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Reciprocals of Integrated Nodes.

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I. Introduction

We survey recent results solving the classical problems of linear optimal control and normalized coprime factorizations for a wide class of infinite-dimensional linear systems called *integrated nodes*. These include well-posed linear systems, but form a subclass of operator nodes that were studied in Staffans [16, Section 4.7]. These results were obtained using the reciprocal approach introduced in Curtain [4], [5] and illustrate its usefulness in solving system theoretic problems for systems with unbounded input and output operators.

As is customary, a system has a state space Z, an input space U and an output space Y; we assume that all three spaces are separable Hilbert spaces. An *operator node* is specified by three generating operators A, B, C and a characteristic function $\mathfrak{G}$. These are assumed to satisfy:

- A is a closed densely defined operator on Z with nonempty resolvent set.
- $C \in \mathcal{L}(D(A), Y)$ is bounded where $D(A)$ is equiped with the graph norm.
- $B^* \in \mathcal{L}(D(A^*), U)$ is bounded where $D(A^*)$ is equiped with the graph norm.
- $\mathfrak{G} : \rho(A) \to \mathcal{L}(U, Y)$ satisfies the following for $\alpha, s \in \rho(A)$

$$\mathfrak{G}(s) - \mathfrak{G}(\alpha) = (\alpha - s)C(sI - A)^{-1}(\alpha I - A)^{-1}B. \quad (1)$$

The *dual* of an operator node is specified by the operators A^*, C^*, B^*, $\mathfrak{G}(\bar{\alpha})^*$.

We remark that the above definition is purely algebraic and to incorporate a concept of dynamics extra assumptions are needed. There are several ways of doing this (for example as in [16]), but for our purposes it is convenient to follow the approach in Opmeer [14]), where he introduced the concept of an integrated node and gave many p.d.e. examples.

Definition 1.1: An *integrated node* is an operator node for which $\rho(A)$ contains a right half-plane and there exist $M > 0$ and $n \in \mathbb{Z}$ such that

$$\|(sI - A)^{-1}\| \leq M|s|^n.$$

The above resolvent estimate is equivalent to the statement: A generates an exponentially bounded integrated semigroup (see Arendt et al. [1, Section 3.2]).

Note that $(sI - A)^{-1}B$, $C(sI - A)^{-1}$ and the characteristic function of an integrated node are polynomially bounded on the same right half-plane as the resolvent.

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We recall that the Laplace transform can be defined for certain Banach space valued distributions and that the image of the set of Laplace transformable distributions is exactly the set of functions defined on some right half-plane that are analytic and polynomially bounded (see Schwartz [15]). This allows us to define the state and output of an integrated node as Laplace transformable distributions (Z-valued and Y-valued, respectively).

Definition 1.2: For an initial state $z_0 \in Z$ and a Laplace transformable distribution u the state and output of an integrated node are defined through their Laplace transforms on some right half-plane by

$$\hat{z}(s) = (sI - A)^{-1}z_0 + (sI - A)^{-1}B\hat{u}(s), \quad (2)$$
$$\hat{y}(s) = C(sI - A)^{-1}z_0 + \mathfrak{G}(s)\hat{u}(s). \quad (3)$$

So the above definition defines the dynamics of the system via the Laplace transforms, which is a neat way of avoiding the technical complications inherent in state space definitions.

The class of integrated nodes is significantly larger than the much studied class of well-posed linear systems (see [16]). In our approach we focus on the subclass of state linear systems that was studied in Curtain and Zwart [2]. $\Sigma(A, B, C, D)$ is a *state linear system* if A is the infinitesimal generator of a strongly continuous semigroup $T(\cdot)$ on Z, $B \in \mathcal{L}(U, Z), C \in \mathcal{L}(Z, Y), D \in \mathcal{L}(U, Y)$, i.e., B and C are *bounded* operators. Although this class is very small, it turns out that a detailed study of its system theoretic properties allows us to analyse those of the very general class of integrated nodes. This is a huge simplification, since the mathematics involved in manipulating bounded operators is much easier than that involved in unbounded operators. The key is the concept of the reciprocal system that has been studied recently in a series of papers by Curtain and Opmeer [4], [5], [6], [7]. Under the generic assumption that $0 \in \rho(A)$ an operator node can be related to a state linear system that is called its reciprocal system.

Definition 1.3: Suppose that the operator node Σ with generating operators A, B, C and characteristic function $\mathfrak{G}$ is such that $0 \in \rho(A)$. Its reciprocal system is the state linear system $\Sigma(A^{-1}, A^{-1}B, -CA^{-1}, \mathfrak{G}(0))$.

