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# Nonlinear observer design using invariant manifolds and applications

Dimitrios Karagiannis, Alessandro Astolfi

*Abstract*— **The problem of constructing (reduced-order) observers for general nonlinear systems is addressed. It is shown that an asymptotic estimate of the unknown states can be obtained by rendering attractive an appropriately selected (invariant) manifold in the extended state space. The proposed methodology is applicable to systems that are not necessarily linear in the unmeasured states. This is illustrated with two practical examples: a single-machine infinite-bus system and a perspective vision system.**

## I. INTRODUCTION

The problem of constructing observers for nonlinear systems has received a lot of attention due to its importance in practical applications, where some of the states may not be available for measurement. In the case of linear systems, a comprehensive theory can be found in [1]. Since then, several attempts have been made to extend these results to nonlinear systems.

In [2], [3] an observer is constructed by first using a nonlinear transformation to linearize the plant up to an output injection term and then applying standard linear observer design techniques. The existence of such a transformation, however, relies on a set of stringent assumptions which are hard to verify in practice. Lyapunov-like conditions for the existence of a nonlinear observer with asymptotically stable error dynamics have been given in [4]. In [5], [6] an observer for uniformly observable nonlinear systems in canonical form has been proposed based on a global Lipschitz condition and a gain assignment technique. Some extensions to this result, which avoid the transformation to canonical form and allow for more flexibility in the selection of the observer gain, have been proposed in [7]. More recently, in [8], [9] conditions for the existence of a nonlinear observer have been given in terms of the (local) solution of a partial differential equation, thus extending Luenberger's early ideas [1] to the nonlinear case. Finally, a globally convergent reduced-order observer for systems in canonical form has been proposed in [10] using the notion of output-to-state stability.

In the present paper the problem of constructing reduced-order observers for general nonlinear systems is formulated as a problem of rendering attractive an appropriately selected invariant manifold in the extended state-space of the plant and the observer. It is shown that a solution to this problem can be obtained (at least locally) without resorting to high-gain designs as in [6], [7]. The proposed methodology is applicable to time-varying systems and to systems that are not necessarily linear in the unmeasured states, thus extending our previous result in [11].

The proposed approach is illustrated with two practical examples. The first is the problem of estimating the (time-varying) total inductance and the rotor angle in a single-machine infinite-bus system. The second is the problem of estimating the three-dimensional motion of an object using two-dimensional images obtained from a single camera. The validity of the proposed designs is tested via simulations.

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## II. PROBLEM FORMULATION

We consider nonlinear, time-varying systems described by equations of the form

$$\dot{\eta} = f(\eta, y, t) \tag{1}$$

$$\dot{y} = h(\eta, y, t)\,, \tag{2}$$

where $\eta \in \mathbb{R}^n$ is the unmeasured state and $y \in \mathbb{R}^m$ is the measurable output. It is assumed that the vector fields $f(\cdot)$ and $h(\cdot)$ are forward complete, *i.e.* trajectories starting at time $t_0$ are defined for all times $t \geq t_0$.

*Definition 1:* The dynamical system

$$\dot{\hat{\eta}} = \alpha(y, \hat{\eta}, t) \tag{3}$$

with $\hat{\eta} \in \mathbb{R}^p$, $p \geq n$, is called a (local) *observer* for the system (1)-(2), if there exist mappings

$$\beta(\cdot) : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p \quad \text{and} \quad \phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^p,$$

with $\phi(\cdot)$ (locally) left-invertible[1], such that the manifold

$$\mathcal{M}_t = \{(\eta, y, \hat{\eta}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p : \beta(y, \hat{\eta}, t) = \phi(\eta)\} \tag{4}$$

has the following properties.

1) All trajectories of the extended system (1)-(2)-(3) that start on the manifold $\mathcal{M}_t$ at time $t$ remain there for all future times $\tau > t$, *i.e.* $\mathcal{M}_t$ is positively invariant.
2) All trajectories of the extended system (1)-(2)-(3) that start in a neighborhood of $\mathcal{M}_t$ asymptotically converge to $\mathcal{M}_t$.

The above definition is in the spirit of the definition given in [4] and implies that an asymptotic estimate of the state $\eta$ is given by $\phi^L(\beta(y, \hat{\eta}, t))$, where $\phi^L$ denotes the left-inverse of $\phi$.

[1] A mapping $\phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^p$ is left-invertible if there exists a mapping $\phi^L(\cdot) : \mathbb{R}^p \to \mathbb{R}^n$ such that $\phi^L(\phi(x)) = x$, for all $x \in \mathbb{R}^n$.

