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On Verification of Controlled Hybrid Dynamics through Ellipsoidal Techniques

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Abstract— **This paper deals with the dynamics of controlled hybrid systems under piecewise open-loop controls restricted by hard bounds. The system equations may be reset when crossing some prespecified domains ("the guards") in the state space. Here the continuous dynamics which govern the motion between the guards are complemented by discrete transitions which govern the resets. A state space model for such systems is proposed and the reach sets for such models are described. A verification problem is considered whose solution indicates whether the reach set (at given time or at some time within a given time interval) intersects or avoids a prespecified target set. The computational side of verification is treated through ellipsoidal techniques that indicate routes for numerical algorithms including parallel calculations.**

I. Introduction

The standard reachability problem is an essential topic in control theory [3], [4],[5], [8]. Recent applications require treating reachability in the more complicated setting of systems with hybrid dynamics. The notion of hybrid system has slightly varying definitions in [1], [2], [10], [9] and [11].

This paper deals with a controlled process governed by an array of linear subsystems, one of which is switched on at each time and determines the system's on-line continuous dynamics. This switching is logically controlled: when the continuous state crosses some preassigned zones ("the guards"), the current subsystem may be switched to another subsystem of the array. The guards are taken here as hyperplanes, each of which allows a uniquely specified reset (a transition from the current subsystem to another one). In addition, the reset may or may not entail a change in the phase coordinates of the process.

Discussed in this paper is the reachability problem for such a hybrid system followed by the problem of verifying whether the system reaches or always avoids a given target set.The solutions are described through HJB-type equations.

Despite the fairly complicated dynamics, efficient computation of reach tubes through ellipsoidal approximations [8], with further numerical solution of related verification problems appears to be available. Ellipsoidal operations for related algorithms are indicated.

II. The Hybrid System

The overall system is governed by an array of subsystems indexed $i = 1, ..., k$,

$$\dot{x} \in A^{(i)}(t)x + B^{(i)}(t)u^{(i)}(t) + C^{(i)}(t)f^{(i)}(t), \qquad (1)$$

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with continuous matrix coefficients $A^{(i)}(t), B^{(i)}(t), C^{(i)}(t)$, and $x \in \mathbb{R}^n$. The $u^{(i)}$ are piecewise *open-loop controls* restricted by inclusions $u^{(i)}(t) \in \mathcal{P}^{(i)}(t) \subset \mathbb{R}^p$, and $\mathcal{P}^{(i)}(t)$ are continuous set-valued functions. The functions $f^{(i)}$ are given.

The bounds on the control values are *ellipsoidal*,

$$\mathcal{P}^{(i)}(t) = \mathcal{E}(p^{(i)}(t), P^{(i)}(t)). \qquad (2)$$

The continuous functions $p(t)$ and $P(t) = P'(t)$ are the center and the "shape" matrix of the ellipsoid $\mathcal{E}(p(t), P(t))$. Recall that the *support function* of $\mathcal{E}(p, P)$, $\rho(l \mid \mathcal{E}(p, P))$ is given by

$$\max\{(l, x) \mid x \in \mathcal{E}(p, P)\} = (l, p) + (l, Pl)^{1/2}.$$

In the phase space $\mathbb{R}^n$ are given k hyperplanes,

$$H_i = \{x \mid (c^{(i)}, x) - \gamma_i = 0\},\ c^{(i)} \in \mathbb{R}^n,\ \gamma_i \in \mathbb{R}$$

for $i = 1, \ldots, k$. These are *the enabling zones* (the guards).

Here is how the system operates.

At time t_0 the motion initiates from a point in the starting set $\mathcal{X}^0 = \mathcal{E}(x^0, X^0)$ and follows subsystem i (take $i = 1$, to be specific), with one of the controls $u^{(1)}(t)$ until (at time τ'_{i_1}) it reaches H_{i_1}, the first of the hyperplanes H_i along its route ($i_1 \neq 1$).

Here a binary operation interferes: the motion either continues along the "old" subsystem $i = 1$ or switches to (*is reset to*) a "new" subsystem $i_1 \neq 1$. (If $i_1 = 1$, we presume there is no reset.)

Before crossing H_{i_1}, the state of the system is denoted as $\{t, x, [1^+]\}$; after the binary operation the state is denoted as either $\{t, x, [1^+, i_1^+]\}$, if there was a reset, or $\{t, x, [1^+, i_1^-]\}$, if there was no reset.

The motion then develops according to the subsystem $i = 1$ or $i = i_1$ until crossing the next hyperplane H_{i_2}, when a similar binary operation takes place: The motion either follows the previous subsystem or is reset to subsystem i_2. After the second crossing the state of the system is $\{t, x, [1^+, i_1^s, i_2^s]\}$, where each boolean index "s" is either $s = +$ or $s = -$.

Thus the state has a memoryless part $\{t, x\}$, which is the current position of the continuous-time variable, and a part $[1^+, i_1^s, i_2^s]$ with memory, related to the discrete event variable i_j^s, which describes the sequence of switchings made earlier by the system. At each new crossing a new term is added to this sequence.

Thus the following general rules are observed:

1) Crossing each hyperplane H_i results either in a reset to subsystem i or there is no reset.

