has exactly two switches on a half of a period.

This property allows formulation of a criterion for absolute stability of system (9) in class $\mathcal{M}$.

Theorem 5.3: [19] System (9) with $n=3$ is absolute stable in class $\mathcal{M}$ if and only if matrices A and B are Hurwitz and for any nonnegative numbers p, q, r the following inequality holds

$$\det(e^{-r(p+q)}e^{Ap}e^{Bq} + I) \neq 0.$$

This result also allows to get an analogue of Perron-Frobenius theorem for linear third-order inclusions.

Theorem 5.4: [19] Assume $n=3$ and differential inclusion

$$dx/dt \in \{Dx : D = A + b\nu c^*,\ \nu \in [0,\mu]\}$$

has a sharp invariant cone. Then the Lyapunov exponent of this inclusion is equal to the maximal real eigenvalue of matrices A and $A + b\mu c^*$.

Consider some computational issues. Assume the roots $\{\mu_1, \mu_2, \mu_3\}$ and poles $\{\lambda_1, \lambda_2, \lambda_3\}$ of function $1 + W(s) = 1 + (c, (A - sI)^{-1}b)$ are all different. Then

$$W(s) = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2} + \frac{\alpha_3}{s-\lambda_3}$$

for some complex numbers α_1, α_2, α_3. In a particular basis the triple (A,b,c) is presented in a form $A = \mathrm{diag}\,\{\lambda_1, \lambda_2, \lambda_3\}$, $b = col(-1,-1,-1)$ and $c = col(\alpha_1, \alpha_2, \alpha_3)$.

Lemma 5.1: [19] The matrix B in this basis has a form

$$B = K^{-1}\mathrm{diag}\,\{\mu_1, \mu_2, \mu_3\}K,$$

where $K = \{k_{ij}\}_{i,j=1}^3$, $\quad k_{ij} = \frac{\alpha_i}{\lambda_i - \mu_j}$.

Denote $\Lambda_1(p) = \mathrm{diag}\,\{e^{\lambda_j p}\}_{j=1}^3$, $\Lambda_2(q) = \mathrm{diag}\,\{e^{\mu_j q}\}_{j=1}^3$. Theorem 5.3 takes the following form

Theorem 5.5: [19] Assume $n=3$. System (9) is absolute stable in class $\mathcal{M}$ if and only if matrices A and B are Hurwitz and the following inequality holds:

$$\min_{p\geq 0,\ q\geq 0\ r\geq 0} \det(K)\det(\Lambda_2(q)K + K\Lambda_1(-p)e^{r(p+q)}) \neq 0.$$

There are no similar results for inclusions of bigger dimension $n \geq 4$.

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