The power of this concept is that certain stability properties of an integrated node are equivalent to the corresponding stability properties of its reciprocal system which has bounded generating operators. These stability concepts need to be chosen with care. The most commonly used concept of stability, exponential stability ($||T(t)|| \leq Me^{-\alpha t}$), is not

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preserved. In fact, if $T(\cdot)$ is exponentially stable, $e^{A^{-1}t}$ will never be exponentially stable and $e^{A^{-1}t}$ need not even be uniformly bounded in norm in t. Although the strong stability ($T(t)z \to 0$ as $t \to \infty$) of A does imply the strong stability of A^{-1} in the case that $T(\cdot)$ is a contraction semigroup, this is not known in general. So the first step was to introduce new system-theoretic stability concepts that are inherited by the reciprocal system (see Section II). The second and hardest step was to generalize the classic system theory results from [2] for state linear systems using these weaker stability concepts. The results on the linear regulator problem are summarized in Section III, and those on coprime factorization in Section IV. The third step that is explained in Section V was to show the preservation of stability properties of the integrated node to corresponding ones of the reciprocal system. The last step was to translate the above mentioned system theoretic problems for an integrated node into equivalent ones for its reciprocal system. Since reciprocal systems are state linear systems, the solutions to these problems are given in Sections II – IV. In Section V we explain how these solutions can be translated back to solutions for the original system node.
The reciprocal approach has also been used to solve spectral factorization problems for well-posed linear systems (see Curtain and Sasane [8]) and to solve the problem of robust stabilization with respect to left-coprime factor perturbations for integrated nodes (c.f. Curtain [7]).

Our standing assumption that $0 \in \rho(A)$ can be relaxed to the assumption that $\rho(A) \cap i\mathbb{R}$ is not empty. It is also possible to eliminate all assumptions on the spectrum of A by relating an integrated node to a discrete-time system using a Cayley-type transform as in Opmeer [14]. However, the nice feature about the reciprocal approach is that it is a continuous-time system.

Finally, we remark that while the reciprocal approach is a powerful technique for many system theoretic problems, it is not a panacea. For example, it cannot be used to derive the complete theory of Riccati equations in Mikkola [12].

II. New stability concepts for state linear systems

In this section, we review the properties of state linear systems from [2], [5] and [6]. $\Sigma(A, B, C, D)$ is a state linear system if A is the infinitesimal generator of a strongly continuous semigroup $T(\cdot)$ on Z, $B \in \mathcal{L}(U, Z), C \in \mathcal{L}(Z, Y), D \in \mathcal{L}(U, Y)$, where Z, U, Y are separable Hilbert spaces.
For an input $u \in \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}^+; U)$ and initial state $z_0 \in Z$ the state $z(t) \in Z$ at time $t \in \mathbb{R}^+$ is defined by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)\, ds. \qquad (4)$$

If u is continuously differentiable and $z_0 \in D(A)$, then z as defined above is continuously differentiable and for each $t \in \mathbb{R}^+$ $z(t) \in D(A)$ and satisfies

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0. \qquad (5)$$

The output of the state linear system is defined by

$$y(t) = Cz(t) + Du(t). \qquad (6)$$

Definition 2.1: The *transfer function* $\mathbf{G}$ of the state linear system $\Sigma(A, B, C, D)$ is defined by: $\mathbf{G} - D$ equals the Laplace transform of $CT(t)B$ on some right half-plane.
The *characteristic function* $\mathfrak{G}$ is defined for all $s \in \rho(A)$ by

$$\mathfrak{G}(s) = D + C(sI - A)^{-1}B.$$

Instead of using the concept of exponential stability, we work with the following weaker stability concepts.

Definition 2.2:
1) A state linear system is input stable if there exists a constant $\beta > 0$ such that for all $u \in \mathbf{L}_2(0, \infty; U)$ $\| \int_0^\infty T(t)Bu(t)\, \mathrm{d}t\|^2 \leq \beta \int_0^\infty \|u(t)\|^2\, \mathrm{d}t$;
2) A state linear system is output stable if there exists a constant $\gamma > 0$ such that for all $z \in Z$ $\int_0^\infty \|CT(t)z\|^2\, \mathrm{d}t \leq \gamma \|z\|^2$;
3) A state linear system is input-output stable if the transfer function $\mathbf{G} \in \mathbf{H}_\infty(\mathcal{L}(U, Y))$.
4) A state linear system is a system-stable if it is input, output and input-output stable.

While the exponential stability of the semigroup implies system-stability, strong stability does not. Similar definitions can be found in Staffans [16], but the essential difference in our definition to previous definitions is we have made no stability assumptions on the semigroup at all and so it may have spectrum in $\mathbb{C}^+$. The concepts of input and output stability are equivalent to the boundedness of the following input and output maps.

Definition 2.3: The output map $\mathcal{C} : Z \to \mathbf{L}_2(0, \infty; Y)$ of an output stable state linear system $\Sigma(A, B, C, D)$ is defined by $(\mathcal{C}z)(t) := CT(t)z$ and the observability Gramian by $L_C := \mathcal{C}^*\mathcal{C}$.