2) The crossing takes place in the direction of the support vector $c^{(i)}$, and at points of crossing we have

$$\min\{(c^{(i)}(t), z) \mid z \in F^{(j)}(t,x)\} \geq \varepsilon > 0, \qquad (3)$$

for all $i, j = 1, ..., k,$, in which

$$F^{(j)}(t,x) =$$

$$A^{(j)}(t)x + B^{(j)}(t)\mathcal{E}(p^{(j)}(t), P^{(j)}(t)) + C^{(j)}(t)f^{(j)}(t).$$

3) The state after j crossings is $\{t, x, [i_1^s, \ldots, i_j^s]\}$, and each index s is $+$ or $-$.
4) Upon crossing hyperplane H_m the sequence $[i_1^s, \ldots, i_j^s]$, describing the "discrete event" part of the state, is augmented by a new term, which is either m^+, if there is a switching to subsystem m, or m^-, if there is no switching.

This notation allows one to trace back the array of subsystems used earlier from any on-line position $\{t, x\}$. If, for example, the state is $\{t, x, [1^+, i_1^-, ..., i_j^-]\}$ with $s = -$ for all $i_1, ..., i_j$, the trajectory did not switch at any of the j crossings, having followed the initial subsystem $i = 1$ throughout the whole process.

Note that at each state $\{t, x, [1^+, i_1^s, ..., i_j^s]\}$ the system follows the subsystem whose number coincides with that of the last term with index $s = +$.

The hybrid system under consideration differs from so-called *switching systems* in that the time instants for crossing are not fixed but are determined by the course of each trajectory. Note, however, that resets result in an instantaneous change of velocity $\dot{x}(t)$, but with *no change* in the current position $x(t)$ of the system in the phase space.(Such a change could be added by demanding in addition to the above that at point $x(\tau)$ of reset (m) the state space variable instantaneously moves from $x(\tau)$ to $x(\tau+) = K_m x(\tau) + k^{(m)}$ for a given $n \times n$ matrix K_m and n-vector $k^{(m)}$. While complicating the formulas, such change would not affect the essential steps of the presented schemes).

The paper is concerned with reachability under piecewise open-loop controls with possible resets of controlled systems at given guards, and in between these resets the control is open-loop. The starting set $\mathcal{X}^0$ is ellipsoidal, $\mathcal{X}^0 = \mathcal{E}(x^0, X^0)$.

III. Reachability under Resets

The reachability problem has two versions.

Problem I. Find the set of all $\{x\}$ reachable from starting position $\{t_0, \mathcal{X}^0\}$ at **given time** t through all possible controls. This is the **reach set** $\mathcal{X}(t; t_0, \mathcal{X}^0)$ **at time** t from $\{t_0, \mathcal{X}^0\}$.

Problem II. Find the set of all $\{x\}$ reachable from starting position $\{t_0, \mathcal{X}^0\}$ at **some time** t within interval $t \in [t', t''] = T$ through all possible controls. This is the **reach set** $\mathcal{X}(t', t''; t_0, \mathcal{X}^0)$ **within interval** T.

One may observe that the problem consists in investigating branching trajectory tubes, in describing their cross-sections ("cuts"), and the unions of such cross-sections. The reach sets may therefore be *disconnected*. We next discuss reach sets at given time t.

Let us first describe the reach set for a given sequence $[i_1^{s_1}, \ldots, i_r^{s_r}]$ of crossings, from position $\{t_0, \mathcal{X}^0, [j]\}$, taking $j = 1$ to be specific. The index s_1 is either $-$ or $+$.

1) The reach set after one crossing: (a) Before reaching H_{i_1}, $i_1 = j$ we have

$$\mathcal{X}^{(1)}[t] = \mathcal{X}^{(1)}(t; t_0, \mathcal{X}^0, [1^+]) \;=\; G^{(1)}(t, t_0)\mathcal{X}^0 +$$

$$+ \int_{t_0}^{t} G^{(1)}(t,s)[B(s)\mathcal{P}^{(1)}(s) + C(s)f^{(1)}(s)]ds, \qquad (4)$$

in which $G^{(i)}(t,s)$ is the transition function for subsystem i.

(b) To be precise, suppose that before reaching H_j we have

$$\max\{(c^{(j)}, x) \mid x \in \mathcal{X}^{(1)}[t]\} = \rho_j^+(t) < \gamma_j.$$

The first instant of time when $\mathcal{X}^{(1)}[t] \bigcap H_j \neq \emptyset$ is τ_j', the smallest positive root of the equation

$$\gamma_j - \rho_j^+(t) = 0.$$

Introducing the function

$$\rho_j^-(t) = \min\{(c^{(j)}, x) \mid x \in \mathcal{X}^{(1)}[t]\},$$

we observe that the condition $\mathcal{X}^{(1)}[t] \bigcap H_j \neq \emptyset$ will hold so long as

$$\rho_j^-(t) \leq \gamma_j \leq \rho_j^+(t),$$

and the point of departure from H_j is the largest positive root τ_j'' of

$$\gamma_j - \rho_j^-(t) = 0.$$

Condition (3)above ensures that τ_j', τ_j'' are *unique*. Note that τ_j'' is the time instant when the entire reach set $\mathcal{X}^{(1)}[t]$ leaves H_j.

Denote $\mathcal{X}^{(1)}[t] \bigcap H_j = \mathcal{Z}_1^{(j)}(t)$.

