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Control of the Inertia Wheel Pendulum by Bounded Torques

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Abstract— A control system for driving the inertia wheel pendulum to the upward position has been reported recently in the literature based on Interconnection and Damping Assignment Passivity–based Control. An important feature of this controller is that it swings up and balances the pendulum in the upward position without switching between two different controllers. This paper, enhances the control system by the practical capability of maintaining the control action (applied actuator torque) inside prescribed limits. The perfomance of the proposed controller is illustrated via simulations, which have been compared with those reported in the literature.

Index Terms—Underactuated system, bounded control, asymptotic stability, passivity.

I. INTRODUCTION

Interconnection and Damping Assigment Passivity–based Control (IDA–PBC), introduced recently in [3], is a control design methodology that assigns a desired dynamic in closed-loop. An important feature of the IDA–PBC technique is the systematic design procedure to yield an energy function in closed-loop which qualifies as a Lyapunov function.

The Inertia Wheel Pendulum is an underactuated mechanical system consisting of a physical pendulum with a symmetric disk attached to the tip, which is free to spin about an axis parallel to the axis of rotation of the pendulum (see Figure 1). Underactuated nature is because it has two degrees–of–freedom and only one actuator located at the disk. Based on the IDA–PBC technique, [3] designed a controller that "almost global" resolved the problem of stabilization —without switching— the upward position of the inertia wheel pendulum. However, this controller may demand a high input torque that makes difficult a practical implementation. The goal of this paper is to propose an IDA–PBC controller that provides bounded torque within desired bounds.

This paper is organized as follows. Section II summarizes the inertia wheel pendulum model and control problem formulation. In Sections III and IV, we introduce the proposed controller and our stability analysis, respectively. Simulation on an inertia wheel pendulum are given in Section V. Finally, we offer some concluding remarks in Section VI.

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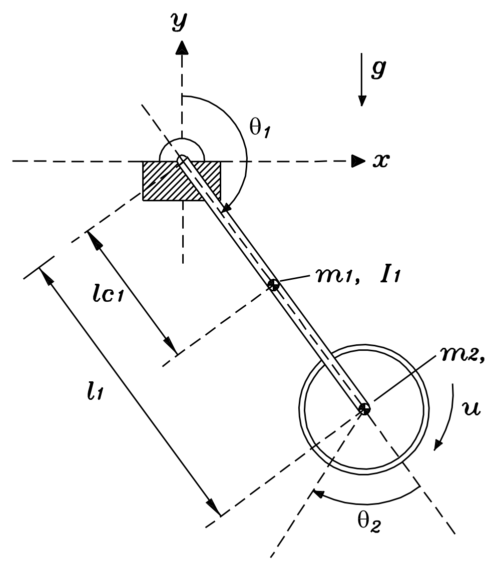

Fig. 1. Inertia wheel pendulum

II. MODEL AND CONTROL PROBLEM FORMULATION

The inertia wheel pendulum depicted in Figure 1, may be modelled as a two-degrees-of-freedom serial mechanism, where the pendulum forms the first link and the rotating disk forms the second link. We assume that the center of mass of the disk is coincident with its axis of rotation and measure the angle of the pendulum clockwise from the vertical. Under these assumptions the Euler–Lagrange's equations of motion can be written as [4]:

$$\begin{bmatrix} m_{11}+m_{22} & m_{22} \\ m_{22} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -m_3\sin(q_1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{1}$$

where θ_1 and θ_2 are the joint positions of the pendulum and the disk, respectively, $m_3 \overset{\triangle}{=} (m_1 g l_{c_1} + m_2 g l_1)$, and u is the control input torque acting between disk and pendulum, $m_{11} = m_1 l_{c_1}^2 + m_2 l_1^2 + I_1$ and $m_{22} = I_2$. The remaining parameters are shown in Table I.
Next, we introduce a global change of coordinates