The input map $\mathcal{B} : \mathbf{L}_2(0, \infty; U) \to Z$ of an input stable state linear system is defined by $\mathcal{B}u := \int_0^\infty T(s)Bu(s)\, ds$ and the controllability Gramian by $L_B := \mathcal{B}\mathcal{B}^*$.

The Laplace transforms of the input and output maps play an important role in the sequel.

Definition 2.4: For an output stable state linear system we define $\widehat{\mathcal{C}} : Z \to \mathbf{H}_2(Y)$ by $\widehat{\mathcal{C}}z := \widehat{\mathcal{C}z}$.
For an input stable state linear system we define $\widehat{\mathcal{B}}$ for $u \in U, z \in Z, s \in \mathbb{C}_0^+$ by $\langle \widehat{\mathcal{B}}(s)u, z\rangle := \langle u, \widehat{\mathcal{B}^*z}(\bar{s})\rangle$.

For an output stable system $\widehat{\mathcal{C}}$ is the analytic extension of $C(sI - A)^{-1}$ to $\mathbb{C}_0^+$ and a similar remark applies to $\widehat{\mathcal{B}}$. The relation between the transfer and characteristic function needs some clarification. While we always have that $\mathbf{G} = \mathfrak{G}$ on some right-half plane, it is known that outside this region, they may differ (see Curtain and Zwart [2, Example 4.3.8] for a counter example).

Lemma 2.5: 1) If the state linear system $\Sigma(A,B,C,D)$ is output stable, then

$$\mathbf{G}(s) = D + \widehat{\mathcal{C}}(s)B \quad \forall s \in \mathbb{C}_0^+ \qquad (7)$$

$$\mathbf{G}(s) = D + C(sI-A)^{-1}B \qquad (8)$$
$$= \mathfrak{G}(s) \quad \forall s \in \mathbb{C}_0^+ \cap \rho(A).$$

2) If the state linear system $\Sigma(A,B,C,D)$ is input stable, then (9) holds and

$$\mathbf{G}(s) = D + C\widehat{\mathcal{B}}(s) \quad \forall s \in \mathbb{C}_0^+. \qquad (9)$$

3) If $\Sigma(A,B,C,D)$ is input or output stable, and either $\sigma(A) \cap i\mathbb{R}$ has measure zero, or U,Y are finite-dimensional, then there exists an almost everywhere defined function $\mathbf{G}_0 : i\mathbb{R} \to \mathcal{L}(U,Y)$ such that for almost all $\omega \in \mathbb{R}$ and all nontangential paths we have

$$\mathbf{G}_0(i\omega) = \lim_{s \to i\omega} \mathbf{G}(s).$$

Moreover, for $i\omega \in i\mathbb{R} \cap \rho(A)$ we have $\mathbf{G}_0(i\omega) = \mathfrak{G}(i\omega)$.

The properties input and output stability are related to the existence of solutions to Lyapunov equations (see Grabowski [10]) and the following lemma vindicates the choice of the concept of output stability.

Lemma 2.6: The state linear system $\Sigma(A,B,C,D)$ is input stable if and only if the following controllability Lyapunov equation has a bounded nonnegative solution $L \in \mathcal{L}(Z)$:

$$ALz + LA^*z = -BB^*z \quad \text{for all } z \in D(A^*). \qquad (10)$$

In this case, the controllability Gramian L_B is the smallest bounded nonnegative solution of (10) and $L_B^{1/2}T(t)^*z \to 0$ as $t \to \infty$ for all $z \in Z$.
The state linear system $\Sigma(A,B,C,D)$ is output stable if and only if the following observability Lyapunov equation has a bounded nonnegative solution $L \in \mathcal{L}(Z)$:

$$A^*Lz + LAz = -C^*Cz \quad \text{for all } z \in D(A). \qquad (11)$$

In this case, the observability Gramian L_C is the smallest bounded nonnegative solution of (11) and $L_C^{1/2}T(t)z \to 0$ as $t \to \infty$ for all $z \in Z$.

III. The linear regulator problem for state linear systems

The linear regulator problem is a building block in systems theory. For the state linear system $\Sigma(A,B,C,D)$ we consider the optimal control problem

$$\min_{u \in \mathbf{L}_2(0,\infty;U)} \int_0^\infty \| y(t) \|^2 + \| u(t) \|^2 \, dt,$$

where y is defined by (4), (6).