## III. Main result

In this section we present the main result of the paper, namely a tool for constructing a nonlinear observer of the form given in Definition 1.

*Proposition 1:* Consider the system (1)-(2)-(3) and suppose that there exist mappings $\beta(\cdot) : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$, $\phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^p$ and a (local) left-inverse $\phi^L(\cdot) : \mathbb{R}^p \to \mathbb{R}^n$ such that the following hold.

(A1) $\det(\frac{\partial \beta}{\partial \hat{\eta}}) \neq 0$.

(A2) The system

$$\dot{z} = \frac{\partial \beta}{\partial y}\left(h(\eta, y, t) - h(\phi^L(\phi(\eta) + z), y, t)\right) - \frac{\partial \phi}{\partial \eta} f(\eta, y, t) + \left.\frac{\partial \phi}{\partial \eta}\right|_{\eta = \phi^L(\phi(\eta)+z)} f(\phi^L(\phi(\eta) + z), y, t) \quad (5)$$

has a (locally) asymptotically stable equilibrium at $z = 0$, uniformly in $\eta, y, t$.

Then there exists a function $\alpha(\cdot)$ such that (3) is an observer for the system (1)-(2).

*Proof:* Consider the variable

$$z = \beta(y, \hat{\eta}, t) - \phi(\eta), \quad (6)$$

where $\beta(\cdot)$ is a continuous mapping such that (A1) holds. Note that $z$ represents the distance at time $t$ of the system trajectories from the manifold $\mathcal{M}_t$ defined in (4). The dynamics of $z$ are given by

$$\dot{z} = \frac{\partial \beta}{\partial y} h(\eta, y, t) + \frac{\partial \beta}{\partial \hat{\eta}} \alpha(y, \hat{\eta}, t) + \frac{\partial \beta}{\partial t} - \frac{\partial \phi}{\partial \eta} f(\eta, y, t).$$

Using (A1), the function $\alpha(\cdot)$ in (3) can be selected as

$$\alpha(y, \hat{\eta}, t) = \left(\frac{\partial \beta}{\partial \hat{\eta}}\right)^{-1}\Big( - \frac{\partial \beta}{\partial y} h(\phi^L(\beta(y, \hat{\eta}, t)), y, t) - \frac{\partial \beta}{\partial t} + \left.\frac{\partial \phi}{\partial \eta}\right|_{\eta = \phi^L(\beta(y, \hat{\eta}, t))} f(\phi^L(\beta(y, \hat{\eta}, t)), y, t)\Big) \quad (7)$$

yielding the dynamics (5). It follows from (A2) that the distance $z$ from the manifold $\mathcal{M}_t$ converges asymptotically to zero. Note, moreover, that $\mathcal{M}_t$ is invariant, *i.e.* if $z(t) = 0$ for some $t$, then $z(\tau) = 0$ for all $\tau > t$. Hence, by Definition 1, the system (3) with $\alpha(\cdot)$ given by (7) is a (reduced-order) observer for (1)-(2). ■

*Remark 1:* The proof of Proposition 1 provides an implicit description of the observer dynamics (3) in terms of the mappings $\beta(\cdot)$, $\phi(\cdot)$ and $\phi^L(\cdot)$ which must then be selected to satisfy (A2).[2] Hence the problem of constructing an observer for the system (1)-(2) is *reduced* to the problem of rendering the system (5) asymptotically stable by assigning the functions $\beta(\cdot)$, $\phi(\cdot)$ and $\phi^L(\cdot)$. This non-standard stabilization problem can be extremely difficult to solve, since, in general, it relies on the solution of a set of partial differential equations. However, in many cases of practical interest, these equations are solvable, as demonstrated in the following two examples.

[2]Note, however, that the function $\alpha(\cdot)$ in (7) renders the manifold $\mathcal{M}_t$ invariant *for any* mappings $\beta(\cdot)$ and $\phi(\cdot)$.

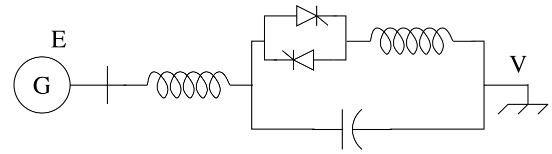


Fig. 1. Diagram of the single-machine infinite-bus system with TCSC.

