Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

A High Gain Observer for a Class of Implicit Systems

Abdellatif El Assoudi
LCPI, Departement GE, ENSEM
Univ. Hassan II Ain Chock
B.P 8118, Oasis
Casablanca Morocco
Email: a.elassoudi@ensem-uh2c.ac.ma

El hassane El Yaagoubi
LCPI, Departement GE, ENSEM
Univ. Hassan II Ain Chock
B.P 8118, Oasis
Casablanca Morocco
Email: e.elyaagoubi@ensem-uh2c.ac.ma

Hassan Hammouri
LAGEP, UMR CNRS 5007
Univ. Lyon I - ESCPE-LYON, Bat. 308G
43 Bd du 11 novembre 1918
69622 Villeurbanne Cedex, France
Email: hammouri@lagep.cpe.fr

Abstract— **The high gain observer for dynamical systems described by ordinary differential equations is widely discussed in the literature, see for instance [1], [2], [3], [4],[5], [6], [7], [8], [9], [10], [11], [12]. The aim of this paper is to extend this observer design to a class of differential-algebraic systems. In practice, the computation of solutions of differential-algebraic equations requires the combination of an ordinary differential equations (O.D.E.) routine together with an optimization algorithm. Therefore, a natural way permitting to estimate the state of such a system is to design a procedure based on a similar numerical algorithm. Beside some numerical difficulties, the drawback of such a method lies in the fact that it is not easy to establish a rigorous proof of the convergence of the observer. The main result of this paper is stated in section 3. It consists in showing that the state estimation problem for a class of differential-algebraic systems can be achieved by using an observer having an O.D.E. structure on some $\mathbb{R}^N$.**

Keywords: Nonlinear system, implicit system, high gain observer.

I. Introduction and Problem statement

In this paper, the following class of implicit systems is considered:

$$\left\{\begin{array}{l} \dot{x} \;\;=\;\; f_0(x,z) + \sum_{i=1}^{m} u_i f_i(x,z) \\ \varphi(x,z) = 0 \\ y \;\;=\;\; h(x,z) \end{array}\right. \tag{1}$$

where $y \in \mathbb{R}^p$, $u = (u_1, \dots, u_m) \in \mathbb{R}^m$, $(x,z) \in \mathbb{R}^n \times \mathbb{R}^d$, the f_i's, h and $\varphi = (\varphi_1, \dots, \varphi_d)^T$ are assumed to be sufficiently smooth and:

$$\left.\frac{\partial \varphi}{\partial z}\right|_{x,z} \text{ is of full rank } \forall (x,z) \in \mathcal{M} \tag{2}$$

where $\mathcal{M}$ is the set of zeros of φ:

$$\mathcal{M} = \left\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^d, \text{ s.t. } \varphi(x,z) = 0\right\} \tag{3}$$

Remark 1: Condition (2) implies:

i) the local uniqueness of solutions z of $\varphi(x,z) = 0$, for every x.

ii) $\mathcal{M}$ is a smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^d$.

In the case where the solution z of $\varphi(x,z) = 0$ can be explicitly expressed as $z = \psi(x)$, system (1) becomes a system of O.D.E. Hence, an observer can be formulated as a system of O.D.E. Otherwise, one may ask if there exists an observer that can be described by ordinary differential equations. In the sequel, we will use the following definition.

Definition 1: A **non initialized (resp. an initialized) exponential observer** for system (1), with input u and output y, is a dynamical system of the form:

$$\left\{\begin{array}{l} \dot{\omega} = \Gamma(\omega, u, y) \\ \omega(0) \in \Omega \subset \mathbb{R}^N \end{array}\right. \tag{4}$$

for which there exists a map $\Xi = (\Xi_1, \Xi_2)$ from $\mathbb{R}^N$ into $\mathbb{R}^n \times \mathbb{R}^d$ such that $\|\Xi_1(\omega(t)) - x(t)\|$ together with $\|\Xi_2(\omega(t)) - z(t)\|$ exponentially converge to 0, as $t \to \infty$, where Ω is such that $\Xi(\Omega)$ contains an open set containing $\mathcal{M}$ (resp. $\Xi(\Omega) = \mathcal{M}$).