We say that $\Sigma(A,B,C,D)$ satisfies the *finite cost condition* if for all initial states $z_0 \in Z$ there exists an input $u \in \mathbf{L}_2(\mathbb{R}^+;U)$ such that the output $y \in \mathbf{L}_2(\mathbb{R}^+;Y)$. It is well-known that, under this assumption, for each $z_0 \in Z$ there exists a unique $u^{\rm opt} \in \mathbf{L}_2(\mathbb{R}^+;U)$ for which the minimum is attained. Moreover, there exists a bounded nonnegative operator $Q^{\rm opt}$ such that the minimal cost is given by $\langle Q^{\rm opt}z_0, z_0\rangle$. The optimal input is a state feedback: $u^{\rm opt}(t) = F^{\rm opt}z^{\rm opt}(t)$, where $F^{\rm opt} := -S^{-1}(D^*C + B^*Q^{\rm opt})$, $S = I + D^*D$.
$Q^{\rm opt}$ is the smallest bounded nonnegative solution to the *control algebraic Riccati* equation on $D(A)$

$$A^*Q + QA + C^*C = (QB + C^*D)S^{-1}(B^*Q + D^*C). \qquad (12)$$

Here we are interested in the properties of this control Riccati equation and its dual.

Theorem 3.1: If the state linear system $\Sigma(A,B,C,D)$ satisfies the finite cost condition, then there exists a bounded nonnegative solution Q of the control Riccati equation (12). Moreover, the right-factor system

$$\Sigma(A_Q, BS^{-1/2}, [C+DF;F], [D;I]S^{-1/2}), \qquad (13)$$

where $A_Q = A + BF$, $F = -S^{-1}(D^*C + B^*Q)$, is output and input-output stable. If, in addition, the dual system $\Sigma(A^*,C^*,B^*,D^*)$ satisfies the finite cost condition, then the right-factor system is system-stable.
If $\Sigma(A^*,C^*,B^*,D^*)$ satisfies the finite cost condition, then there exists a bounded nonnegative solution to the filter Riccati equation on $D(A^*)$

$$AP + PA^* + BB^* = (PC^* + BD^*)R^{-1}(CP + DB^*), \qquad (14)$$

where $R = I + DD^*$ and $L = -(PC^* + BD^*)R^{-1}$. Moreover, the left-factor system

$$\Sigma(A_P, R^{-1/2}[B+LD, L], C, R^{-1/2}[D,I]), \qquad (15)$$

where $A_P = A + LC$, is input and input-output stable. If, in addition,
$\Sigma(A,B,C,D)$ satisfies the finite cost condition, then the left-factor system is system-stable.

The *optimal right-factor system* (with $Q = Q^{\rm opt}$) has a special property.

Theorem 3.2: Suppose that the state linear system $\Sigma(A,B,C,D)$ satisfies the finite cost condition, and let Q^{opt} denote the smallest bounded nonnegative solution to the control Riccati equation (12). Then the optimal right factor system

$$\Sigma(A_{Q^{opt}}, BS^{-1/2}, [C+DF^{opt};F^{opt}], [D;I]S^{-1/2}) \qquad (16)$$

corresponding to Q^{opt} has an inner transfer function.

Dually, the optimal left-factor system

$$\Sigma(A_{P^{opt}}, R^{-1/2}[B+L^{opt}D, L^{opt}], C, R^{-1/2}[D,I]) \qquad (17)$$

corresponding to P^{opt}, the smallest bounded nonnegative solution to the filter Riccati equation (14), has a co-inner transfer function.

It is clear that the control Riccati equation is the same as the observability Lyapunov equation of the right-factor system

(12). Its controllability Lyapunov equation has solutions connected to those of the two Riccati equations.

Theorem 3.3: Suppose that the state linear system $\Sigma(A,B,C,D)$ and its dual satisfy the finite cost condition and P,Q are arbitrary bounded nonnegative solutions of the Riccati equations (14), (12), respectively.

1) The controllability and observability Lyapunov equations of the right-factor system (13) have solutions $L_1=(I+PQ)^{-1}P$ and $L_2=Q$, respectively.
2) The observability gramian L_C of the right-factor system (13) is Q^{opt}, the smallest bounded nonnegative solution of the control Riccati equation (12).
3) The controllability gramian L_B of right-factor system (13) is $P^{opt}(I+QP^{opt})^{-1}$, where P^{opt} is the smallest bounded nonnegative solution of the filter Riccati equation (14).
4) $r(L_BL_C) \le r(L_1L_2) < 1$, where r denotes the spectral radius.

IV. Coprime factorizations for state linear systems

In this section we collect various properties of the transfer function $[\mathbf{N}^Q;\mathbf{M}^Q]$ of the right factor (13) and of the transfer function $[\mathbf{N}^{opt};\mathbf{M}^{opt}]$ of the optimal right factor (16). First we recall some definitions of coprimeness.