(c) After the crossing we have to envisage two branches:

($-$) with no reset—then nothing changes and

$$\mathcal{X}(t; t_0, \mathcal{X}^0, [1^+, j^-]) = \mathcal{X}^{(1)}(t; t_0, \mathcal{X}^0, [1^+]);$$

($+$) with reset—then we consider the union

$$\mathcal{X}(t; t_0, \mathcal{X}^0, [1^+, j^+]) =$$

$$\bigcup\{\mathcal{X}^{(j)}(t; s, \mathcal{Z}_1^{(j)}(s)) \mid s \in [\tau_j', \tau_j'']\}, \quad t \geq \tau_j''.$$

Thus, in case ($-$) the reach tube develops further along the "old" subsystem (1), while in case (+) it develops along the "new" subsystem ($j = i_1$).

(d) For each new crossing we repeat this procedure and obtain the reach set $\mathcal{X}(t; t_0, \mathcal{X}^0, [1^+, i_1^{(s_1)}, \ldots, i_r^{(s_r)}])$ for the branch $[1^+, i_1^{(s_1)}, \ldots, i_r^{(s_r)}]$.

We further impose the following condition.

Assumption 3.1: The intervals $[\tau_i', \tau_i'']$, $i = 1, \ldots, r$, do not intersect.

For any interval $[t_0, t]$, with $\tau''_{i_m} \le t \le \tau'_{i_{m+1}}$ and $\tau''_{i_j} \le t^* \le \tau'_{i_{j+1}}$, $i_j < i_m$, one may observe the following semigroup-like property:

$$\mathcal{X}(t; t_0, \mathcal{X}^0, [1^+, i_1^{s_1}, \dots, i_m^{s_m}]) = \tag{5}$$

$$\mathcal{X}(t; t^*, \mathcal{X}(t^*; t_0, \mathcal{X}^0, [1^+, i_1^{s_1}, \dots, i_j^{s_j}]), [i_{j+1}^{s_j+1}, \dots, i_m^{s_m}]).$$

Assumption 3.1 is not a necessary requirement and can be omitted. However under this assumption the explanation of the general scheme is more transparent and relieved of unimportant details.

2) The one-stage crossing transformation: Recall that the continuous-time transition between crossings along subsystem j, from position $\{\tau, \mathcal{X}\}$ with $\tau \ge \tau''_j$ to the position at time $t \le \tau'_{j+1}$ is $\mathcal{X}^{(j)}(t; \tau, \mathcal{X})$.

On the other hand, we may define a "one-stage crossing" transformation from position (state) $\{\tau'_j, \mathcal{X}, [1^+]\}$ at the first time (τ'_j) of crossing H_j to the last time (τ''_j) of crossing H_j:

$$T_j^s\{\tau'_j, \mathcal{X}, [1^+]\} =$$

$$\{\tau''_j, \mathcal{X}^{(1)}(\tau''_j; \tau'_j, \mathcal{X}, [1^+, j^-])\}, \text{ if } s = -,$$

$$T_j^s\{\tau'_j, \mathcal{X}, [1^+]\} =$$

$$\{\tau''_j, \bar{\mathcal{Z}}_1^{(j)}[\tau''_j],\ [1^+, j^+]\}, \quad \text{if } s = +.$$

Above,

$$\bar{\mathcal{Z}}_1^{(j)}[\tau''_j] = \cup\{\mathcal{X}^{(j)}(\tau''_j; t, H_j \cap \mathcal{X}^{(1)}(t; \tau'_j, \mathcal{X})) |\ t \in [\tau'_j, \tau''_j]\}.$$

We can now represent a branch $[1^+, i_1^{s_1}, \dots, i_m^{s_m}]$ through a sequence of alternating operations of type T_j^s and $\mathcal{X}^{(j)}$.

For example, the reach set for the branch $[1^+, i_1^+, i_2^-]$ from starting position $\{\tau, \mathcal{X}, [1^+]\}$, $\tau \le \tau'_{i_1}$, at time $t \in [\tau''_{i_2}, \tau'_{i_3}]$ requires the following sequence of mappings:

$$T_{i_1}^+\{\tau'_{i_1}, \mathcal{X}^{(1)}(\tau'_{i_1}; \mathcal{X}, \tau), [1^+]\} = \{\tau''_{i_1}, \bar{\mathcal{Z}}_1^{(i_1)}[\tau''_{i_1}], [1^+, i_1^+]\},$$

$$\mathcal{X}^{(i_1)}[\tau'_{i_2}] \ = \ \mathcal{X}^{(i_1)}(\tau'_{i_2}; \tau''_{i_1}, \bar{\mathcal{Z}}_1^{(i_1)}[\tau''_{i_1}]),$$

$$T_{i_2}^-\{\tau'_{i_2}, \mathcal{X}^{(i_1)}[\tau'_{i_2}], [1^+, i_1^+]\} =$$

$$\{\tau''_{i_2}, \mathcal{X}^{(i_2)}(\tau''_{i_2}; \tau'_{i_2}, \mathcal{X}^{(i_1)}[\tau'_{i_2}]), [1^+, i_1^+, i_2^-]\};$$

and then, for $t \in [\tau''_{i_2}, \tau'_{i_3}]$, the desired set of positions is given as

$$\{t; \mathcal{X}^{(i_2)}(t, \tau'_{i_2}, \mathcal{X}^{(i_1)}[\tau'_{i_2}]), [1^+, i_1^+, i_2^-]\}.$$

Thus $\mathcal{X}^{(i_2)}(t; \tau'_{i_2}, \mathcal{X}^{(i_1)}[\tau'_{i_2}])$ is one branch of the overall reach set $\mathcal{X}(t; \mathcal{X}, \tau)$

Lemma 3.1: The branch $\mathcal{X}(t; t_0, \mathcal{X}^0, [1, i_1^{(s_1)}, \dots, i_k^{(s_k)}])$ is given by the composition of alternating one-stage crossing transformations T_j^s and continuous maps $\mathcal{X}^{(j)}$, $j = 1, \dots, k$.