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \tag{2}$$

which can be expressed equivalently as

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}. \tag{3}$$

TABLE I
Parameters

Description	Notation	Units
Length of the pendulum	l_1	m
Distance at the center of mass of the pendulum	l_{c1}	m
Mass of the pendulum	m_1	kg
Mass of the disk	m_2	kg
Moment of inertia of the pendulum	I_1	kg.m^2
Moment of inertia of the disk	I_2	kg.m^2
Gravity acceleration	g	m/s^2

This leads to a simplified model

$$\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -m_3 \sin(q_1) \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} u, \quad (4)$$

The model (4) can be written through Hamilton's equations of motion as

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{p_1}{m_{11}} \\ \frac{p_2}{m_{22}} \\ m_3 \sin(q_1) - u \\ u \end{bmatrix}, \quad (5)$$

with a Hamiltonian function given by

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}\left[\frac{p_1^2}{m_{11}} + \frac{p_2^2}{m_{22}}\right] + m_3 \cos(q_1), \quad (6)$$

where $\boldsymbol{q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$ and $\boldsymbol{p} = \begin{bmatrix} p_1 & p_2 \end{bmatrix}^T = \begin{bmatrix} m_{11}\dot{q}_1 & m_{22}\dot{q}_2 \end{bmatrix}^T$ are the generalized positions and momenta respectively.

We can now formulate the control problem under actuator torque constraints addressed in this work. Consider the inertia wheel pendulum model (5). Assume the actuator is able to supply a known maximum torque u^{max} such that

$$|u(t)| \leq u^{max}. \quad (7)$$

We also assume that the maximum torque satisfies the following condition:

$$u^{max} > m_3 \quad (8)$$

Formally, the control objective can be expressed as to find a control law such that

$$\lim_{t\to\infty} \mathrm{dist}(q_1(t), \Gamma) = 0 \text{ and } |u(t)| \leq u^{max}, \quad (9)$$

where $\mathrm{dist}(q_1(t), \Gamma)$ denote the smaller distance between q_1 and every element in

$$\Gamma = \{\cdots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \cdots\}.$$

III. CONTROLLER DESIGN

A. Previous work

Based on IDA–PBC methodology, the following control law was proposed in [3],

$$u = \underbrace{\gamma_1 \sin(q_1) + k_p(q_2 + \gamma_2 q_1)}_{u_{es}} + \underbrace{k_v k_2(\dot{q}_2 + \gamma_2 \dot{q}_1)}_{u_{di}} \quad (10)$$

where

$$\gamma_1 = \tfrac{a_2}{a_1+a_2} m_3, \qquad \gamma_2 = -\tfrac{m_{11}(a_2+a_3)}{m_{22}(a_1+a_2)}, \qquad k_2 = -\tfrac{m_{22}(a_1+a_2)}{a_1 a_3 - a_2^2} > 0, \quad (11)$$

and k_p and k_v are positive arbitrary constants. The remaining constant a_1, a_2 and a_3 must satisfy

$$a_1 > 0, \quad a_1 a_3 - a_2^2 > 0, \quad a_1 + a_2 < 0, \qquad \tfrac{u^{max}}{m_3} > \tfrac{a_2}{a_1+a_2}, \quad (12)$$

As it was pointed out in [3], the control law (10) is the sum of two terms

$$u = u_{es} + u_{di}$$

where

$$\begin{aligned} u_{es} &= \gamma_1 \sin(q_1) + k_p(q_2 + \gamma_2 q_1) \\ u_{di} &= k_v k_2(\dot{q}_2 + \gamma_2 \dot{q}_1). \end{aligned} \quad (13)$$

The term u_{es} is designed to achieve the energy shaping and u_{di} injects the damping to the closed–loop system.