Definition 4.1: $[\mathbf{N};\mathbf{M}] \in \mathbf{H}_\infty(\mathcal{L}(U,Y\oplus U))$ is right coprime if there exist $\tilde{\mathbf{X}},\tilde{\mathbf{Y}}$ such that $[\tilde{\mathbf{X}},\tilde{\mathbf{Y}}]\in\mathbf{H}_\infty(\mathcal{L}(U\oplus Y,U))$ and for all $s\in\mathbb{C}_0^+$ there holds

$$\tilde{\mathbf{X}}(s)\mathbf{M}(s)-\tilde{\mathbf{Y}}(s)\mathbf{N}(s)=I. \tag{18}$$

$[\tilde{\mathbf{N}},\tilde{\mathbf{M}}]\in\mathbf{H}_\infty(\mathcal{L}(Y\oplus U,Y))$ is left coprime over $\mathbb{C}_0^+$ if there exist $\mathbf{X},\mathbf{Y}$ such that $[\mathbf{X};\mathbf{Y}]\in\mathbf{H}_\infty(\mathcal{L}(Y,Y\oplus U))$ and for all $s\in\mathbb{C}_0^+$ there holds

$$\tilde{\mathbf{M}}(s)\mathbf{X}(s)-\tilde{\mathbf{N}}(s)\mathbf{Y}(s)=I. \tag{19}$$

Of particular interest are normalized pairs.

Definition 4.2: We call $[\mathbf{N};\mathbf{M}] \in \mathbf{H}_\infty(\mathcal{L}(U,Y\oplus U)$ normalized if it is inner, i.e., for almost all $\omega\in\mathbb{R}$ the following holds

$$\mathbf{M}(i\omega)^*\mathbf{M}(i\omega)+\mathbf{N}(i\omega)^*\mathbf{N}(i\omega)=I. \tag{20}$$

We call $[\tilde{\mathbf{N}},\tilde{\mathbf{M}}]\in\mathbf{H}_\infty(\mathcal{L}(Y\oplus U,Y)$ normalized if it is co-inner, i.e., for almost all $\omega\in\mathbb{R}$ the following holds

$$\tilde{\mathbf{M}}(i\omega)\tilde{\mathbf{M}}(i\omega)^*+\tilde{\mathbf{N}}(i\omega)\tilde{\mathbf{N}}(i\omega)^*=I. \tag{21}$$

We now define coprime factorizations of a function $\mathbf{G}$.

Definition 4.3: The function $\mathbf{G}$ has a right-coprime factorization if there exist $[\mathbf{N};\mathbf{M}]\in\mathbf{H}_\infty(\mathcal{L}(U,Y\oplus U)$ that are right coprime, $\mathbf{M}$ has an inverse on some right half-plane and $\mathbf{G}(s)=\mathbf{N}(s)\mathbf{M}(s)^{-1}$ on some right half-plane.
It has a left-coprime factorization if there exist $[\tilde{\mathbf{N}};\tilde{\mathbf{M}}]\in\mathbf{H}_\infty(\mathcal{L}(Y\oplus U,Y)$ that are left coprime, $\tilde{\mathbf{M}}$ has an inverse on some right half-plane and $\mathbf{G}(s)=\tilde{\mathbf{M}}(s)^{-1}\tilde{\mathbf{N}}(s)$ on some right half-plane.

We remark that some authors require that the inverses of $\mathbf{M}$ and of $\tilde{\mathbf{M}}$ are well-posed, i.e., bounded in norm on some right-half plane. In fact, if $\mathbf{G}$ is well-posed and it has a right-coprime factorization according to our definition, then from (18) we obtain $\mathbf{M}^{-1}=\tilde{\mathbf{X}}-\tilde{\mathbf{Y}}\mathbf{G}$, which is well-posed.

Definition 4.4: A transfer function has a doubly coprime factorization if it has a right-coprime factorization $[\mathbf{N};\mathbf{M}] \in \mathbf{H}_\infty(\mathcal{L}(Y\oplus U,U)$ with Bezout factor $[\tilde{\mathbf{X}},\tilde{\mathbf{Y}}]\in\mathbf{H}_\infty(\mathcal{L}(U\oplus Y,U))$ and a left-coprime factorization $[\tilde{\mathbf{N}},\tilde{\mathbf{M}}]\in\mathbf{H}_\infty(\mathcal{L}(Y\oplus U,Y)$, with Bezout factor $[\mathbf{X};\mathbf{Y}]\in\mathbf{H}_\infty(\mathcal{L}(Y,Y\oplus U))$ such that

$$\begin{bmatrix}\tilde{\mathbf{X}} & -\tilde{\mathbf{Y}}\\ -\tilde{\mathbf{N}} & \tilde{\mathbf{M}}\end{bmatrix}^{-1}=\begin{bmatrix}\mathbf{M} & \mathbf{Y}\\ \mathbf{N} & \mathbf{X}\end{bmatrix} \tag{22}$$

holds on $\mathbb{C}_0^+$.

