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Stability of formations with delays in information exchange.

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Abstract— **The effect of delays in information exchange on the stability of a formation is studied. Upper limits for allowable delays are obtained through the use of two methods applicable to a class of systems.**

I. INTRODUCTION

Formations of vehicles may often be a means of accomplishing tasks which an individual vehicle would not be capable of performing. Examples include a group of space vehicles acting as a space antenna, a group of UAV's cooperating in a search mission, platoons of underwater vehicles in a formation etc.

The information flow between a set of vehicles in a formation may be considered the backbone of the formation as it is this information flow that allows the individual vehicles to coordinate their actions, be it to control their relative positions or to accomplish some task which may be the objective of the formation as a whole. The information flow, and the actions individual vehicles take based on it, must be such that the overall formation remains stable.

Several papers have recently addressed this issue. For example in [4] the authors formulate the problem where the vehicles possess potentials with respect to virtual leaders and one another. The virtual leaders can be used to direct the motion of the formation as a whole and the potential between the vehicles allows for maintaining the overall shape of the formation. In [5] structural potential functions are used to stabilize formations, and in [6] graph-theoretic methods are used to define formations and to study splitting and rejoining formations of vehicles.

In [1] the role of the Laplacian of a graph on the stability of a formation is studied and criteria for stability are derived. These criteria are based on the assumption that each vehicle has instantaneous and complete information about the states of its neighbors.

Information flow among the vehicles in a formation is considered in [2]. In this work it is shown how the information flow can be designed in such a way as to converge to a steady state that can be used as a reference by all vehicles. The results are based on a discrete model for the dynamics of the vehicles as well as for the flow of information.

In the papers mentioned above, it is generally assumed that each vehicle has full and up to date information about its neighbors at any given instant. Even when this is possible, however, there may be an interest in minimizing the rate of information exchange. In addition, communication between vehicles may be of limited precision due to e.g. limited transmission power of vehicles. It is therefore important to understand the dynamics of information flow in a formation and how it relates to the dynamics of the vehicles of the formation. Ultimately such understanding will enable the design of "minimal" information exchange topologies that result in stable formations.

Limited information flow under the assumption that exchange of information between vehicles takes place at specified discrete times was considered in [3], and the effect of the resulting delay in information transfer on the stability of the formation was considered.

In the current work a formation of vehicles is considered where the transfer of information between vehicles takes finite time. This information is assumed to be used by the individual vehicles to maintain their positions with respect to the formation. It is easy to convince oneself that too much delay in the transfered information, i.e. large transfer times for the information will lead to instability. On the other hand some delay in the transfered information may sometimes be unavoidable especially in the case where the information is relayed between several vehicles before it reaches a vehicle that needs to use it. The goal in this paper is to study how the delay in information transfer affects the stability of a formation and to obtain a limit on this delay for stability.

The problem is approached in two general ways. The first approach makes use of the theory of retarded functional differential equations and may be described as a direct stability analysis. The other approach is to consider the Schur transformation of a Laplace transform of the representative dynamic equations. It turns out that this leads to the problem of determining the stability of a number of disjoint linear systems, the stability of which guarantees the stability of the original formation.

The rest of this paper is organized as follows: In section 2, the main problem, a type of formation with delay in information transfer, is described. In section 3 a method of direct stability analysis is studied and an example is given in section 3.1. In section 4 an indirect method of stability analysis is described. An example of this is given in section 4.1 . Concluding remarks are given in section 5.

II. PROBLEM STATEMENT

Consider a set of n vehicles with vehicle i having dynamics given by

$$\dot{x}_i = Ax_i + Bu_i, \tag{1}$$

$$y_i = C_1 x_i, \tag{2}$$

$$z_{ij} = C_2(x_i - x_j(t - \tau_{ij})) \tag{3}$$

where τ_{ij} is the time delay in the transfer of information from vehicle j to vehicle i. As in [1], each vehicle is assumed to act on an error signal z_i that is defined by

$$z_i = \frac{1}{|J_i|} \sum_{j \in J_i} z_{ij}, \tag{4}$$

where J_i is the set of vehicles that vehicle i obtains information from. The control action is given through

$$\dot{v}_i = Fv_i + G_1 y_i + G_2 z_i \tag{5}$$

$$u_i = Hv_i + K_1 y_i + K_2 z_i \tag{6}$$

Defining the formation-wide state vector x by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{7}$$

and using the notation $\widehat{M}$ for a block diagonal matrix that consists of the matrix M on its diagonal we obtain the formation wide dynamic equation

$$\dot{x} = (\widehat{A} + B\widehat{K_1 C_1})x + \widehat{BH}v + B\widehat{K_2 C_2}\bar{L}x \tag{8}$$

where the operator $\bar{L}$ is defined through

$$\bar{L}x = \begin{bmatrix} x_1 - \frac{1}{|J_1|}\sum_{j \in J_1} x_j(t - \tau_{1j}) \\ x_2 - \frac{1}{|J_2|}\sum_{j \in J_2} x_j(t - \tau_{2j}) \\ \vdots \\ x_n - \frac{1}{|J_n|}\sum_{j \in J_n} x_j(t - \tau_{nj}) \end{bmatrix} \tag{9}$$