## IV. Example 1: A single-machine infinite-bus system with TCSC

In this section the approach outlined in Proposition 1 is used to design an observer for a single-machine infinite-bus system driven by a thyristor-controlled series capacitor (TCSC). These devices are a special class of Flexible AC Transmission Systems (FACTS), which are extensively used for enhancing transient stability in power systems, see [12], [13] for more details. The proposed observer yields asymptotic estimates of the machine angle and the (time-varying) inductance using measurements of the rotor velocity. This partly extends the result in [14], where the problem of controlling the TCSC using measurements of both angle and velocity has been addressed based on the reduced-order observer proposed in [11].

Assuming a first-order model for the TCSC, the dynamics of the single-machine infinite-bus system, depicted in Figure 1, are described by the equations [15]

$$\begin{aligned} \dot{\delta} &= \omega \\ \dot{\omega} &= \frac{1}{T}\left(P_m - D\omega - EV\lambda \sin(\delta)\right) \\ \dot{\lambda} &= \frac{1}{T_{dc}}\left(-\lambda + \lambda_* + u\right), \end{aligned} \quad (8)$$

where $\delta \in [0, 2\pi)$ is the rotor angle, $\omega$ is the angular velocity of the rotor, $\lambda$ is the total inductance of the bus, $T$, $P_m$, $D$, $E$ and $V$ are constants representing the inertia, mechanical power, damping coefficient, generator voltage and bus voltage respectively, $T_{dc}$ is the time constant and $u$ is the control signal. It is assumed that only $\omega$ is available for measurement. The operating point for the system (8) is defined as $(\delta_*, 0, \lambda_*)$. Note that, for this point to be an equilibrium, $\lambda_*$ must satisfy the condition $\lambda_* = P_m/(EV \sin(\delta_*))$.

To motivate the observer design we first develop a state feedback control law using arguments pertaining to Hamiltonian systems and passivity-based control. To this end, define the state vector $x = [\delta, \omega, \lambda]$ and note that the system (8) can be put in port-controlled Hamiltonian form

$$\dot{x} = (J(\delta) - R)\frac{\partial H(x)}{\partial x} + g\left(\frac{1}{T_{dc}}u - EV\sin(\delta)\omega\right), \quad (9)$$

where

$$J(\delta) = \begin{bmatrix} 0 & \frac{1}{T} & 0 \\ -\frac{1}{T} & 0 & -\frac{EV}{T}\sin(\delta) \\ 0 & \frac{EV}{T}\sin(\delta) & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{D}{T^2} & 0 \\ 0 & 0 & \dfrac{1}{T_{dc}} \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\begin{aligned} H(x) \; &= \; -P_m\left(\delta - \delta_*\right) - EV\lambda_*\left(cos(\delta) - cos(\delta_*)\right) \\ &\quad + \frac{1}{2}T\omega^2 + \frac{1}{2}\left(\lambda - \lambda_*\right)^2 . \end{aligned} \tag{10}$$

Note that the Hamiltonian function (10) has a minimum at the desired equilibrium $(\delta_*, 0, \lambda_*)$. Hence, the control law

$$u = T_{dc}EV\sin(\delta)\omega - K\left(\lambda - \lambda_*\right) \tag{11}$$

with $K > -1$ is such that the closed-loop system (8)-(11) has an asymptotically stable equilibrium at $(\delta_*, 0, \lambda_*)$. However, the controller (11) requires measurement of $\sin(\delta)$ and $\lambda$ which are assumed unknown.

### A. Observer design

We now proceed to the design of an observer for the unmeasured states $\delta$ and $\lambda$. Let

$$\phi(\delta, \lambda) = \left[\; \sin(\delta) \quad \cos(\delta) \quad \lambda \;\right]^T \tag{12}$$

which satisfies the invertibility condition in Proposition 1, and define the error variable

$$z = \beta(\omega, \hat{\eta}) - \phi(\delta, \lambda), \tag{13}$$

where $\hat{\eta} \in \mathbb{R}^3$ is the observer state and

$$\beta(\omega, \hat{\eta}) = \left[\; \beta_1(\omega, \hat{\eta}) \quad \beta_2(\omega, \hat{\eta}) \quad \beta_3(\omega, \hat{\eta}) \;\right]^T$$

is a mapping to be defined. The dynamics of $z$ are given by

$$\begin{aligned} \dot{z} \; &= \; \frac{\partial\beta}{\partial\hat{\eta}}\dot{\hat{\eta}} + \frac{\partial\beta}{\partial\omega}\frac{1}{T}\Big(P_m - D\omega \\ &\quad - EV\left(\beta_3(\omega,\hat{\eta}) - z_3\right)\left(\beta_1(\omega,\hat{\eta}) - z_1\right)\Big) \\ &\quad - \begin{bmatrix} \omega\left(\beta_2(\omega,\hat{\eta}) - z_2\right) \\ -\omega\left(\beta_1(\omega,\hat{\eta}) - z_1\right) \\ \dfrac{1}{T_{dc}}\left(z_3 - \beta_3(\omega,\hat{\eta}) + \lambda_* + u\right) \end{bmatrix}. \end{aligned}$$