An alternative scheme for calculating reach sets is through value functions of optimization problems. Its advantage is that it is not restricted to linear systems.

IV. Reachability through value functions

As shown in [7], the reach sets for ordinary (non-hybrid) systems may be calculated as level sets of solutions to HJB (Hamilton-Jacobi-Bellman) equations for certain optimization problems. We will follow this scheme for the hybrid system under consideration. Consider first a one-stage crossing.

(a) *Before crossing* $H_{i_1} = H_j$, we assume, as in Section II, that the system operates from position $\{t_0, \mathcal{X}^0, [1]\}$. Then, for $t < \tau'_j$, we have

$$\mathcal{X}^{(1)}[t] = \{x \mid V^{(1)}(t, x) \le 0\},$$

wherein

$$V^{(1)}(t, x) = \min_{u^{(1)}}\{d^2(x^{(1)}(t_0), \mathcal{X}^0) \mid x^{(1)}(t) = x\},$$

and $x^{(1)}(t) = x(t)$ is the trajectory of subsystem 1. We also write $V^{(1)}(t, x) = V^{(1)}(t, x_1, x_2, \dots, x_n)$, for $x = (x_1, \dots, x_n)$.

(b) *At the crossing* we have $\mathcal{X}^{(1)}(t; \tau'_j, \mathcal{X}) \cap H_j = \mathcal{Z}_1^{(j)}(t)$, which can be calculated as follows. Without loss of generality we may take $c_1^{(j)} = 1$. Then

$$\mathcal{Z}_1^{(j)}(t) = \{x \mid V^{(1)}(t, \zeta(x), x_2, \dots, x_n)) \le 0\},$$

$$\zeta(x) = \gamma_j - \sum_{i=2}^n c_i^{(j)} x_i.$$

In particular, if the hyperplane $H_j = \{x \mid x_j = \gamma_j\}$,

$$\mathcal{Z}_1^{(j)}(t) = \{x \mid V^{(1)}(t, x_1, \dots, \gamma_j, \dots, x_n) \le 0\},$$

where γ_j replaces x_j and the set $\mathcal{Z}_1^{(j)}(t) \ne \emptyset$ iff $\rho_j^-(t) \le \gamma_j \le \rho_j^+(t)$, wherein

$$\rho_j^+(t) \ = \ \max\{(c^{(j)}, x) \mid V^{(1)}(t, x) \le 0\},$$
$$\rho_j^-(t) \ = \ \min\{(c^{(j)}, x) \mid V^{(1)}(t, x) \le 0\}.$$

This happens within the time interval $[\tau'_j, \tau''_j]$, $\rho_j^+(\tau'_j) = \rho_j^-(\tau''_j) = 0$.

(c) *After crossing* $H_{i_1} = H_j$, we envisage two branches: $(-)$ with no reset: then $\mathcal{X}(t; t_0, \mathcal{X}^0, [1^+, j^-]) = \mathcal{X}^{(1)}(t; t_0, \mathcal{X}^0, [1^+])$,

$(+)$ with reset: then for $t \in \tau''_j \le t \le \tau'_{j+1}$ we have to calculate the union

$$\bigcup\{\mathcal{X}^{(j)}(t; s, \mathcal{Z}_1^{(j)}(s)) \mid s \in [\tau'_j, \tau''_j]\} = \mathcal{X}(t; \tau''_j, \bar{\mathcal{Z}}_1^{(j)}[\tau_j]) =$$

$$\mathcal{X}(t; t_0, \mathcal{X}^0, [1^+, j^+]), \ \ \tau''_j \le t \le \tau'_{j+1}.$$

For $t > \tau''_j$ this union may be calculated as the level set for the function

$$\mathcal{V}(t, x \mid [1^+, j^+]) \ = \ \min_s\{V^{(j)}(t, s, x) \mid s \in [\tau'_j, \tau''_j]\},$$

wherein

$$V^{(j)}(t,s,x) \;=\; \min_{u^{(j)}}\{V^{(j)}(s,\zeta(x(s)),x_2(s),\ldots,x_n(s)) \mid u^{(j)}(\xi)\in\mathcal{P}^{(i)}(\xi),\ \xi\in[s,t],\ x(t)=x\},$$

so that

$$\mathcal{X}(t;t_0,\mathcal{X}^0,[1^+,j^+]) = \{x \mid \mathcal{V}(t,x\mid[1^+,j^+])\le 0\} = \bigcup\{\mathcal{X}(t,s,\mathcal{Z}_1^{(j)}(s)) \mid s\in[\tau_j',\tau_j'']\} = \mathcal{X}^{(j)}[t],$$

and

$$\mathcal{X}^{(j)}[t] \;=\; \cup\{\mathcal{X}(t,s,\mathcal{Z}_1^{(j)}(s)) \mid s\in[\tau_j',\tau_j'']\},$$
$$\mathcal{X}(t,s,\mathcal{Z}_1^{(j)}(s)) \;=\; \{x : V^{(j)}(t,s,x)\le 0\},$$

with $\mathcal{X}^{(j)}[t] \;=\; \{x:\mathcal{V}(t,x\mid[1^+,j^+])\le 0\}$.