In the case where all the τ_{ij}'s are zero, i.e. when information is exchanged instantaneously, the operator $\bar{L}$ reduces to the (dimensionally extended) Laplacian of the formation (see [1]).

III. DIRECT STABILITY INVESTIGATIONS

A direct method of investigating the stability of a linear system with delays has been described in [8]. This method is applicable to a linear autonomous (and homogeneous) retarded functional differential equation (RFDE) of the form

$$\dot{x}(t) = \mathcal{L}(x_t) \tag{10}$$

where $x_t \in \mathbf{B}$ is defined by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-h, 0] \tag{11}$$

and the functional $\mathcal{L} : \mathbf{B} \to \mathbf{R^n}$ is continuous and linear, where $\mathbf{B}$ denotes the vector space of continuous and bounded functions mapping the interval $[-h, 0]$ into $\mathbf{R}^n$.

According to the Riesz Representation Theorem the RFDE can be written in the form

$$\dot{x}(t) = \int_{-\infty}^{0} [d\eta(\theta)]x(t + \theta) \tag{12}$$

where η is a matrix of functions of bounded variation on $(-\infty, 0]$ and the integral is a Riemann-Stiltjes one.

The characteristic function corresponding to the linear autonomous *RFDE* (10) is given by

$$D(s) = \det(sI - \int_{-\infty}^{0} e^{s\theta} d\eta(\theta)), s \in \mathbf{C} \tag{13}$$

where $\mathbf{C}$ is the set of complex numbers.

The definition of stability used in [8] is that the roots of the characteristic function have strictly negative real parts. It is also assumed that $D(s)$ does not have any zeros on the imaginary axis. This in turn assures the asymptotic stability of the origin under the dynamics defined by the *RFDE*. However (see also [1]) in the study of the stability of a formation, it may happen that some of the roots of the characteristic function are zero; these roots may correspond to the "unobservable" part of the dynamics of the formation which represents the motion of the formation "as a whole". For this reason, it is necessary to make an extension of the result in [8] for it to be valid in the case of zero roots of the characteristic function. The goal in this section is to make this extension.

Using Cauchy's argument principle and the fact that the characteristic function D has no poles in the right hand plane (see [8]) the number N of zeros of D in the right hand plane can be expressed as

$$N = \frac{1}{2\pi i} \lim_{H \to \infty, \varepsilon \to 0} \sum_{k=1}^{4} \oint_{(g_k)} \frac{1}{D(\lambda)} \frac{dD(\lambda)}{d\lambda} d\lambda \tag{14}$$

where (g_k) $(k = 1, 2, 3, 4)$ are given in the complex plane as follows:

$(g_1) : s = He^{i\phi}, H \in \mathbf{R}_+, \phi$ ranges from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$;
$(g_2) : s = i\omega$, where ω ranges from H to ε;
$(g_3) : s = \varepsilon e^{i\phi}, \varepsilon \in \mathbf{R}_+, \phi$ ranges from $+\frac{\pi}{2}$ to $-\frac{\pi}{2}$;
$(g_4) : s = i\omega$, where ω ranges from $-\varepsilon$ to $-H$;

For the first leg of the integration contour, (g_1), we have (see [8])

$$\frac{1}{2\pi i} \oint_{(g_1)} \frac{1}{D(\lambda)} \frac{dD(\lambda)}{d\lambda} d\lambda = \frac{n}{2} \tag{15}$$

where n is the dimension of the state space.

To determine the contribution from the third leg of the integration contour (g_3), one can make use of a Taylor expansion of $D(\lambda)$ for small λ under the assumption that $D(0) = 0$. It is then straightforward to show that the contribution is

$$\frac{1}{2\pi i}\oint_{(g_3)}\frac{1}{D(\lambda)}\frac{dD(\lambda)}{d\lambda}d\lambda = -\frac{1}{2}. \tag{16}$$