Provided that the Jacobian matrix $\partial\beta/\partial\hat{\eta}$ is invertible, the observer dynamics can be selected as in the proof of Proposition 1, namely

$$\begin{aligned} \dot{\hat{\eta}} = &-\left(\frac{\partial\beta}{\partial\hat{\eta}}\right)^{-1}\frac{\partial\beta}{\partial\omega}\frac{1}{T}\Big(P_m - D\omega - EV\beta_3(\omega,\hat{\eta})\beta_1(\omega,\hat{\eta})\Big) \\ &+\left(\frac{\partial\beta}{\partial\hat{\eta}}\right)^{-1}\begin{bmatrix} \omega\beta_2(\omega,\hat{\eta}) \\ -\omega\beta_1(\omega,\hat{\eta}) \\ \dfrac{1}{T_{dc}}\left(-\beta_3(\omega,\hat{\eta}) + \lambda_* + u\right) \end{bmatrix}, \end{aligned} \tag{14}$$

yielding the error dynamics

$$\begin{aligned} \dot{z} \; &= \; \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & -\dfrac{1}{T_{dc}} \end{bmatrix} z \\ &\quad + \frac{EV}{T}\frac{\partial\beta}{\partial\omega}\left[\; \beta_3(\omega,\hat{\eta}) \quad 0 \quad \sin(\delta) \;\right] z\,. \end{aligned}$$

The above system is simplified by selecting

$$\beta_2(\omega,\hat{\eta}) = \hat{\eta}_2\,, \qquad \beta_3(\omega,\hat{\eta}) = \hat{\eta}_3\,,$$

which yields

$$\dot{z} = \begin{bmatrix} \dfrac{\partial\beta_1}{\partial\omega}\dfrac{EV}{T}\hat{\eta}_3 & \omega & \dfrac{\partial\beta_1}{\partial\omega}\dfrac{EV}{T}\sin(\delta) \\ -\omega & 0 & 0 \\ 0 & 0 & -\dfrac{1}{T_{dc}} \end{bmatrix} z\,. \tag{15}$$

Selecting the function $\beta_1(\cdot)$ as

$$\beta_1(\omega,\hat{\eta}) = \hat{\eta}_1 - \gamma\frac{T}{EV}\hat{\eta}_3\omega \tag{16}$$

with $\gamma < 4/T_{dc}$ ensures that (A1) in Proposition 1 is satisfied and the equilibrium $z = 0$ is globally stable. Moreover, using the candidate Lyapunov function $V(z) = \frac{1}{2}|z|^2$ it can be easily shown that

$$\lim_{t\to\infty} z_1 = 0\,, \qquad \lim_{t\to\infty} z_3 = 0\,,$$

provided that $\lambda$ is bounded away from zero. As a result, from (12)-(13) an asymptotic estimate of $\sin(\delta)$ and $\lambda$ is given by $\beta_1(\omega,\hat{\eta})$ and $\beta_3(\omega,\hat{\eta}) = \hat{\eta}_3$ respectively.

### B. Simulation results

To test the performance of the proposed observer, the system (8)-(14) with $u = 0$ (which implies $\lambda(t) > 0$ for all $t$) has been simulated using the parameters given in [13], namely $P_m = 0.8$ p.u., $D = 1$ p.u., $E = 1.06679$ p.u., $V = 1$ p.u., $T = 0.05$ s, and $T_{dc} = 0.05$ s. The rotor angle at the operating point is $\delta_* = \pi/13.671$ rad. The observer gain has been set to $\gamma = 50$ and the initial conditions to $\hat{\eta}(0) = [0, 1, 1]^T$. The system is considered to be at an equilibrium at $t = 0$ s. At $t = 0.4$ s a three-phase fault occurs at the generator bus and it is cleared at $t = 0.6$ s.