Noticing that in practice, an initialized observer only works if the measurements are not noisy, and that the initial state of the observer satisfies the constraint $\varphi(\Xi(\omega)) = 0$.

This paper is organized as follows: In section 2, we will give an initialized high gain observer. This observer construction is based on a triangular structure containing this proposed in [8], [12]. In section 3, we robustify the above observer in order to obtain a non initialized one.

II. Initialized high gain observer based on a triangular structure

Given a nonlinear system:

$$\left\{\begin{array}{l} \dot{\zeta} = F(\zeta, u) = F_0(\zeta) + \sum_{i=1}^{m} u_i F_i(\zeta) \\ y = H(\zeta) \end{array}\right. \tag{5}$$

where the input $u \in \mathbb{R}^m$, the state $\zeta \in \mathcal{N}$ a smooth manifold of dimension n, the output $y \in \mathbb{R}^p$, F is a smooth vector field with respect to these arguments.

Noticing that the class of systems (1) forms a particular class of (5). Indeed, solutions $(x(t), z(t))$ of system (1) are identical to those of the system :

$$\left\{\begin{array}{l}\dot{\zeta} = F(\zeta, u)\\ y = H(\zeta) = h(x,z)\\ \zeta = \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{M}\end{array}\right. \tag{6}$$

where,

$$F(\zeta, u) = \begin{pmatrix} F^1(\zeta, u) \\ F^2(\zeta, u) \end{pmatrix} \tag{7}$$

with

$F^1(\zeta, u) = f_0(x, z) + \sum_{i=1}^{m} u_i f_i(x, z)$

$F^2(\zeta, u) = -(\frac{\partial \varphi}{\partial z} \mid_{x,z})^{-1}(\frac{\partial \varphi}{\partial x} \mid_{x,z})[f_0(x, z) + \sum_{i=1}^{m} u_i f_i(x, z)]$

Based on a triangular structure, in this section we will give a sufficient condition which allows to design an initialized high gain observer for system (5).

Recall that system (5) is said to be **uniformly observable** (see [8]) if for every initial state $\zeta(0) \neq \zeta'(0)$ and every input u defined on any interval $[0, T]$; there exist $t \in [0, T]$, such that $H(\zeta(t)) \neq H(\zeta'(t))$. For the single output systems, the authors in [8], [9] have shown that uniformly observable systems can be characterized by a canonical form. This canonical form is next used to design a high gain observer. Many extensions to multi-output systems are established in the literature (see for instance [2], [5], [6], [7], [10], [12]).

In this section, we restrict ourselves to the class of system (5) that can be transformed by a diffeomorphism to the following triangular structure:

$$\left\{\begin{array}{lcl}\dot{\xi}^1 & = & A_1\xi^1 + \psi^1(\xi, u)\\ \vdots & & \\ \dot{\xi}^j & = & A_j\xi^j + \psi^j(\xi, u)\\ \vdots & & \\ \dot{\xi}^p & = & A_p\xi^p + \psi^p(\xi, u)\\ y & = & C\xi = \left[C_1\xi^1, \dots, C_p\xi^p\right]^T\end{array}\right. \tag{8}$$

where

$$\xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^p \end{pmatrix};\ \xi^j = \begin{pmatrix} \xi^j_1 \\ \vdots \\ \xi^j_{n_j} \end{pmatrix} \in \mathbb{R}^{n_j},\ where\ n_j \geq 2$$

$\psi^j(\xi, u) = \tilde{\psi}^j(\xi) + \bar{\psi}^j(\xi)u$, with

$$\tilde{\psi}^j(\xi) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{\psi}^j_{n_j}(\xi) \end{pmatrix}.$$

and the $n_j \times m$ matrices $\bar{\psi}^j(\xi)$ satisfy the following structure:

$$\text{For}\ \ 1 \leq j \leq p;\ \ 1 \leq i \leq n_j - 1,\ \ \bar{\psi}^j_i = \bar{\psi}^j_i(\xi^j_1, \cdots, \xi^j_i) \tag{9}$$

Otherwise, $\bar{\psi}^j_{n_j}$ may depend on all components of ξ.