Lastly, for legs (g_2) and (g_4) we have

$$\frac{1}{2\pi i}\oint_{(g_1)\cup(g_4)}\frac{1}{D(\lambda)}\frac{dD(\lambda)}{d\lambda}d\lambda = \frac{1}{\pi}\lim_{H\to\infty,\varepsilon\to 0}\Delta_{[\varepsilon,H]}\zeta \tag{17}$$

where ζ is the argument of $D(i\omega)$. Putting together (15), (16), and (17) one obtains

$$N = \frac{n-1}{2} + \frac{1}{\pi}\lim_{H\to\infty,\varepsilon\to 0}\Delta_{[\varepsilon,H]}\zeta \tag{18}$$

The last term in (18), i.e. the value of the change in the argument of ζ, can be evaluated in much the same way as the method used in [8]. One starts by denoting the real and imaginary parts of the zeros of $D(i\omega)$ by $R(\omega)$ and $S(\omega)$. It is assumed that the dimension of the state space n is even (a similar approach can be used for the case where n is odd).

If one denotes the number of zeros of R in the range $[0,\infty)$ by r, it can be shown (see [8]) that when n is even r is finite. Next, let the zeros of R be denoted by $\rho_r = 0, \rho_{r-1}, \rho_{r-2}, ..., \rho_1$ where $0 < \rho_{r-1} < \rho_{r-2} < ... < \rho_1$. In the interval $[0,\rho_1]$ the zeros of S are denoted by $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_s = 0$ where $\rho_1 > \sigma_1$. Now, defining r^* such that $\rho_{r^*} > \sigma_{s-1}$ and $\rho_{r^*}+1 < \sigma_{s-1}$, and using Lemma 2.7 in [8] one has

$$\Delta_{[\sigma_{s-1},\infty]}\zeta = \pi\mathrm{sgn}R(\rho_1+0)\sum_{k=1}^{r^*}(-1)^{k+1}\mathrm{sgn}S(\rho_k) \tag{19}$$

Also, we note that

$$\lim_{\varepsilon\to 0}\Delta_{[\varepsilon,\sigma_{s-1}]}\zeta = \lim_{\varepsilon\to 0}\zeta(\varepsilon) - \zeta(\sigma_{s-1}), \tag{20}$$

or,

$$\lim_{\varepsilon\to 0}\Delta_{[\varepsilon,\sigma_{s-1}]}\zeta] = \lim_{\varepsilon\to 0}\arctan(\frac{S(\varepsilon)}{R(\varepsilon)}) - \zeta(\sigma_{s-1}). \tag{21}$$

This gives the final formula for the total number of zeros in the right hand plane

$$N = \frac{n}{2} - \frac{1}{2} + \mathrm{sgn}R(\rho_1+0)\sum_{k=1}^{r^*}(-1)^{k+1}\mathrm{sgn}S(\rho_k) + \dots \\ \frac{1}{\pi}(\lim_{\varepsilon\to 0}\arctan(\frac{S(\varepsilon)}{R(\varepsilon)}) - \zeta(\sigma_{s-1})). \tag{22}$$

A. Example

Consider a one-dimensional formation of three vehicles that has dynamics given by

$$\dot{x}_i = \mathcal{A}x_i + \mathcal{B}u_i; \quad x_i \in R^2 \tag{23}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{24}$$

and

$$\mathcal{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{25}$$

The control u_i is defined through

$$u_i = \sum_{j\in J_i} b_{ij}(x_i - x_j(t-\tau)), \tag{26}$$

where each b_{ij} is a 1×2 matrix.

Defining the formation-wide state vector by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{27}$$

the overall dynamic system can be written

$$\dot{x} = Ax + Bx(t-\tau) \tag{28}$$

where the matrices A and B are defined through

$$A = \left[\begin{matrix} 0 & 1 & 0 \\ \sum_{j\in J_1} b_{1j}(1) & \sum_{j\in J_1} b_{1j}(2) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sum_{j\in J_2} b_{2j}(1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}\right. \cdots \\ \left.\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ \sum_{j\in J_2} b_{2j}(2) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \sum_{j\in J_3} b_{3j}(1) & \sum_{j\in J_3} b_{3j}(2) \end{matrix}\right] \tag{29}$$

and

$$B = \left[\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -b_{12}(1) \\ 0 & 0 & 0 \\ -b_{21}(1) & -b_{21}(2) & 0 \\ 0 & 0 & 0 \\ -b_{31}(1) & -b_{31}(2) & -b_{32}(1) \end{matrix}\right. \cdots \\ \left.\begin{matrix} 0 & 0 & 0 \\ -b_{12}(2) & -b_{13}(1) & -b_{13}(2) \\ 0 & 0 & 0 \\ 0 & -b_{23}(1) & -b_{23}(2) \\ 0 & 0 & 0 \\ -b_{32}(2) & 0 & 0 \end{matrix}\right] \tag{30}$$

For the above system $D(s)$ will take the form

$$D(s) = \det(sI - A - Be^{-\tau s}). \tag{31}$$

Thus the determinant will have a parametric dependence on the time delay τ. Denoting the real and imaginary parts of $D(i\omega)$ by $R(\omega;\tau)$ and $S(\omega;\tau)$, respectively, (18) can now be used to determine the values of τ for which $N = 0$.