The indicated union set may be nonconvex.

(d) Repeating the procedure for each new crossing, we obtain the reach set $\mathcal{X}(t;t_0,\mathcal{X}^0,[1^+,i_1^{(s_1)},\ldots,i_k^{(s_k)}])$ for the branch $[1^+,i_1^{(s_1)},\ldots,i_k^{(s_k)}]$.

This is the general scheme for successively applying the transformations given above. We now consider the verification problem.

V. The verification problem

Let $\mathcal{M}=\mathcal{E}(m,M),\ M=M'>0$, be a given target set and $\mathcal{M}\cap H_i=\emptyset,\quad i=1,\ldots,k$.

Problem 3. Given starting position $\{t_0,\{\mathcal{X}^0,1^+\}\}$, target set $\mathcal{M}$ and time $t>t_0$, verify whether there exists a branch $\mathcal{I}(l)=[1^+,i_1^{(s_1)},\ldots,i_l^{(s_l)}]$ such that one of the following conditions is fulfilled:

$$\text{(i) } \mathcal{X}^{(l)}[t] \;=\; \mathcal{X}(t;t_0,\mathcal{X}^0,[1^+,\ldots,i_l^{(s_l)}])\cap\mathcal{M}=\emptyset,$$
$$\text{(ii) } \mathcal{X}^{(l)}[t] \;=\; \mathcal{X}(t;t_0,\mathcal{X}^0,[1^+,\ldots,i_l^{(s_l)}])\cap\mathcal{M}\neq\emptyset,$$
$$\text{(iii) } \mathcal{X}^{(l)}[t] \;=\; \mathcal{X}(t;t_0,\mathcal{X}^0,[1^+,\ldots,i_l^{(s_l)}])\subseteq\mathcal{M}.$$

Let us investigate the solution to this problem for a fixed branch $\mathcal{I}(k)=[1^+,i_1^{(s_1)},\ldots,i_k^{(s_k)}]$, assuming $l=k,\ t>\tau_k''$.

Theorem 5.1: (A) Suppose the union set $\mathcal{X}^{(k)}[t]$ is *convex*. Then case (i) of Problem 3 will hold iff

$$\max\{-\rho(-l|\mathcal{M})-\rho(l|\mathcal{X}^{(j)}[t]) \mid (l,l)\le 1\}=\delta>0. \tag{6}$$

Case (ii) will hold iff

$$\max\{-\rho(-l|\mathcal{M})-\rho(l|\mathcal{X}^{(j)}[t]) \mid (l,l)\le 1\}\le 0. \tag{7}$$

Case (iii) will hold iff

$$\min\{\rho(l|\mathcal{M})-\rho(l|\mathcal{X}^{(j)}[t]) \mid (l,l)\le 1\}\ge 0. \tag{8}$$

(B) Suppose the set $\mathcal{X}^{(k)}[t] \;=\; \bigcup\{\mathcal{X}^{(j)}[t,s] \mid s\in[\tau_j',\tau_j'']\}$ is *nonconvex.*

Then case (i) of Problem 3 will hold *for all* s iff

$$\min_s\max_l\{-\rho(-l|\mathcal{M})-\rho(l|\mathcal{X}^{(j)}[s]) \mid (l,l)\le 1,\ s\in[\tau_j',\tau_j'']\}=\delta>0. \tag{9}$$

Case (ii) will hold *for some* s iff

$$\min_s\max_l\{-\rho(-l|\mathcal{M})-\rho(l|\mathcal{X}^{(j)}[s]) \mid (l,l)\le 1,\ s\in[\tau_j',\tau_j'']\}\le 0. \tag{10}$$

Case (iii) will hold *for some* s iff

$$\min_s\min_l\{\rho(l|\mathcal{M})-\rho(l|\mathcal{X}^{(j)}[s]) \mid (l,l)\le 1,\ s\in[\tau_j',\tau_j'']\}\ge 0. \tag{11}$$

Finally, if we want to verify the same properties for nonconvex sets $\mathcal{X}^{(k)}[\vartheta',\vartheta''] \;=\; \bigcup\{\mathcal{X}[t] \mid t\in[\vartheta',\vartheta'']\}$, reachable within a given interval $[\vartheta',\vartheta''],\ \vartheta\ge\tau_j''$, we have to perform additional operations similar to (9)-(11) over sets $\mathcal{X}[t]$ with $t\in[\vartheta',\vartheta'']$.

If we have more than one crossing, then each crossing adds a new parameter $s=s_j$, with range in an interval of type $[\tau_j',\tau_j'']$, related to this crossing. Thus, after k such intervals, we will have to calculate the related nonconvex unions of convex sets by optimizing parametrized value functions over k parameters $s_j,\ j=1,\ldots,k$.