The damping injection term u_{di} can be written in terms of the generalized coordinates $\boldsymbol{q}$ and momenta $\boldsymbol{p}$ as

$$u_{di} = k_v k_3(p_2 + k_4 p_1) \quad (14)$$

where

$$k_3 = -\tfrac{a_1+a_2}{a_1 a_3 - a_2^2} > 0, \quad k_4 = -\tfrac{a_2+a_3}{a_1+a_2}. \quad (15)$$

Hence, the control law (10) can be rewritten as

$$u = \underbrace{\gamma_1 \sin(q_1) + k_p(q_2 + \gamma_2 q_1)}_{u_{es}} + \underbrace{k_v k_3(p_2 + k_4 p_1)}_{u_{di}}. \quad (16)$$

Closed–loop system: A detailed stability analysis —in Lagrangian formulation— of the equilibria of the closed–loop system formed by the control law (16) and the Hamiltonian system (5), and given by (17), which is shown in the next page, was presented in [1]. It is easy to note that the equilibrium set is

$$E = \begin{bmatrix} q_{1*} \\ q_{2*} \\ p_{1*} \\ p_{2*} \end{bmatrix} = \begin{bmatrix} n\pi \\ -\gamma_2 n\pi \\ 0 \\ 0 \end{bmatrix} \quad (18)$$

where $n \in \mathbb{N}$. We can see from Figure 1 that q_{1*} with n even correspond to the desired upward position, while the ones with odd n are with pendulum hanging.

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{p_1}{m_{11}} \\ \frac{p_2}{m_{22}} \\ m_3 \sin(q_1) - [\gamma_1 \sin(q_1) + k_p(q_2 + \gamma_2 q_1) + [k_v k_3(p_2 + k_4 p_1)]] \\ \gamma_1 \sin(q_1) + k_p(q_2 + \gamma_2 q_1) + [k_v k_3(p_2 + k_4 p_1)] \end{bmatrix}. \tag{17}$$

B. Proposed controller

By bearing in mind, the actuator torque constraints control problem, addressed in this work, we can modify the control law (16) in order to have bounded all its terms.

By following similar steps to those in [3], we can propose the new energy shaping term

$$u_{es} = \gamma_1 \sin(q_1) + k_p \tanh(q_2 + \gamma_2 q_1)$$

with γ_1 and γ_2 given in (11) and k_p a positive constant suitable selected. This new u_{es} is the result of proposing the following desired potential energy function:

$$V_d(\boldsymbol{q}) = \frac{m_{11}m_3}{a_1 + a_2}\cos(q_1) + k_1 \ln[\cosh(q_2 + \gamma_2 q_1)]. \tag{19}$$

Besides, because we require bounded control action, we propose the damping injection term u_{di} as:

$$u_{di} = k_v \tanh(k_3(p_2 + k_4 p_1)) \tag{20}$$

which still preserves the passivity property of the closed–loop system ([5]). The constants k_3 and k_4 are given in (15) and k_v is a properly selected positive constant.

Therefore, we propose the following control law:

$$\begin{aligned} u &= \underbrace{\gamma_1 \sin(q_1) + k_p \tanh(q_2 + \gamma_2 q_1)}_{u_{es}} \\ &\quad + \underbrace{k_v \tanh(k_3(p_2 + k_4 p_1))}_{u_{di}}. \end{aligned} \tag{21}$$

The positive constants k_p and k_v must be chosen sufficiently small. More specifically, they must satisfy

$$u^{max} - \gamma_1 \quad > k_p + k_v.$$

It is worth noticing that all terms in (21) are bounded; furthermore, the control law is bounded by

$$\begin{aligned} |u| &\leq \gamma_1 + k_p + k_v \\ &\leq u^{max}. \end{aligned} \tag{22}$$

IV. STABILITY ANALYSIS

A feature of the IDA–PBC methodology is that the desired energy function $H_d(\boldsymbol{q}, \boldsymbol{p})$ qualifies as a Lyapunov function candidate $V(\boldsymbol{q}, \boldsymbol{p})$. Hence, to carry out the stability analysis we propose

$$\begin{aligned} V(\boldsymbol{q}, \boldsymbol{p}) &= H_d(\boldsymbol{q}, \boldsymbol{p}) \\ &= \Big[\frac{1}{2\Delta}[a_3 p_1^2 - 2a_2 p_1 p_2 + a_1 p_2^2] \\ &\quad + \frac{m_{11}m_3}{a_1 + a_2}\cos(q_1) \\ &\quad + k_1 \ln[\cosh(q_2 + \gamma_2 q_1)]\Big] \end{aligned} \tag{23}$$

where $\Delta = a_1 a_3 - a_2^2$.