Finally,

$$A_j = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ \vdots & & & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix};\ C_j = (1\ 0\ \dots\ 0) \tag{10}$$

A geometric condition permitting to characterize control affine system (5) that can be transformed into the above triangular structure is as follows:

F_i, $i = 0, 1, \cdots, m$, are the vector fields of system (5), there exist p integers $n_1 \geq 2$, $\cdots$, $n_p \geq 2$ such that:

1) $\zeta \xrightarrow{\Phi} \begin{pmatrix} H_1(\zeta) \\ \vdots \\ L^{n_1-1}_{F_0}(H_1)(\zeta) \\ \vdots \\ H_p(\zeta) \\ \vdots \\ L^{n_p-1}_{F_0}(H_p)(\zeta) \end{pmatrix}$ is a diffeomorphism from $\mathcal{N}$ into $\Phi(\mathcal{N})$.

2) $\left\{\begin{array}{l} For 1 \leq i \leq n_j - 1; 1 \leq j \leq m; 1 \leq l \leq p \\ dL_{F_j}(L^{i-1}_{F_0}(H_l)) \wedge dH_l \ldots \wedge dL^{i-1}_{F_0}(H_l) \ \ = \ \ 0 \end{array}\right.$

where $\wedge$ denotes the exterior product of differential forms.

Claim 1: Under the conditions 1), 2), the map Φ transforms system (5) into the triangular structure (8)-(9)-(10). The proof of the claim is straightforward.

A hight gain observer based on the above triangular structure can be synthesized as follows:

Assume that the ψ^j's are global Lipschitz w.r.t. ξ (ie $\|\psi^j(\xi, u) - \psi^j(\bar{\xi}, u)\| \leq c\|\xi - \bar{\xi}\|$, for some positive constant c), then an exponential observer for system (8) takes the form:

$$\left\{\begin{array}{lcl}\dot{\hat{\xi}}^1 & = & A_1\hat{\xi}^1 + \psi^1(\hat{\xi}, u) + \Delta^1_\theta K_1(C_1\hat{\xi}^1 - y_1)\\ \vdots & & \\ \dot{\hat{\xi}}^p & = & A_p\hat{\xi}^p + \psi^p(\hat{\xi}, u) + \Delta^p_\theta K_p(C_p\hat{\xi}^p - y_p)\end{array}\right. \tag{11}$$

where $\Delta^j_\theta = \begin{pmatrix} \theta^{\delta_j} & & 0 \\ & \ddots & \\ 0 & & \theta^{n_j\delta_j} \end{pmatrix}$, K_j is such that $A_j + K_jC_j$ is Hurwitz and θ is a positive constant which may be large. $\delta_1 > 0$, $\cdots$, $\delta_p > 0$ are integer numbers which satisfy the following linear program:

$$\left\{\begin{array}{lcl} -n_j\delta_j + n_i\delta_i & < & \delta_j \\ 1 \leq i, j \leq p & & \end{array}\right. \tag{12}$$

More precisely, we can state:

Theorem 1: Let u be any bounded input, then there exists a constant $\theta_0 > 0$ such that for every $\theta \geq \theta_0$, we have:

$$\|\hat{\zeta}(t) - \zeta(t)\| \leq \tilde{\lambda}\|\hat{\zeta}(0) - \zeta(0)\| \exp(-\mu t)$$

where $\tilde{\lambda} > 0$ and $\mu > 0$ are constant.
The proof of theorem 1 is based on similar technics that we will use for the non initialized observer (see proof of theorem 2 below).
Now an initialized high gain observer for system (6)-(7) can be designed as follows:
Let Φ be the diffeomorphism which transforms (6)-(7) into the form (8)-(9)-(10), here $\mathcal{N} = \mathcal{M} = \{(x, z);\ \varphi(x, z) = 0\}$. Set $\Upsilon(x)$ the implicit solution of $\varphi(x, z) = 0$ and $\tilde{\Phi}(x) = \Phi(x, \Upsilon(x))$, we obtain:

Lemma 1: Assuming that $\|\Phi(\zeta) - \Phi(\bar{\zeta})\| \geq \lambda\|\zeta - \bar{\zeta}\|$, for some constant $\lambda > 0$ and that the nonlinear terms ψ^j's are global Lipschitz w.r.t. ξ, then an initialized high gain observer for system (6)-(7) takes the following form:

$$\left\{\begin{array}{lcl} \dot{\hat{x}} & = & f_0(\hat{x}, \hat{z}) + \sum_{i=1}^{m} u_i f_i(\hat{x}, \hat{z}) \\ & & +(\frac{\partial \tilde{\Phi}}{\partial x}\mid_{\hat{x}})^{-1}\Delta_\theta K(h(\hat{x}, \hat{z}) - y) \\ \dot{\hat{z}} & = & -(\frac{\partial \varphi}{\partial z}\mid_{\hat{x},\hat{z}})^{-1}(\frac{\partial \varphi}{\partial x}\mid_{(\hat{x},\hat{z})})\dot{\hat{x}} \\ \varphi(\hat{x}(0), \hat{z}(0)) = 0 & & \end{array}\right. \tag{13}$$

where

$$K = \begin{pmatrix} K_1 & & 0 \\ & \ddots & \\ 0 & & K_p \end{pmatrix},\ \Delta_\theta = \begin{pmatrix} \Delta^1_{\theta^{\delta_1}} & & 0 \\ & \ddots & \\ 0 & & \Delta^p_{\theta^{\delta_p}} \end{pmatrix}$$

with K_j, Δ^j_θ are defined above.
The proof of this lemma is straightforward. Indeed, let $\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = \zeta(t) = \Phi^{-1}(\xi(t))$ be an unknown trajectory of system (6)-(7). From Theorem (1), we know that $\hat{\xi}(t)$ exponentially converges to $\xi(t)$.
Now set $\begin{pmatrix} \hat{x}(t) \\ \hat{z}(t) \end{pmatrix} = \hat{\zeta}(t) = \Phi^{-1}(\hat{\xi}(t))$, it is easy to verify that $\hat{\zeta}(t)$ satisfies (13).
Finally, the condition $\|\Phi(\zeta) - \Phi(\bar{\zeta})\| \geq \lambda\|\zeta - \bar{\zeta}\|$ implies that $\hat{\zeta}(t)$ exponentially converges to the unknown trajectory $\zeta(t)$ of system (6)-(7).

Remark 2: Although $\Upsilon(x)$ cannot be explicitly expressed, $(\frac{\partial \tilde{\Phi}}{\partial x}\mid_{\hat{x}})^{-1}$ can be calculated. Indeed, one can verify that the computation of $(\frac{\partial \tilde{\Phi}}{\partial x}\mid_{\hat{x}})$ only requires the knowledge of $\frac{\partial \varphi}{\partial x}\mid_{(\hat{x},\hat{z})}$, $\frac{\partial \varphi}{\partial z}\mid_{(\hat{x},\hat{z})}$, $\frac{\partial \Phi}{\partial x}\mid_{(\hat{x},\hat{z})}$ and $\frac{\partial \Phi}{\partial z}\mid_{(\hat{x},\hat{z})}$.

III. NON INITIALIZED HIGH GAIN OBSERVER

In the above section, an initialized high gain observer has been proposed. This observer works if the following conditions hold:

i) The constraint $\varphi(\hat{x}(0), \hat{z}(0)) = 0$ is satisfied.
ii) The system is not disturbed and the measurements are not noisy.

In order to robustify the above observer, we will propose a non initialized observer: the state of our candidate observer can be initialized outside the manifold $\mathcal{M}$.
Using the same notations as in section 2 and considering the extension of system (6)-(7) on the wall space $\mathbb{R}^n \times \mathbb{R}^d$:

$$\left\{\begin{array}{lcl} \dot{\zeta} & = & F(\zeta, u) = F_0(\zeta) + \sum_{j=1}^{m} u_j f_j(\zeta) \\ y & = & H(\zeta) = h(x, z) \\ \zeta & \in & \mathbb{R}^n \times \mathbb{R}^d \end{array}\right. \tag{14}$$

The following map plays an important role in the non initialized observer construction:

$$\Sigma(\zeta) = \begin{pmatrix} \Phi(\zeta) \\ \varphi(\zeta) \end{pmatrix} \tag{15}$$

where, $\Phi(\zeta) = \begin{pmatrix} \Phi^1(\zeta) \\ \vdots \\ \Phi^p(\zeta) \end{pmatrix}$ with $\Phi^j = \begin{pmatrix} H_j \\ \vdots \\ L^{n_j-1}_{F_0}(H_j) \end{pmatrix}$; n_j are positive integers such that $n_1 + \ldots + n_p = n$.
In order to design a non initialized high gain observer, we will make the following assumption:

H1)

i) Σ is a diffeomorphism from a tubular neighborhood $\Omega_\varepsilon = \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^d\ /\ \|\varphi(x, z)\| < \varepsilon\}$ of $\mathcal{M}$
ii) Φ transforms the restriction of system (14) to $\mathcal{M}$ into the form (8), (9) and (10).
iii) $\|\Sigma(\zeta, \eta) - \Sigma(\bar{\zeta}, \bar{\eta})\| \geq \sigma\|(\zeta - \bar{\zeta}, \eta - \bar{\eta})\|$ for some constant $\sigma > 0$.

Remark 3: Assumption **H1)-ii)** has been assumed in section 2. This means that the restriction of system (14) to the manifold $\mathcal{M}$ is uniformly observable. But it doesn't imply in general uniform observability of system (14) restricted of Ω_ε.
Now combining assumptions **H1)-i)** and **H1)-ii)**, it follows that Σ transforms system (14) restricted to Ω_ε into the following form:

$$\left\{\begin{array}{lcl} \dot{\xi}^1 & = & A_1\xi^1 + \psi^1(\xi, \eta, u) \\ \vdots & & \\ \dot{\xi}^p & = & A_p\xi^p + \psi^p(\xi, \eta, u) \\ \dot{\eta} & = & 0 \\ y & = & \left[\begin{array}{ccc} C_1\xi^1 & \ldots & C_p\xi^p \end{array}\right]^T \end{array}\right. \tag{16}$$

where $\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Sigma(\zeta)$ with $\xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^p \end{pmatrix}$; $\xi^j = \Phi^j(\zeta)$ and $\eta = \varphi(\zeta)$. The A_j's and C_j's are as in (10).
Our candidate non initialized observer for system (6)-(7)

takes the following form:

$$\begin{cases} \dot{\hat{x}} &=& f_0(\hat{x},\hat{z}) + \sum_{i=1}^{m} u_i f_i(\hat{x},\hat{z}) \\ && +\widetilde{\Sigma}_{11}(\hat{x},\hat{z})\Delta_\theta K(h(\hat{x},\hat{z})-y) \\ && -\widetilde{\Sigma}_{12}(\hat{x},\hat{z})\Lambda\varphi(\hat{x},\hat{z}) \\ \dot{\hat{z}} &=& -(\frac{\partial\varphi}{\partial z}\mid_{\hat{x},\hat{z}})^{-1}(\frac{\partial\varphi}{\partial x}\mid_{\hat{x},\hat{z}})[f_0(\hat{x},\hat{z}) \\ && +\sum_{i=1}^{m} u_i f_i(\hat{x},\hat{z})] \\ && +\widetilde{\Sigma}_{21}(\hat{x},\hat{z})\Delta_\theta K(h(\hat{x},\hat{z})-y) \\ && -\widetilde{\Sigma}_{22}(\hat{x},\hat{z})\Lambda\varphi(\hat{x},\hat{z}) \end{cases} \tag{17}$$

where $y(t)$ is the output measurement of system (6)-(7).
$K = \begin{pmatrix} K_1 & & 0 \\ & \ddots & \\ 0 & & K_p \end{pmatrix}$ such that $A_j + K_j C_j$ is Hurwitz,
$\Delta_\theta = \begin{pmatrix} \Delta_\theta^1 & & 0 \\ & \ddots & \\ 0 & & \Delta_\theta^p \end{pmatrix}$; $\Delta_\theta^j = \begin{pmatrix} \theta^{\delta_j} & & 0 \\ & \ddots & \\ 0 & & \theta^{n_j\delta_j} \end{pmatrix}$,
θ is a positive constant. $\delta_1 > 0, \cdots, \delta_p > 0$ are as in (12) and Λ is a $d \times d$ symmetric positive definite (S.P.D) matrix.
Finally, $\begin{pmatrix} \widetilde{\Sigma}_{11}(\hat{x},\hat{z}) & \widetilde{\Sigma}_{12}(\hat{x},\hat{z}) \\ \widetilde{\Sigma}_{21}(\hat{x},\hat{z}) & \widetilde{\Sigma}_{22}(\hat{x},\hat{z}) \end{pmatrix} = [\frac{\partial\Sigma}{\partial(x,z)}\mid_{\hat{x},\hat{z}})]^{-1}$.
As in the high gain observer theory [2], [8], [10], [9],[11], [12], the proof of the convergence of our candidate observer requires the following technical assumption:
H2) The nonlinear terms $\psi^j(\xi,\eta,u)$ are assumed to be global Lipschitz functions with respect to (ξ,η), namely:
For every bounded input u; for every (ξ,η), $(\overline{\xi},\overline{\eta})$ belonging to some bounded subset of $\mathbb{R}^d$, there exists a constant a such that: $\|\psi^j(\xi,\eta,u) - \psi^j(\overline{\xi},\overline{\eta},u)\| \le a(\|\xi-\overline{\xi}\| + \|\eta-\overline{\eta}\|)$.
Now, we can state our main result.