The obvious candidates for normalized right- (left-) coprime factors are the transfer functions of the right (left) factor system (13), respectively (15). While they are always in $\mathbf{H}_\infty$, they are not necessarily normalized and the coprime property is not that obvious. The following properties do follow fairly easily from the Riccati equation theory in Section III.

Theorem 4.5: If $\Sigma(A,B,C,D)$ is output stabilizable with transfer function $\mathbf{G}$, then

1) $[\mathbf{N}^Q;\mathbf{M}^Q]\in\mathbf{H}^\infty(U,Y\oplus U)$.
2) $\mathbf{M}^Q$ is invertible on some right half-plane and its inverse is the transfer function of the state linear system $\Sigma(A,B,-S^{1/2}F,S^{1/2})$.
3) on some right half-plane there holds $\mathbf{G}=\mathbf{N}^Q{\mathbf{M}^Q}^{-1}$.
4) $[\mathbf{N}^{opt};\mathbf{M}^{opt}]$ is a normalized factorization of $\mathbf{G}$.
5) If the spectrum of A on the imaginary axis has measure zero, then $[\mathbf{N}^Q;\mathbf{M}^Q]$ is a normalized factorization of $\mathbf{G}$.

The following condition for coprimeness is important.

Lemma 4.6: Let $\mathbf{G}$ be a transfer function with a normalized right factorization $[\mathbf{N};\mathbf{M}]$. Then it is coprime if and only if the Hankel norm of $[\mathbf{N};\mathbf{M}]$ is strictly less than 1.

Combined with Theorem 3.3 , it leads to the main existence result on coprime factorizations (see [6]).

Theorem 4.7: If the state linear system $\Sigma(A,B,C,D)$ and its dual satisfy the finite cost condition, then its transfer function has a normalized right-coprime factorization given by $[\mathbf{N}^{opt};\mathbf{M}^{opt}]$, the transfer function of the optimal right factor system (16). The transfer function $[\mathbf{N}^Q;\mathbf{M}^Q]$ of an arbitrary right factor system (13) is also a right-coprime factorization. If $\sigma(A)\cap i\mathbb{R}$ has measure zero, then it is also normalized.

This theorem has an obvious dual and so we deduce the existence of a doubly coprime factorization for an input and output stabilizable system. However, in the Youla-Bongiorno parameterization of all stabilizing controllers one needs explicit formulas for the Bezout factors. These were obtained in Curtain and Opmeer [6] by solving the Nehari problem for $[-\mathbf{N}^Q,\mathbf{M}^Q]$.

Theorem 4.8: Suppose that the state linear system $\Sigma(A, B, C, D)$ and its dual satisfy the finite cost condition and σ is any number satisfying $1 > \sigma > r^{1/2}(PQ(I + PQ)^{-1}$.

1) If either $\sigma(A) \cap i\mathbb{R}$ has measure zero or $Q = Q^{opt}$, the smallest nonnegative solution to the control Riccati equation, then the transfer function $[\mathbf{N}^Q; \mathbf{M}^Q]$ is a normalized right-coprime factorization of $\mathbf{G}$ with Bezout factors $[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]$, the transfer function of the system-stable state linear system $\Sigma(\tilde{A}_L, [B + \tilde{L}D, \tilde{L}], S^{-1/2}B^*Q, S^{-1/2}[I, -D^*])$, where $\tilde{A}_L = A + \tilde{L}C$, $\tilde{L} = -(\sigma^{-2}\tilde{W}PC^* + BD^*)R^{-1}$ and $\tilde{W} = (I + PQ - \sigma^{-2}PQ)^{-1}$.
2) If either $\sigma(A) \cap i\mathbb{R}$ has measure zero or $P = P^{opt}$, the smallest nonnegative solution to the filter Riccati equation, then the transfer function $[\tilde{\mathbf{N}}^P, \tilde{\mathbf{M}}^P]$ is a normalized left-coprime factorization of $\mathbf{G}$ with Bezout factors $[\mathbf{X}; \mathbf{Y}]$, the transfer function of the system-stable state linear system $\Sigma(\tilde{A}_F, PC^*R^{-1/2}, [C + D\tilde{F}; \tilde{F}], [I, -D^*]R^{-1/2})$, where $\tilde{A}_F = A + B\tilde{F}$ and $\tilde{F} = -S^{-1}(\sigma^{-2}B^*Q\tilde{W} + D^*C)$.

Remark 4.9: If A_Q generates an exponentially stable semigroup, then with $\sigma = 1$ the Bezout factors are in $\mathbf{H}_\infty$, and they reduce to the usual finite-dimensional formulas. But, in general, we only know that the candidate Bezout factors are in $\mathbf{H}_2$ (modulo a constant).

Remark 4.10: If $[\mathbf{N}^Q; \mathbf{M}^Q]$ is not normalized, then it is a right-coprime factorization, but in general, we do not know if the candidate Bezout factors from Theorem 4.8 will be in $\mathbf{H}_\infty$. Although we can always deduce suitable Bezout factors from those for the optimal right factor, this leads to messy formulas.