We have now observed that the necessary numerical procedures require operations over convex sets, their geometric and algebraic sums and their intersections. This brings us to the use of the ellipsoidal calculus in approximating the solution elements by parametrized families of ellipsoids.

VI. Ellipsoidal techniques—reachability

We first calculate the reach set after a one-stage crossing. Here we consider external ellipsoidal approximations.

(a) Starting with $\mathcal{X}^0=\mathcal{E}(x^0,X^0), i=1$, the reach set $\mathcal{X}(t;t_0,\mathcal{X}^0,[1^+])$ is described by the set-valued integral (4), for which there is external *ellipsoidal approximation* [6], [8]:

$$\mathcal{X}(t;t_0,\mathcal{X}^0,[1])\subset\mathcal{E}(x^{(1)}(t),X_+^{(1)}(t)),$$

in which

$$\dot{X}_+^{(1)}=A^{(1)}(t)X_+^{(1)}+X_+^{(1)}A^{(1)\prime}(t)+\pi(t)X_+^{(1)}+(\pi(t))^{-1}B^{(1)}(t)P^{(1)}(t)B^{(1)\prime}(t), \tag{12}$$
$$\dot{x}^{(1)}=A^{(1)}(t)x^{(1)}+B^{(1)}(t)p^{(1)}(t)+C^{(1)}(t)f^{(1)}(t),$$
$$X_+^{(1)}(t_0)=X_-^{(1)}(t_0)=X^0,\ x^{(1)}(t_0)=x^0,$$

and $\pi(t)>0$ are parametrizing functions. These approximations will be *tight* along a given *direction* $l(t)=G^{(1)\prime}(t_0,t)l$, $l\in\mathbb{R}^n$, ($G^{(i)}(t_0,t)$ is the transition matrix of the homogeneous subsystem i), if $\pi(t)=\pi_l(t)$ (see [8]), and

$$\pi_l(t)=(l(t),B^{(1)}(t)P^{(1)}(t)B^{(1)\prime}(t)l(t))^{\frac{1}{2}}(l,X_+^{(1)}(t)l)^{-\frac{1}{2}}.$$

This yields the important equality

$$\mathcal{X}(t;t_0,\mathcal{X}^0,[1]) = \bigcap\{\mathcal{E}(x^{(1)}(t),X_+^{(1)}(t)) \mid (l,l)\le 1\}. \quad (13)$$

Each of the matrices $X_+^{(1)}(t)$ and therefore each of the ellipsoids $\mathcal{E}(x^{(1)}(t),X_+^{(1)}(t))$ depends on the parametrizing function π which, for the tight ellipsoids that we need, in its turn depends on $l\in\mathbb{R}^n$. To emphasize this dependence, we include l in the arguments, e.g. $X_+^{(1)}(t)=X_+^{(1)}(t|l)$).

(b) We now discuss the crossing $\mathcal{E}(x^{(1)}(t),X_+^{(1)}(t|l))\cap H_j$. Let $\{e^{(i)}\}$ be the orthonormal basis of the original coordinate system and $c^{(j)}=\sum_{i=1}^n \alpha_i^{(j)}e^{(i)}$, assuming, without loss of generality, that $\alpha_j^{(j)}\neq 0$. Then there exists a nondegenerate linear map $\mathbf{T}$, such that

$$\mathbf{T}c^{(j)}=\mathbf{e}^{(j)},\ \mathbf{T}e^{(i)}=\mathbf{e}^{(i)},$$

$$\mathbf{T}e^{(i)}=\mathbf{e}^{(i)},\ i=2,\dots,k,\ i\neq j,$$

wherein $\{\mathbf{e}^{(i)}\}$ is the basis for the new coordinate system. The hyperplane H_j then transforms into $\{x: x_j=\gamma_j\}$.

Now, in the new coordinates, keeping the former notations, we have

$$E_1^{(j)}(t|l)=\mathcal{E}(z^{(j)}(t),Z_{1+}^{(j)}(t|l))=$$

$$\mathcal{E}(x^{(1)}(t),X_+^{(1)}(t|l))\cap H_j=\{x: V_+^{(1)}(t,x|l)\le 1,\ x_j=\gamma_j\},$$

where

$$V_+^{(1)}(t,x|l)=(x-x^{(j)}(t),(X_+^{(1)})^{-1}(t|l)(x-x^{(j)}(t))). \quad (14)$$

and $x_j=\gamma_j$.

The intersection $E_1^{(j)}(t|l)$ is a degenerate ellipsoid with support function

$$\rho(q\ |E_1^{(j)}(t|l))=(q,z^{(j)}(t))+(q,Z_{1+}^{(j)}(t|l)q)^{\frac{1}{2}},$$

and $z^{(j)}(t),Z_{1+}^{(j)}(t|l$ is found with a standard calculation.