Figure 2 shows the time histories of the rotor angle and velocity and of the observation errors. We see that $z_1$ and $z_3$ converge asymptotically to zero. Note, however, that during the fault the stability properties of $z_1$ and $z_2$ are lost, since the dynamics of the machine become

$$\dot{\delta} = \omega\,, \qquad \dot{\omega} = \frac{1}{T}\left(P_m - D\omega\right),$$

hence $\delta$ becomes undetectable. However, after the fault has been cleared, $z_1$ converges asymptotically to zero.

## V. Example 2: A perspective vision system

A classical problem in machine vision is to determine the position of an object moving in the three-dimensional space by observing the motion of its projected feature on the two-dimensional image space of a charge-coupled device (CCD) camera. The case where the motion of the object is described by linear (possibly time-varying) dynamics with known parameters has received particular attention, see *e.g.* [16], [17], [18]. The systems that arise in this case are known as perspective dynamical systems and the problem

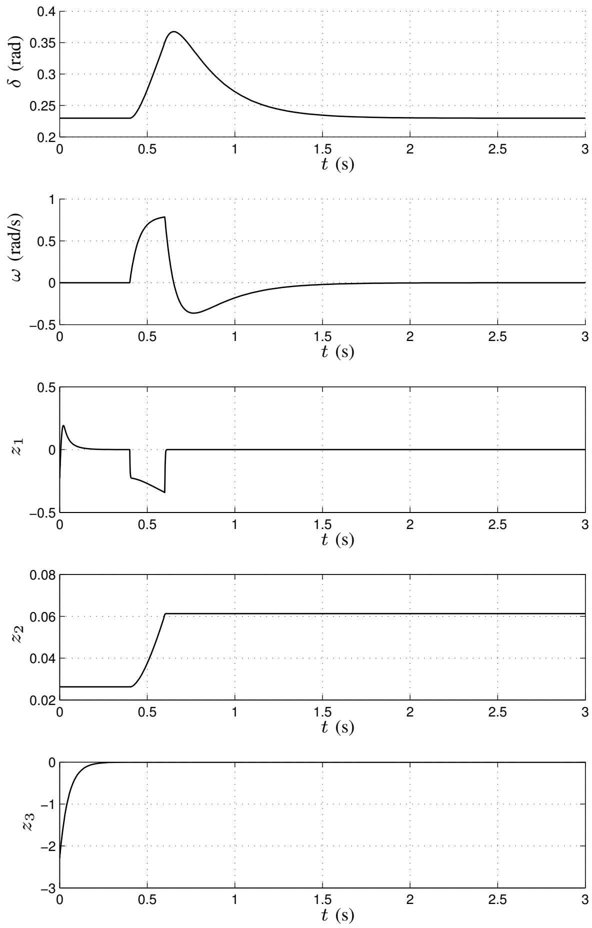


Fig. 2. Time histories of the rotor angle $\delta$, rotor velocity $\omega$ and of the observation errors $z_1$, $z_2$ and $z_3$.

of determining the object space co-ordinates reduces to the problem of estimating the depth (or range) of the object.

In this section a solution to the range identification problem is presented based on the nonlinear observer design of Section III. The proposed scheme achieves asymptotic convergence of the observation error to zero and is considerably simpler than the fourth-order asymptotic observer proposed in [18], as well as the fifth-order approximate observer in [17]. Moreover, it can be easily tuned to achieve the desired convergence rate. As a result, the performance is greatly enhanced.

The motion of an object undergoing rotation, translation and linear deformation can be described by the affine system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad (17)$$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$ are the unmeasurable co-ordinates of the object in an inertial reference frame with $x_3$ being perpendicular to the camera image space, as shown in Figure 3. The motion parameters $a_{ij}$, $b_i$ are possibly time-varying and are assumed known. Using the perspective (or "pinhole")

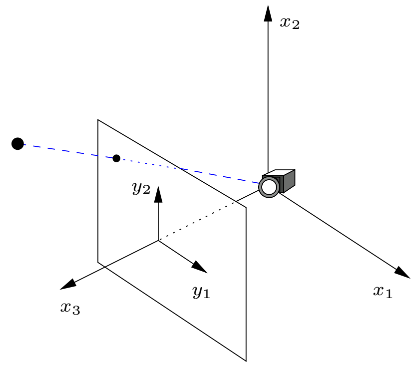


Fig. 3. Diagram of the perspective vision system.

model for the camera, the measurable co-ordinates on the image space are given by

$$y = \begin{bmatrix} y_1, & y_2 \end{bmatrix}^T = \epsilon \begin{bmatrix} \frac{x_1}{x_3}, & \frac{x_2}{x_3} \end{bmatrix}^T, \quad (18)$$

where $\epsilon$ is the focal length of the camera, *i.e.* the distance between the camera and the origin of the image-space axes. Without loss of generality, we assume that $\epsilon = 1$.