Theorem 2: Let u be a bounded input and assume that hypotheses **H1)** and **H2)** hold, then system (17) becomes a non initialized observer for system (6)-(7). More precisely, let $\varepsilon_0 > 0$ be such that the following neighborhood of $\mathcal{M}$ defined by $\widetilde{\Omega}_{\varepsilon_0}$=$\{(\varphi(x,z))^T\Lambda\varphi(x,z) < \varepsilon_0\}$ is contained in the set Ω_ε given in **H1)**, then:
$\exists\theta_0 > 0$; $\forall\theta \geq \theta_0$; $\exists\sigma_1 > 0$; $\exists\sigma_2 > 0$, such that $\forall(\hat{x}(0),\hat{z}(0)) \in \widetilde{\Omega}_{\varepsilon_0}$, we have:
$\|(\hat{x}(t)-x(t),\hat{z}(t)-z(t))\| \le \sigma_1 e^{-\sigma_2 t}\|(\hat{x}(0)-x(0),\hat{z}(0)-z(0))\|$
The following notations will be used:
$\xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^p \end{pmatrix}$, $\hat{\xi} = \begin{pmatrix} \hat{\xi}^1 \\ \vdots \\ \hat{\xi}^p \end{pmatrix}$, $e^k = (\Delta_\theta^k)^{-1}(\hat{\xi}^k - \xi^k)$, where
$e = \begin{pmatrix} e^1 \\ \vdots \\ e^k \end{pmatrix}$.
The proof of the theorem will be based on the following technical lemma:

Lemma 2: Under hypotheses, **H1)**, **H2)** and (12), the following holds:
$\forall r > 0$; $\forall u \in \mathbb{R}^m$, $\|u\| \le r$; $\exists R > 0$; $\forall\theta \geq 1$; $\forall\xi$; $\forall\hat{\xi}$, we have : $\|(\Delta_\theta^k)^{-1}(\psi^k(\hat{\xi},0,u) - \psi^k(\xi,0,u))\| \le R\|e^k\|$
From hypothesis **H1-ii))**, the ith component of $(\Delta_\theta^k)^{-1}(\psi^k(\hat{\xi},0,u) - \psi^k(\xi,0,u))$ takes the form $\theta^{-i\delta_k}(\psi_i^k(\hat{\xi}_1^k,\dots\hat{\xi}_i^k,0,u) - \psi_i^k(\xi_1^k,\dots\xi_i^k,0,u))$. Now, let a be the Lipschitz constant of ψ (see hypothesis **H2)**), it follows that:
For $1 \le i \le n_k - 1$, we have:
$|\theta^{-i\delta_k}(\psi_i^k(\hat{\xi}_1^k,\dots\hat{\xi}_i^k,0,u) - \psi_i^k(\xi_1^k,\dots\xi_i^k,0,u))| \le a\theta^{-i\delta_k}\sqrt{(\hat{\xi}_1^k-\xi_1^k)^2 + \ldots + (\hat{\xi}_i^k-\xi_i^k)^2} \le a\sqrt{(e_1^k)^2 + \ldots + (e_i^k)^2}$,
since $\theta \geq 1$.
For $i = n_k$, we obtain:
$|\theta^{-n_k\delta_k}(\psi_{n_k}^k(\hat{\xi},0,u) - \psi_i^k(\xi,0,u))| \le a\theta^{-n_k\delta_k}\|\hat{\xi}-\xi)\| \le a\theta^{(-n_k\delta_k + \max\{n_j\delta_j;\ 1\le j\le p,\ j\neq k\})}\|e\|$.
From inequality (12), it follows that $(-n_k\delta_k + \max\{n_j\delta_j;\ \ 1 \le j \le p,\ \ j \neq k\}) < 0$. Now taking $\theta \geq 1$, this ends the proof of lemma (2).