Remark 4.11: In Curtain [7] an interesting interpretion to the Bezout factors is given in Theorem 4.8. The controller $\mathbf{K} = \mathbf{Y}\mathbf{X}^{-1}$ stabilizes $\mathbf{G}$ robustly with respect to left-coprime factor perturbations with robustness margin $\sqrt{1-\sigma^2}$ (see Glover and MacFarlane [9]). The stability referred to here is in the input-output sense: $(I - \mathbf{KG})^{-1}, \mathbf{G}(I - \mathbf{KG})^{-1}, (I - \mathbf{KG})^{-1}\mathbf{K}$ and $(I - \mathbf{GK})^{-1}$ are all in $\mathbf{H}_\infty$.

V. Integrated nodes and reciprocals

First we introduce some stability concepts for integrated nodes in terms of the frequency domain maps (2), (3) from Section I.

Definition 5.1: An integrated node is *output stable* if $C(sI - A)^{-1}z$ has an analytic extension to a function $\hat{\mathcal{C}}z \in \mathbf{H}_2(Y)$ for all $z \in Z$;
input-output stable if $\mathfrak{G}$ has an analytic extension to a transfer function $\mathbf{G} \in \mathbf{H}_\infty(\mathcal{L}((U, Y))$;
input stable if its dual node is output stable.
An integrated node is called *system-stable* if it is input, output and input-output stable.

Remark 5.2: In finite-dimensional systems theory, the concepts of characteristic function and transfer function coincide. As remarked in Section 1, for infinite-dimensional systems, this is not always the case.

Remark 5.3: Of course, through the Paley-Wiener theorem Definition 5.1 is equivalent to our earlier Definition 2.2 of stability for state linear systems.
In Grabowski [10] it is shown that in the case that A is the generator of a strongly continuous semigroup, output stability is equivalent to the existence of a bounded nonnegative solution $L_C \in \mathcal{L}(Z)$ to the observation Lyapunov equation (11)

$$A^*L_C z + L_C A z = -C^*Cz \quad \text{for all } z \in D(A).$$

Moreover, the observability gramian $L_C = \mathcal{C}^*\mathcal{C}$ is the smallest nonnegative solution. The key step in the developing the concept of a reciprocal system is to notice that if $0 \in \rho(A)$, then (11) has a solution if and only if the following Lyapunov equation does

$$A^{-*}L + LA^{-1} = -A^{-*}C^*CA^{-1}.$$

This is the observability Lyapunov equation for the pair of bounded operators A^{-1}, CA^{-1}. Similarly, the control Lyapunov equation for the infinite-time admissible B operator has a solution if and only if the control Lyapunov equation for the pair of bounded operators $A^{-1}, A^{-1}B$ has. Next, we substitute $\beta = 0$ in (1) and obtain

$$\begin{aligned}\mathfrak{G}(s) &= \mathfrak{G}(0) + sC(sI - A)^{-1}A^{-1}B \\ &= \mathfrak{G}(0) - CA^{-1}(\frac{1}{s}I - A^{-1})^{-1}A^{-1}B,\end{aligned}$$

which is the characteristic function of the state linear system $\Sigma(A^{-1}, A^{-1}B, -CA^{-1}, \mathfrak{G}(0))$. This is the motivation behind the Definition 1.3 of a reciprocal system that was introduced in [4] for well-posed linear systems. It applies equally well to operator nodes.
The power of this concept is that the stability properties of an operator node are equivalent to the stability properties of a state linear system with bounded generating operators.

Theorem 5.4: Suppose that A, B, C are the generating operators of an integrated node Σ with transfer function $\mathbf{G}$ and zero is in the resolvent set of A. Denote the characteristic function of its reciprocal system by $\mathfrak{G}_r$ and the transfer function of its reciprocal system by $\mathbf{G}_r$. Then

1) $\mathfrak{G}(s) = \mathfrak{G}_r(\frac{1}{s})$ whenever s is in the resolvent set of A.
2) Σ is output stable if and only if its reciprocal system is output stable.
3) Σ is input stable if and only if its reciprocal system is input stable.
4) The integrated node is a system-stable if and only if its reciprocal system is a system-stable. In this case, we have $\mathbf{G}(s) = \mathbf{G}_r(\frac{1}{s})$ for $s \in \mathbb{C}_0^+$.

Moreover, the *regulator problem* for an integrated node is equivalent to the regulator problem for its reciprocal system. The *finite cost condition* for an integrated node is defined naturally in frequency domain terms: for every $z_0 \in Z$ there exists a $\hat{u} \in \mathbf{H}_2(U)$ such that $\hat{y} \in \mathbf{H}_2(Y)$, where $\hat{u}, \hat{y}$ are

defined by (2), (3).
This is equivalent to the time-domain definition given in Section III by the Paley-Wiener theorem.