The perspective system (17) must satisfy the following assumption.

*Assumption 1:* The parameters $a_{ij}$, $b_i$ in (17) and the co-ordinates $y_1$, $y_2$ in (18) are bounded functions of time, *i.e.* $a_{ij}(t), b_i(t) \in \mathcal{L}_\infty, \forall i, j = 1, 2, 3$ and $y(t) \in \mathcal{L}_\infty$. Moreover, $a_{ij}(t)$ and $b_i(t)$ are first-order differentiable and $x_3(t) > \epsilon, \forall t$, where $\epsilon$ is an arbitrarily small positive constant.

*Remark 2:* Assumption 1 is motivated by the physical properties of the perspective system, see [16], [18]. Note that in [18] it is further assumed that the functions $b_i(t)$ are twice differentiable and that $x_3(t) \in \mathcal{L}_\infty$.

The design objective is to reconstruct the co-ordinates $x_1$, $x_2$, $x_3$ from measurements of the image-space co-ordinates $y_1$, $y_2$.

### A. Observer design

As in [17], [18], the first step is to define the (unmeasurable) variable

$$\eta = \frac{1}{x_3}$$

and rewrite the system (17)-(18) in the form (1)-(2), namely

$$\begin{aligned} \dot{\eta} &= -\left(a_{31}y_1 + a_{32}y_2 + a_{33}\right)\eta - b_3\eta^2 && (19) \\ \dot{y} &= \begin{bmatrix} a_{11} - a_{33} & a_{12} \\ a_{21} & a_{22} - a_{33} \end{bmatrix} y + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ &\quad - yy^T \begin{bmatrix} a_{31} \\ a_{32} \end{bmatrix} + \begin{bmatrix} b_1 - b_3 y_1 \\ b_2 - b_3 y_2 \end{bmatrix} \eta \,. && (20) \end{aligned}$$

Note that, if $\eta$ is known, then the co-ordinates $x_1, x_2, x_3$ can be directly obtained from (18), hence the problem reduces to constructing an asymptotic observer for the state $\eta$.

To this end, consider a reduced-order observer of the general form (3) and the error variable

$$z = \beta(y, \hat{\eta}, t) - \eta \,, \quad (21)$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ are continuous mappings to be defined. Note that in this case we have taken $\phi(\cdot)$ to be the identity. The dynamics of $z$ are given by

$$\begin{aligned}\dot{z} = \frac{\partial\beta}{\partial y}\Bigg(&\left[\begin{array}{cc} a_{11}-a_{33} & a_{12} \\ a_{21} & a_{22}-a_{33}\end{array}\right] y + \left[\begin{array}{c} a_{13} \\ a_{23}\end{array}\right] \\ &-yy^T \left[\begin{array}{c} a_{31} \\ a_{32}\end{array}\right] + \left[\begin{array}{c} b_1 - b_3 y_1 \\ b_2 - b_3 y_2\end{array}\right] (\beta(y,\hat{\eta},t) - z)\Bigg) \\ &+ (a_{31}y_1 + a_{32}y_2 + a_{33})(\beta(y,\hat{\eta},t) - z) \\ &+ b_3(\beta(y,\hat{\eta},t) - z)^2 + \frac{\partial\beta}{\partial\hat{\eta}}\alpha(y,\hat{\eta},t) + \frac{\partial\beta}{\partial t},\end{aligned}$$

where

$$\frac{\partial\beta}{\partial y} = \left[\begin{array}{cc} \frac{\partial\beta}{\partial y_1}, & \frac{\partial\beta}{\partial y_2}\end{array}\right].$$

Noting that

$$(\beta(y,\hat{\eta},t) - z)^2 = \beta(y,\hat{\eta},t)^2 - \beta(y,\hat{\eta},t)z - \eta z$$

and provided that $\partial\beta/\partial\hat{\eta}$ is invertible, the function $\alpha(\cdot)$ can be selected as in (7), namely