Proof: **Proof of theorem (2)**
From above, we know that system (6)-(7) coincides with the restriction of system (14) to $\mathcal{M}$ and that Σ transforms the restriction of system (14) to $\mathcal{M}$ (ie system (6)-(7) into the restriction of system (16) to the affine space $\eta = 0$). Moreover (Σ) transforms system (17) into the following form:

$$\begin{cases} \dot{\hat{\xi}}^1 &=& A_1\hat{\xi}^1 + \psi^1(\hat{\xi},\hat{\eta},u) + \Delta_\theta^1 K^1(C_1\hat{\xi}^1 - y_1) \\ \vdots && \\ \dot{\hat{\xi}}^p &=& A_p\hat{\xi}^p + \psi^p(\hat{\xi},\hat{\eta},u) + \Delta_\theta^p K^p(C_p\hat{\xi}^p - y_p) \\ \dot{\hat{\eta}} &=& -\Lambda\hat{\eta} \end{cases} \tag{18}$$

Consequently, to show that system (17) forms an exponential observer for system (6)-(7), it suffices to show that system (18) forms an exponential observer for system (16) restricted to the space $\eta = 0$.
Let $\xi(t)$ be a trajectory of system (16) in which $\eta = 0$ and $\hat{\xi}(t)$ its corresponding estimation following from system (18).
Set $e^j(t) = ((\Delta_\theta^k)^{-1}(\hat{\xi}^j(t) - \xi^j(t))$, and $\delta\psi^j = \psi^j(\hat{\xi},\hat{\eta},u) - \psi^j(\xi,0,u)$, we get:

$$\dot{e}^j = \theta^{\delta_j}(A_j + K_jC_j)e^j + (\Delta_\theta^j)^{-1}\delta\psi^j \tag{19}$$

Since $(A_j + K_jC_j)$ is Hurwitz, there exists a S.P.D. matrix P^j which is the solution of the algebraic Lyapunov equation:

$$(A_j + K_jC_j)^T P^j + P^j(A_j + K_jC_j) = -I_{n_j} \tag{20}$$

Set $V(t) = V_1(t) + \ldots + V_p(t)$ with $V_j(t) = (e^j(t))^T P^j e^j(t)$ and $W = \|\hat{\eta}(t)\|^2$, in the sequel, we will show that there exists $\theta_0 \geq 1$ such that for every $\theta \geq \theta_0$ and every S.P.D. matrix Λ, the functions $V(t)$ and $W(t)$ exponentially converge to 0.

1- For $W = \|\hat{\eta}(t)\|^2$:
From (18), we obtain:

$$\begin{array}{rcl} \dot{W} & = & -2\hat{\eta}^T \Lambda \hat{\eta} \\ & \leq & -2\lambda_{min}(\Lambda) W \end{array}$$

where $\lambda_{min}(\Lambda)$ denotes the smallest eigenvalue of Λ. Thus,

$$\|\hat{\eta}(t)\|^2 \leq e^{-\beta t} \|\hat{\eta}(0)\|^2 \quad (21)$$

where $\beta = 2\lambda_{min}(\Lambda)$.