The following theorem was stated in [13] for well-posed linear systems, but its proof applies to integrated nodes as well.

Theorem 5.5: An integrated node Σ with $0 \in \rho(A)$ satisfies the finite-cost condition if and only if its reciprocal system does. Moreover, if the finite-cost condition is satisfied, then there exist unique optimal controls for Σ and its reciprocal system and the optimal costs are equal.

Since the reciprocal system is a state linear system, the results stated earlier in this article are applicable to it. First we consider the regulator problem. If we apply Theorem 3.1 to the reciprocal system, we obtain the optimal cost operator for its regulator problem as the smallest bounded nonnegative solution to the following reciprocal control Riccati equation.

$$A^{-*}Q + QA^{-1} + A^{-*}C^*CA^{-1} = (QA^{-1}B - A^{-*}C^*D_r)S_r^{-1}(B^*A^{-*}Q - D_r^*CA^{-1}),$$

where $D_r = \mathfrak{G}(0)$ and $S_r = I + D_r^*D_r$. According the Theorem 5.5, the optimal cost operator for the regulator problem for an integrated node is the smallest bounded nonnegative bounded solution to the above reciprocal control Riccati equation. This is in stark contrast to the theory for the smaller class of well-posed linear systems, where one obtains very abstract Riccati equations with odd additional terms (see Mikkola [12]). The reciprocal Riccati equation has bounded operators and is easy to work with. The dual filter Riccati equation is

$$A^{-1}P + PA^{-*} + A^{-1}BB^*A^{-*} = (PA^{-*}C^* - A^{-1}BD_r^*)R_r^{-1}(CA^{-1}P - D_rB^*A^{-*}),$$

where $R_r = I + D_rD_r^*$.
Next we consider the problem of *coprime factorizations* for an integrated node.

Theorem 5.6: Let Σ be an integrated node with $0 \in \rho(A)$. Assume that the finite-cost condition for Σ and for its dual system are both satisfied. Then the characteristic function of Σ has a normalized doubly coprime factorization.

Proof: It follows from Theorem 5.5 that the reciprocal system Σ_r of Σ and its dual satisfy the finite-cost condition. So we can apply Theorems 4.7 and 4.8 to the reciprocal system Σ_r to show that its transfer function has a normalized doubly coprime factorization. Denote the coprime factors and Bezout factors by $\mathbf{M}_r$, $\mathbf{N}_r$, $\tilde{\mathbf{M}}_r$, $\tilde{\mathbf{M}}_r$, $\mathbf{X}_r$, $\mathbf{Y}_r$, $\tilde{\mathbf{X}}_r$, $\tilde{\mathbf{Y}}_r$ and define $\mathbf{M}(s) = \mathbf{M}_r(1/s)$, $\mathbf{N}(s) = \mathbf{N}_r(1/s)$, $\tilde{\mathbf{M}}(s) = \tilde{\mathbf{M}}_r(1/s)$, $\tilde{\mathbf{M}}(s) = \tilde{\mathbf{M}}_r(1/s)$, $\mathbf{X}(s) = \mathbf{X}_r(1/s)$, $\mathbf{Y}(s) = \mathbf{Y}_r(1/s)$, $\tilde{\mathbf{X}}(s) = \tilde{\mathbf{X}}_r(1/s)$, $\tilde{\mathbf{Y}}(s) = \tilde{\mathbf{Y}}_r(1/s)$. Note that $\mathbf{Q}_r \in \mathbf{H}_\infty$ if and only if $\mathbf{Q} \in \mathbf{H}_\infty$. Moreover, $[\mathbf{M}_r; \mathbf{N}_r]$ is normalized if and only if $[\mathbf{M}; \mathbf{N}]$ is normalized. So it follows that $[\mathbf{M}; \mathbf{N}]$ provides a normalized doubly coprime factorization of the characteristic function of Σ. ■

Remark 5.7: From the proof of Theorem 5.6 and Theorem 4.8 one can deduce explicit formulas for the coprime factors and Bezout factors. These are in terms of the generating operators of the reciprocal system. The leads to a complete Youla-Bongiorno type parameterization of all stabilizing controllers (c.f. Curtain, Weiss and Weiss [3]).

Remark 5.8: We remark that the solutions to the problems for integrated systems have been given in terms of the generating operators of the reciprocal systems. For some systems it is possible to give formulas for the solutions to the problems considered here which look exactly like the finite-dimensional ones. We have seen that this is the case for state linear systems and it will also be true for well-posed linear systems which are exponentially stabilizable and detectable and for which B and C operators are not too unbounded (c.f. [17], [3]). However, in general, these formulas will not make any sense. On the other hand, the reciprocal formulas we have derived are comprised of bounded operators that are always well-defined.

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