$$\begin{aligned}\alpha(y,\hat{\eta},t) = \left(\frac{\partial\beta}{\partial\hat{\eta}}\right)^{-1}\Bigg(&-(a_{31}y_1 + a_{32}y_2 + a_{33})\beta(y,\hat{\eta},t) \\ &-b_3(\beta(y,\hat{\eta},t))^2 - \frac{\partial\beta}{\partial t} \\ &-\frac{\partial\beta}{\partial y}\Bigg(\left[\begin{array}{cc} a_{11}-a_{33} & a_{12} \\ a_{21} & a_{22}-a_{33}\end{array}\right] y + \left[\begin{array}{c} a_{13} \\ a_{23}\end{array}\right] \\ &-yy^T\left[\begin{array}{c} a_{31} \\ a_{32}\end{array}\right] + \left[\begin{array}{c} b_1 - b_3y_1 \\ b_2 - b_3y_2\end{array}\right]\beta(y,\hat{\eta},t)\Bigg)\Bigg), \quad (22)\end{aligned}$$

yielding the error dynamics

$$\begin{aligned}\dot{z} \;=\; &-\Bigg(a_{31}y_1 + a_{32}y_2 + a_{33} + 2b_3\eta \\ &+\frac{\partial\beta}{\partial y}\left[\begin{array}{c} b_1 - b_3y_1 \\ b_2 - b_3y_2\end{array}\right]\Bigg) z - b_3 z^2. \quad (23)\end{aligned}$$

Note that the foregoing selection for the function $\alpha(\cdot)$ ensures that the manifold $z = 0$ is invariant. It remains to find a mapping $\beta(\cdot)$ such that it is also attractive, *i.e.* the system (23) is asymptotically stable. The procedure can be outlined in the following statement.

*Proposition 2:* Consider the system (19)-(20)-(3), where $\alpha(\cdot)$ is given by (22), and suppose that

$$(b_1 - b_3y_1)^2 + (b_2 - b_3y_2)^2 > \delta \quad (24)$$

for some $\delta > 0$. Then there exists a function $\beta(y,\hat{\eta},t)$ (with $\partial\beta/\partial\hat{\eta}$ invertible) such that the system (23) is asymptotically stable.

*Proof:* To begin with, note that by Assumption 1 there exists a positive constant $c$ such that

$$|\; a_{31}y_1 + a_{32}y_2 + a_{33} + 2b_3\eta \;| < c\,.$$

Consider now a function of the form

$$\beta(y,\hat{\eta},t) = \hat{\eta} + f(y,t)\,, \quad (25)$$

where $f(\cdot)$ satisfies the partial differential equation

$$\frac{\partial f}{\partial y_1}(b_1 - b_3y_1) + \frac{\partial f}{\partial y_2}(b_2 - b_3y_2) = \kappa(t)$$

with $\kappa(t) \geq c$. A solution to the above equation is given by

$$f(y,t) = \frac{\lambda}{2}\left[\left(-y_1^2 - y_2^2\right)b_3 + 2b_1y_1 + 2b_2y_2\right] \quad (26)$$

with

$$\kappa(t) = \lambda\left[(b_1 - b_3y_1)^2 + (b_2 - b_3y_2)^2\right],$$

where $\lambda > 0$ is a design parameter such that $\lambda\delta \geq c$. It remains to show that, for any set of initial conditions $z(0)$, there exists $\lambda$ such that the trajectories $z(t)$, $t \geq 0$, are bounded and asymptotically converge to zero.

To this end, consider again the system (23) with $\beta(\cdot)$ defined by (25)-(26) and the candidate Lyapunov function

$$V(z) = \frac{1}{2}z^2,$$

whose time-derivative along the system trajectories satisfies

$$\begin{aligned}\dot{V} &\leq -(\lambda\delta - c)z^2 - b_3z^3 \\ &= -z^2\left[(\lambda\delta - c) + b_3z\right].\end{aligned}$$

As a result, the origin $z = 0$ is a uniformly asymptotically stable equilibrium for the system (23) with a region of attraction containing the invariant set

$$B = \left\{\; z \in \mathbb{R} \;:\; |z| < \frac{\lambda\delta - c}{b_3^+} \;\right\}, \quad (27)$$

where

$$b_3^+ = \max_{t\geq 0}|b_3|\,.$$

The proof is completed by noting that, for any set of initial conditions $z(0)$, there exists $\lambda$ (sufficiently large) such that $z(0) \in B$. ■

*Remark 3:* In order to obtain a more practically useful condition on the parameter $\lambda$, suppose that the observer is initialized according to

$$\beta(y(0),\hat{\eta}(0),0) \geq \frac{1}{\epsilon} = 1\,.$$

Then from (21) and Assumption 1 we have

$$|z(0)| = |\beta(y(0),\hat{\eta}(0),0) - \eta(0)| < \beta(y(0),\hat{\eta}(0),0)\,.$$

Hence, the selection

$$\lambda \geq \frac{b_3^+\beta(y(0),\hat{\eta}(0),0) + c}{\delta} \quad (28)$$

is such that $z(0) \in B$, where $B$ is given by (27).