In the hyperplane $\mathcal{H}_j=\{x\mid x_j=\gamma_j\}$ we may consider an array of ellipsoids $\mathcal{E}(z^{(j)}(t),Z_{1+}^{(j)}(t))$, $t\in[\tau_j',\tau_j'']=\mathcal{T}=\cap\{\mathcal{T}_l|(l,l)\le 1\}$, and

$$\mathcal{T}_l=\{t:(x-x^{(1)}(t),(X_+^{(1)})^{-1}(t|l)(x-x^{(1)}(t))\le 1,\ x_j=\gamma_j\ \}.$$

Thus, for all $l\in\mathbb{R}^n, t\in\mathcal{T}_l$ we have

$$\mathcal{E}(z^{(j)}(t),Z_{1+}^{(j)}(t|l))\supseteq \mathcal{Z}_1^{(j)}(t).$$

Moreover, formula (13) implies

$$\bigcap\{\mathcal{E}(z^{(j)}(t),Z_{1+}^{(j)}(t|l))\mid (l,l)\le 1\}\ =\ \mathcal{Z}_1^{(j)}(t).$$

When propagated after the reset along the new subsystem j, each ellipsoid $\mathcal{E}(z^{(j)}(t),Z_{1+}^{(j)}(t|l))$ evolves as

$$\mathcal{X}_+^{(j)}[\vartheta,t\mid l]=\mathcal{X}_+^{(j)}(\vartheta,t,\mathcal{E}(z^{(j)}(t),Z_{1+}^{(j)}(t|l)))=$$

$$G^{(j)}(\vartheta,t)\mathcal{E}(z^{(j)}(t),Z_{1+}^{(j)}(t|l))+$$

$$\int_t^\vartheta G^{(j)}(\vartheta,s)(B(s)\mathcal{E}(p^{(j)}(s),P^{(j)}(s))+C(s)f^{(1)}(s))ds.$$

Here $\vartheta\ge\tau''$. Though generated by an ellipsoid, the set $\mathcal{X}_+^{(j)}[\vartheta,t\mid l]$ need not be an ellipsoid. It requires, in its turn, to be approximated by an array of ellipsoids. Namely, the exact reach set

$$\mathcal{X}_+^{(j)}[\vartheta,t\mid \mathcal{Z}_1^{(j)}(t)]=\bigcap\{\mathcal{X}_+^{(j)}[\vartheta,t\mid l]\mid (l,l)\le 1\},$$

and for each l the set

$$\mathcal{X}_+^{(j)}[\vartheta,t\mid l]\subseteq\mathcal{E}_q(x^{(j)}(\vartheta),X_+^{(j)}(\vartheta|l,q)),\ \forall q\in\mathbb{R}^n.$$

Elements $x^{(j)},X_+^{(j)}$ satisfy equations

$$\dot X_+^{(j)}=A^{(j)}(t)X_+^{(j)}+X_+^{(j)}A^{(j)\prime}(t)+$$

$$\pi_q(t)X_+^{(j)}+(\pi_q(t))^{-1}B^{(j)}(t)P^{(j)}(t)B^{(j)\prime}(t), \quad (15)$$

$$\dot x^{(j)}=A^{(j)}(t)x^{(j)}+B^{(j)}(t)p^{(j)}(t)+C^{(j)}(t)f^{(j)}(t),$$

with starting condition

$$X_+^{(j)}(t)=Z_{1+}^{(j)}(t|l),\ x^{(j)}(t)=z^{(j)}(t).$$

Functions π_q are for subsystem j of (15) and are defined similarly to π_l. Note that matrices $X_+^{(j)}(\vartheta)=X_+^{(j)}(\vartheta|l,q)$ depend on two parameters: $l\in\mathbb{R}^n$ (through the starting condition) and $q\in\mathbb{R}^n$ (through the parametrizing function π_q).

We thus come to the conclusion.

Theorem 6.1: The following equality holds:

$$\mathcal{X}_+^{(j)}[\vartheta,t\mid\mathcal{Z}_1^{(j)}(t)]\ = \quad (16)$$

$$\bigcap\{\mathcal{E}(x^{(j)}(\vartheta),X_+^{(j)}(\vartheta,t\mid l,q))\mid (l,l)\le 1,(q,q)\le 1\}.$$

The last formula indicates the possibility of *parallel calculation* through identical modules.

Finally the nonconvex reach set after one crossing is

$$\mathcal{X}(\vartheta,t_0,\mathcal{X}^0)\ =\ \bigcup\{\mathcal{X}_+^{(j)}[\vartheta,t\mid\mathcal{Z}_j^{(1)}(t)]\mid t\in[\tau_j',\tau_j'']\}, \quad (17)$$

which is the union of an intersection of ellipsoids.

Since

$$\mathcal{E}(x^{(j)}(\vartheta),X_+^{(j)}(\vartheta,t\mid l,q))=\{x:V(t,\vartheta,x|l,q)\le 1\}$$

and

$$V(t,\vartheta,x|l,q)$$

$$=\{x:(x-x^{(j)},X_+^{(j)}(\vartheta,t\mid l,q))(x-x^{(j)})\le 1\}, \quad (18)$$

we may define

$$\mathbf{V}(\vartheta,t_0,x)=\min_t\max_{l,q}\{V(t,\vartheta,x|l,q)\mid$$

$$(l,l)\le 1,\ (q,q)\le 1,\ t\in\mathcal{T}\}. \quad (19)$$

Theorem 6.2: The reach set $\mathcal{X}(\vartheta,t_0,\mathcal{X}^0)$ is the level set

$$\mathcal{X}(\vartheta,t_0,\mathcal{X}^0)=\{x:\mathbf{V}(\vartheta,t_0,x)\le 1\}. \quad (20)$$

Having described the schemes for reachability, we now propagate them to the verification problem.