The closed-loop system is obtained combining the control law (21) and the open–loop system (5), which yields (24), shown in the next page.

Notice that the equilibrium set is

$$\mathcal{E} = \begin{bmatrix} q_{1*} \\ q_{2*} \\ p_{1*} \\ p_{2*} \end{bmatrix} = \begin{bmatrix} n\pi \\ -\gamma_2 n\pi \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

where $n \in \mathbb{N}$. We have omitted, due to paper length reasons, the proof that verify the local positive definiteness of $V(\boldsymbol{q}, \boldsymbol{p})$ for n even.

The time derivate of function (23) along the trajectories of the closed-loop equation (24) becomes

$$\begin{aligned} \dot{V}(\boldsymbol{q}, \boldsymbol{p}) &= \frac{1}{\Delta}[a_3 p_1 \dot{p}_1 - a_2 \dot{p}_1 p_2 - a_2 p_1 \dot{p}_2 + a_1 p_2 \dot{p}_2] \\ &\quad - \frac{m_{11}m_3}{a_1 + a_2}\sin(q_1)\dot{q}_1 \\ &\quad + k_1 \tanh(q_2 + \gamma_2 q_1)[\dot{q}_2 + \gamma_2 \dot{q}_1] \\ &= -k_v[k_3(p_2 + k_4 p_1)]\tanh(k_3(p_2 + k_4 p_1)) \end{aligned} \tag{26}$$

which is a negative semidefinite function, because by design $k_v > 0$.

Because the closed-loop equation (24) is autonomous, we can use the LaSalle's invariance principle to demonstrate that the equilibrium for n even in the set $\mathcal{E}$ are asymptotically stable ([2]). To this end, let us define the set Ω as

$$\begin{aligned} \Omega &= \{\boldsymbol{q} \in \mathbb{R}^2, \boldsymbol{p} \in \mathbb{R}^2 : \dot{V}(\boldsymbol{q}, \boldsymbol{p}) = 0\} \\ &= \{\boldsymbol{q} \in \mathbb{R}^2 \ \& \ \boldsymbol{p} \in \mathbb{R}^2 : \\ &\qquad k_3[p_2 + k_4 p_1]\tanh(k_3[p_2 + k_4 p_1]) = 0\}. \end{aligned}$$

In accordance with LaSalle's invariance principle, we need to determine the largest invariant set in Ω. To this

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{p_1}{m_{11}} \\ \frac{p_2}{m_{22}} \\ m_3 \sin(q_1) - [\gamma_1 \sin(q_1) + k_p \tanh(q_2 + \gamma_2 q_1) + k_v \tanh(k_3(p_2 + k_4 p_1))] \\ \gamma_1 \sin(q_1) + k_p \tanh(q_2 + \gamma_2 q_1) + k_v \tanh(k_3(p_2 + k_4 p_1)) \end{bmatrix}. \quad (24)$$

end, we have that any trajectory in Ω must satisfy

$$k_3[p_2(t) + k_4 p_1(t)] \equiv 0 \quad (27)$$

and in its turn,

$$k_3[\dot{p}_2(t) + k_4 \dot{p}_1(t)] \equiv 0. \quad (28)$$

Further, (27) can be written as

$$k_2[\dot{q}_2(t) + \gamma_2 \dot{q}_1(t)] \equiv 0 \quad (29)$$

with k_2 and γ_2 defined in (11). Integrating (29), we obtain the following:

$$q_2(t) + \gamma_2 q_1(t) \equiv k \quad (30)$$

for any constant k. By using $\dot{p}_1$ and $\dot{p}_2$ from (24) in (28), yields

$$\begin{aligned} & k_3[\sin(q_1(t))[\gamma_1 + k_4[m_3 - \gamma_1]] \\ & + k_p \tanh(q_2(t) + \gamma_2 q_1(t))[1 - k_4] \\ & + k_v \tanh(k_3(p_2(t) + k_4 p_1(t)))[1 - k_4]] \equiv 0. \end{aligned} \quad (31)$$

Considering (27) and (30), (31) becomes

$$\begin{aligned} & k_3[\sin(q_1(t))[\gamma_1 + k_4[m_3 - \gamma_1] \\ & + k_p \tanh(k)[1 - k_4]] \equiv 0 \end{aligned} \quad (32)$$

which imply that $q_1(t)$ must be constant and as consequence $\dot{q}_1(t) \equiv 0$. Next, from (29) we have that $\dot{q}_1(t) \equiv 0 \Rightarrow \dot{q}_2(t) \equiv 0$. Besides, because (27) and (29) are equivalents, we have

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (33)$$

Incorporating (33) in (24), we obtain

$$\begin{aligned} & m_3 \sin(q_1(t)) - [\gamma_1 \sin(q_1(t)) \\ & + k_p \tanh(q_2(t) + \gamma_2 q_1(t)) \\ & + k_v \tanh(k_3(p_2(t) + k_4 p_1(t)))] \equiv 0, \end{aligned} \quad (34)$$

$$\begin{aligned} & \gamma_1 \sin(q_1(t)) + k_p \tanh(q_2(t) + \gamma_2 q_1(t)) \\ & + k_v \tanh(k_3(p_2(t) + k_4 p_1(t))) \equiv 0, \end{aligned} \quad (35)$$

whose solutions are

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \equiv \begin{bmatrix} n\pi \\ -\gamma_2 n\pi \end{bmatrix} \quad (36)$$

with $n \in \mathbb{N}$. This development and particularly the conclusions (33) and (36) allow to establish that the largest invariant set in Ω is $\mathcal{E}$. From LaSalle's invariance principle, we conclude that the equilibria in the set $\mathcal{E}$ for n even are asymptotically stable (see similar details in [1]).

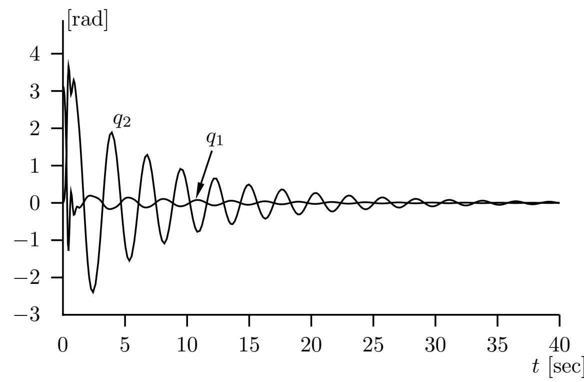

Fig. 2. Joint positions q_1 and q_2: Proposed control law (21).

So we have proven the following:

Proposition. The inertia wheel pendulum (5) in closed-loop with the bounded control law (21) has an infinite number of isolated locally asymptotically stable equilibria, which belong to equilibria set (25) with $n \in \mathbb{N}$ even. Furthermore, the applied torque is bounded by

$$|u| \leq u^{max}.$$

$\diamond \diamond \diamond$

V. SIMULATIONS

In this section we present the simulation results obtained on the inertia wheel pendulum by using the parameters given in [3]. These correspond to $m_3 = 10$, $m_{11} = 0.1$, $m_{22} = 0.2$. We suppose the maximum torque that can supply the actuator is $u^{max} = 45$ Nm. The remaining parameters were selected as $a_1 = 1$, $a_2 = -1.5$ and $a_3 = 6$, which satisfy (12) and in accordance with (11) yields $\gamma_1 = 30$, $\gamma_2 = 4.5$ and $k_2 = 0.0266$. The gains were chosen as $k_p = 3.75$ and $k_v = 10$. Substituting these values in the control law (21), we have