### *B. Simulation results*

In this section the proposed controller is tested via numerical simulations and compared with the one in [18]. Consider the example given in [17], [18] of the perspective system

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3\end{array}\right] = \left[\begin{array}{ccc} -0.2 & 0.4 & -0.6 \\ 0.1 & -0.2 & 0.3 \\ 0.3 & -0.4 & 0.4\end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \\ x_3\end{array}\right] + \left[\begin{array}{c} 0.5 \\ 0.25 \\ 0.3\end{array}\right]$$

with the initial conditions

$$\left[\begin{array}{ccc} x_1(0) & x_2(0) & x_3(0)\end{array}\right]^T = \left[\begin{array}{ccc} 1 & 1.5 & 2.5\end{array}\right]^T.$$

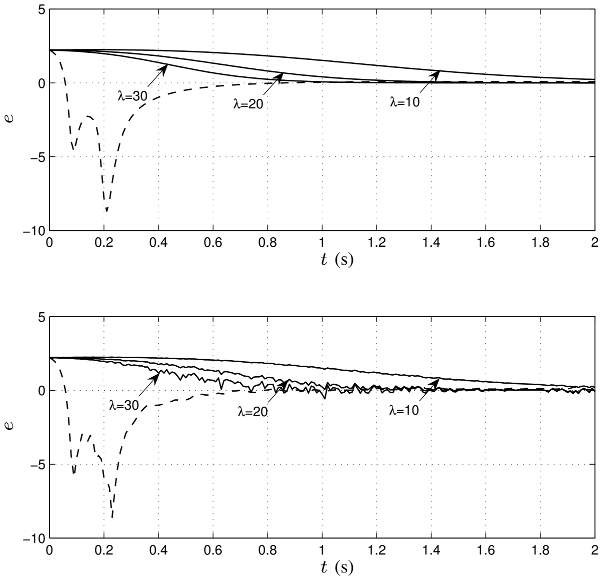


Fig. 4. Time history of the observation error for ideal measurements (top graph) and for measurements corrupted by noise (bottom graph). Solid line: Proposed observer. Dashed line: Observer in [18].

Note that the proposed first-order observer is described by the equations (3) and (22), where the mapping $\beta(\cdot)$ is given by (25)-(26). The estimate of the range $x_3$ is given by[3]

$$\hat{x}_3 = \frac{1}{\beta(y, \hat{\eta}, t)} .$$

The constant $\lambda$ in (26) has been selected sufficiently large so as to satisfy the constraints set out in the proof of Proposition 2. In this case, for $\delta = 0.1$, $c = 0.5$ and $\beta(y(0), \hat{\eta}(0), 0) = 1$, from (28) we obtain $\lambda \geq 8$.

Figure 4 shows the time history of the observation error $e = x_3 - \hat{x}_3$ for different values of the gain $\lambda$, namely $\lambda = 10$, $\lambda = 20$ and $\lambda = 30$, and for the observer proposed in [18], for the case of ideal measurements and for the case when the measurements of $y_1$ and $y_2$ are corrupted by $1\%$ random noise. We see that the transient performance of the proposed observer is significantly superior to the one in [18]. Moreover, the convergence rate can be arbitrarily increased simply by increasing the parameter $\lambda$. However, the sensitivity to the presence of noise also increases in this case.

## VI. Conclusions

The problem of constructing (reduced-order) observers for general nonlinear systems has been addressed using the notion of invariant manifolds. The proposed methodology consists in finding a manifold in state-space which can be expressed in the form (4), parameterized by a mapping $\beta(\cdot)$, and then designing the observer dynamics so that the manifold is invariant and selecting the mapping $\beta(\cdot)$ so that it is also attractive. A constructive proof of existence for such an invariant manifold has been given. However, the design of an appropriate mapping $\beta(\cdot)$ that renders the manifold attractive remains an open issue, since it relies on the solution of a set of partial differential equations (or inequalities), which in general can be extremely difficult to solve. The method has been applied on two practical problems: a state estimation problem for single-machine infinite-bus systems and a range identification problem for perspective vision systems. The efficacy of the proposed designs has been validated via simulations.

[3]The boundedness of $\beta(\cdot)$ away from zero is ensured by the initial condition $\beta(y(0), \hat{\eta}(0), 0) \geq 1$ and the fact that $z$ is decreasing.

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