VII. Ellipsoidal techniques – verification

In section V we described the optimization problems which solve the problem of verification, when $\mathcal{X}^{(k)}[t]$ is the union over variable s of convex sets $\mathcal{X}^{(k)}[t,s]$. We now specify these solutions bearing in mind that the latter are intersections of ellipsoids and the target set is also an ellipsoid.

Let $\mathcal{X}^{(k)}[t,s] = \bigcap\{E[t,s,l] \mid (l,l) \leq 1\}$, where $E[t,s,l]$ stands for a continuous ellipsoid-valued function defined for $s \in [\tau', \tau''],\ (l,l) \leq 1$.

For fixed s the intersection $\mathcal{X}^{(k)}[t,s] \cap \mathcal{E}(m,M) = \emptyset$ iff

$$d_1[t,s] = \max_l \max_q \{-\rho(-q \mid E[t,s,l]) - \rho(q \mid \mathcal{M}) \mid$$
$$|(q,q) \leq 1, (l,l) \leq 1\} \geq \delta > 0$$

and $\mathcal{X}^{(k)}[t] \cap \mathcal{E}(m,M) = \emptyset$ *for all* s iff

$$d_1[t] = \min_s \{d_1[t,s] \mid s \in [\tau', \tau'']\} \geq \delta_1 > 0. \quad (21)$$

For fixed s the intersection $\mathcal{X}^{(k)}[t,s] \cap \mathcal{E}(m,M) \neq \emptyset$ iff

$$d_2[t,s] = \max_l \max_q \{-\rho(-q \mid E[t,s,l]) - \rho(q \mid \mathcal{M}) \ |q = l,$$
$$(q,q) \leq 1, (l,l) \leq 1\} \leq 0$$

and $\mathcal{X}^{(k)}[t] \cap \mathcal{E}(m,M) \neq \emptyset$ *for some* s iff

$$d_2[t] = \min_s \{d_2[t,s] \mid s \in [\tau', \tau'']\} \leq 0. \quad (22)$$

For fixed s the intersection $\mathcal{X}^{(k)}[t,s] \subseteq \mathcal{E}(m,M)$ if

$$d_3[t,s] = \min_l \min_q \{\rho(q|\mathcal{M}) - \rho(q|E[t,s,l]) \mid (l,l) \leq 1\} \geq 0,$$

and $\mathcal{X}^{(k)}[t] \subseteq \mathcal{E}(m,M)$ for all t if

$$d_3[t] = \min_t \{d_3[t,s] \mid s \in [\tau', \tau'']\} \geq 0. \quad (23)$$

These formulas thus allow calculation of the union (17).

Remark 7.1: If the objective is to calculate

$$\bigcup\{\mathcal{X}(\vartheta, t_0, \mathcal{X}^0) \mid \vartheta \in [\vartheta', \vartheta'']\}\ ,\ \vartheta' \geq \tau'',$$

the previous schemes have to be applied once more with yet a new array of external ellipsoids. The total procedure has to be repeated at each new crossing, thus forming a branching process that could be effectively parallelized.

A possible algorithmic scheme.

1. Starting with position $\{t_0, \mathcal{X}_0, [1^+]\}$, with nearest crossing H_j, find approximations

$$\mathcal{X}(\tau_j'; t_0, \mathcal{X}^0, [1]) \subseteq \mathcal{E}(x^{(1)}(t), X_+^{(1)}(t|l)) = E_1^{(j)}(t|l), \quad (24)$$

which are *tight* along directions $l(t)$ generated by selected $l \in \mathbb{R}^n,\ (l,l) = 1$.

2. Calculate the one-stage crossing transformation $T_j^s E_+(t|l)$:

for $s = -$, there is no reset and we continue with formulas (12);

for $s = +$, calculate for $t \in \mathcal{T}_l$ the array $T_j^s E_+(t|l) = \mathcal{E}(z^{(j)}(t), Z_{1+}^{(j)}(t|l))$.

3. Proceed further on, so that:

if $s = -$, continue with formulas (12),

if $s = +$, calculate the arrays $\{\mathcal{E}(x^{(j)}(\vartheta), X_+^{(j)}(\vartheta, t \mid l, q))\}$.

4. Calculate the set $\mathcal{X}(\vartheta, t_0, \mathcal{X}^0)$ through (18)-(20).

5. Apply results to verification, following the discussion at the beginning of this section.

Remark 7.2: Proceeding *in parallel* with *arrays of ellipsoids* corresponding to arrays of directions l, q, an increase in number of directions would allow eventually to approach the exact solution with any degree of accuracy.

VIII. Conclusion

This paper studies the reachability problem for a hybrid system whose dynamics at each instant of time is governed by one of the linear subsystems in a given array, with possible resets from one subsystem to another when crossing any of the given "guards" specified as hyperplanes.

The paper introduces the state space variable for such a hybrid system and describes its reach sets through a branching process.

This description further allows to solve the verification problem, through an array of operations on ellipsoids intertwined with solution of some optimization problems for quadratic functions. The approach permits parallel calculation of an array of ellipsoidal approximations. Lastly, by increasing the number of elements in the array, one may approach the exact solution to any degree of accuracy.

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