2- Now let us show that $V(t)$ exponentially converges to 0.
From (19) and (20), we get:

$$\begin{array}{ll} \dot{V}_j & = -\theta^{\delta_j} \|e^j\|^2 + 2(e^j)^T P^j (\Delta_\theta^j)^{-1} \delta\psi^j \\ & \leq -\theta^{\delta_j} \|e^j\|^2 + 2\|P^j\| \|e^j\| \|(\Delta_\theta^j)^{-1} \delta\psi^j\| \end{array} \quad (22)$$

Now using the global Lipschitz condition of ψ (see hypothesis **H2)** together with lemma (2) and inequality (21), then the following inequalities hold for $\theta \geq 1$:

$$\begin{array}{ll} \|(\Delta_\theta^j)^{-1} \delta\psi^j\| & \leq \|(\Delta_\theta^j)^{-1} (\psi^j(\hat{\xi}, \hat{\eta}, u) - \psi^j(\hat{\xi}, 0, u))\| \\ & + \|(\Delta_\theta^j)^{-1} (\psi(\hat{\xi}, 0, u) - \psi(\xi, 0, u))\| \\ & \leq c_1 \|\eta\| + c_2 \|e\| \end{array} \quad (23)$$

where c_2 is a constant which doesn't depend on θ. Thus,

$$\begin{array}{rcl} \dot{V}_j & \leq & -\theta^{\delta_j} \|e^j\|^2 + 2c_1 \|P^j\| \|e^j\| \|\eta\| \\ & & +2c_2 \|P^j\| \|e^j\|^2 \end{array} \quad (24)$$

Set $\delta = min\{\delta_j, \ 1 \leq j \leq p\}$, we obtain:

$$\begin{array}{rcl} \dot{V} & \leq & \sum_{j=1}^p (-\theta^{\delta_j} \|e^j\|^2 + 2c_1 \|P^j\| \|e^j\| \|\eta\| \\ & & +2c_2 \|P^j\| \|e^j\|^2) \\ & \leq & -\varepsilon \|e\|^2 + \lambda \|e\| \|\eta\| \end{array} \quad (25)$$

But $\|\eta(t)\|$ exponentially converges to 0. This ends the proof.

■

IV. Conclusion

Based on triangular structure, a high gain observer is proposed for a class of multi-output uniformly observable systems. This observer synthesis can be used to solve the observer problem for a class of implicit systems. Two observers are proposed. The first one is an initialized observer which requires an initialisation condition $\varphi(\hat{x}, \hat{z}) = 0$ which render it non robust with respect to disturbances and noises. The second one is a non initialized observer can be initialized from a tubular neighborhood of the submanifold $\mathcal{M} = \{(x, z)/\varphi(x, z) = 0\}$ of $\mathbb{R}^n \times \mathbb{R}^d$.

References

[1] O. Bernard, G. Sallet, and A. Sciandra, "Nonlinear observers for a class of biological systems: application to validation of a phytoplanktonic growth model," *IEEE Trans. Automat. Contr.*, vol. 43, no. 8, pp. 1056–1065, 1998.

[2] G. Bornard and H. Hammouri, "A high gain observer for a class of nonlinear systems under locally regular inputs," in *Proceedings of the 30th CDC IEEE*, Brighton, GB, 1991.

[3] G. Ciccarella, M. Dalla, and A. Germani, "A luenberger-like observer for nonlinear systems," *IJC*, vol. 57, no. 3, pp. 537–556, 1993.

[4] F. Deza, "Contribution to synthesis of exponential observers," PhD thesis, INSA, Rouen France, 1991.

[5] F. Deza, E. Busvelle, and J. Gauthier, "High gain estimation for nonlinear systems," *Systems and Control Letters*, vol. 18, pp. 295–299, 1992.

[6] F. Deza, J. G. E. Busvelle, and D. Rakotopara, "Exponential observers for nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 38, no. 3, 1993.

[7] E. ElYaagoubi, A. ElAssoudi, and H. Hammouri, "High gain observer: Attenuation of the peak phenomena," in *American Control Conference*, Boston, Massachusetts, 2004.

[8] J. Gauthier, H. Hammouri, and S. Othman, "A simple observer for nonlinear systems, application to bioreactors," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 875–880, 1992.

[9] J. Gauthier and I. Kupka, "Observability and observers for nonlinear systems," *SIAM J. Cont. and Optim*, vol. 32, pp. 975–997, 1994.

[10] ——, *Deteministic Observation Theory and Applications.* Cambridge University Press, 2001.

[11] H. Hammouri and M. Farza, "Nonlinear observers for locally uniformly observable systems," *ESAIM: Control, Optimization and Calculus of Variations*, vol. 9, pp. 353–370, March 2003.

[12] J. Schaffner and M. Zeitz, "Decentralized block diagonal triangular observer design for nonlinear systems," in *ECC*, 1997.