$$|u| < \gamma_1 + k_p + k_v = 43.75 \text{ Nm} < u^{max}, \quad (37)$$

The rest initial configuration was $[q_1(0) \quad q_2(0) \quad p_1(0) \quad p_2(0)]^T = [3.14 \quad 0 \quad 0 \quad 0]^T$. Figures 2 and 3 shown joint positions q_1 and q_2 with control laws (21) and (16), respectively.

A simple observation shows that Figures 2 and 3 are very similar and both converge toward the equilibrium point $\begin{bmatrix} q_1 & q_2 & p_1 & p_2 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$.

The applied bounded and non-bounded control inputs u, are shown in Figures 4 and 5, respectively. Notice that the bounded torque clearly evolve inside the prescribed

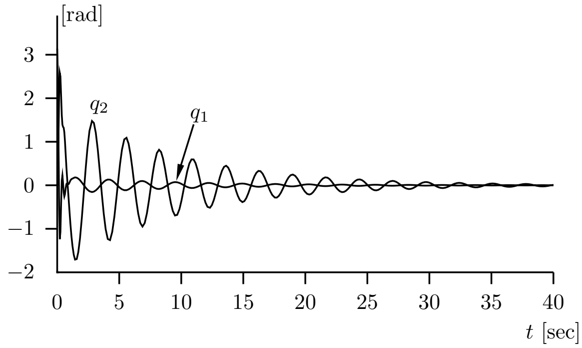

Fig. 3. Joint positions q_1 and q_2: Original control law (16).

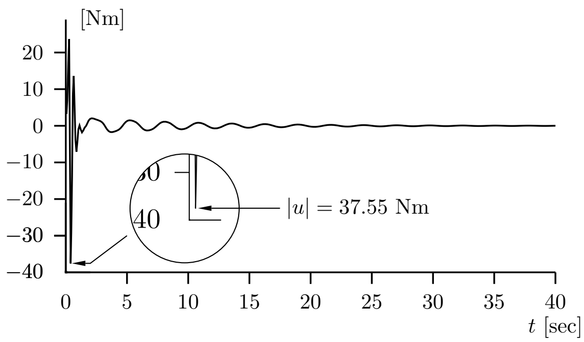

Fig. 4. Applied bounded torque u: Proposed control law (21).

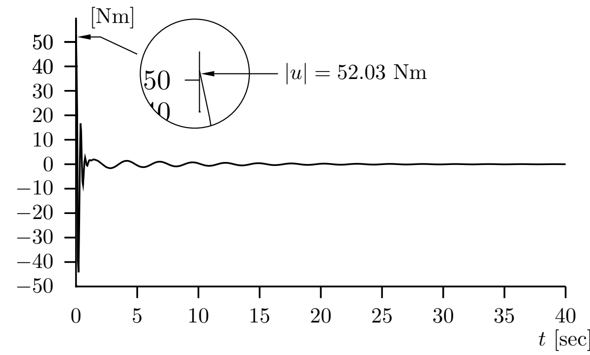

Fig. 5. Applied non-bounded torque u: Original control law (16).

limit given in (37) with a maximum torque of 37.55 Nm, while the non-bounded torque achieves values greater than $u^{max} = 45$ Nm.

VI. CONCLUDING REMARKS

Motivated by practical matter on IDA–PBC, this paper has proposed a control scheme with bounded torque for the inertia wheel pendulum. The proposed controller was designed following the IDA–PBC methodology, but now taking care in the actuator torque restriction. Conditions on the controller gains are provided to guarantee that the control action remains within the prescribed limit.

VII. ACKNOWLEDGMENTS

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