It is straightforward to show (through numerical calculations) that $\frac{1}{\pi}(\lim_{\varepsilon\to 0}\frac{S(\varepsilon;\tau)}{R(\varepsilon;\tau)} - \zeta(\sigma_{s-1})) = -\frac{1}{2}$ for all τ. Then, with m=3, the condition for $N = 0$ is that

$$\mathrm{sgn}R(\rho_1 + 0;\tau)\sum_{k=1}^{r^*=2}(-1)^{k+1}\mathrm{sgn}S(\rho_k;\tau) = 2 \tag{32}$$

To study this last condition contour plots of $S(\omega;\tau)$ and $R(\omega;\tau)$ in the $\omega\tau$-plane are used as shown in Figure 1. $S(\omega;\tau) = 0$ on the dashed curve and $R(\omega;\tau) = 0$ on the solid curves. Both $S(\omega;\tau)$ and $R(\omega;\tau)$ are also zero for $\omega = 0$ for all values of τ. In addition $R(\omega\tau) < 0$ in the shaded region and $R(\omega\tau) > 0$ in the non-shaded region. It then follows directly from (32) that $D(i\omega)$ is stable for all $\tau < 1.5$.

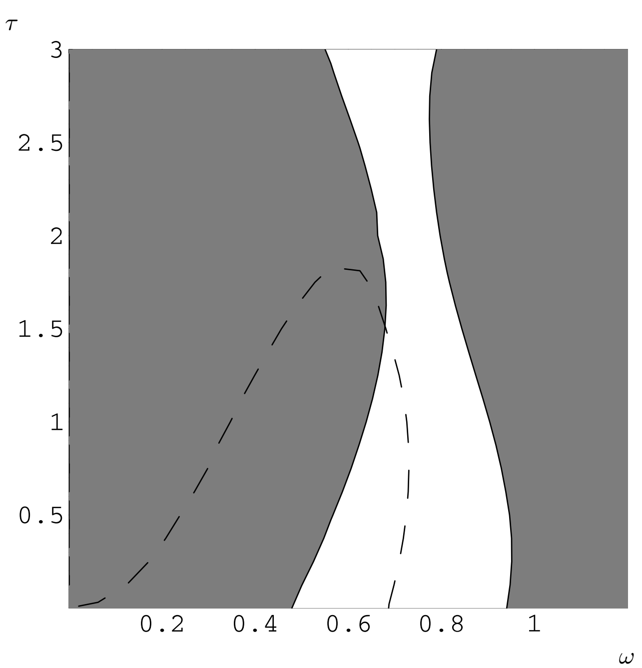

Fig. 1. Plots of $R(\omega) = 0$ and $S(\omega) = 0$ in $\tau\omega$-plane. The system is unstable for $\tau > 2.2$

As an illustration of the result above, the x-coordinates of the vehicles as functions of time are shown in Figure 2 for the case $\tau = 0.5$, i.e. the stable case. In Figure 3, the same x-coordinates are shown for the case $\tau = 1.6$which is unstable as expected. The initial values used for the $x-$coordinates of the vehicles are in both cases $x_1(0) = 1$, $\dot{x}_1(0) = -0.1$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$, $x_3(0) = 0$, $\dot{x}_3(0) = 0.1$.

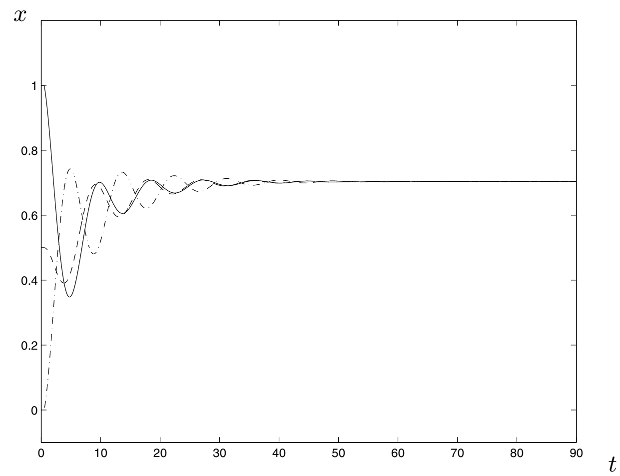

Fig. 2. Plots of x_1 (solid) and x_2 (dashed) and x_3 (dashed-dotted as functions of time for $\tau = .5$

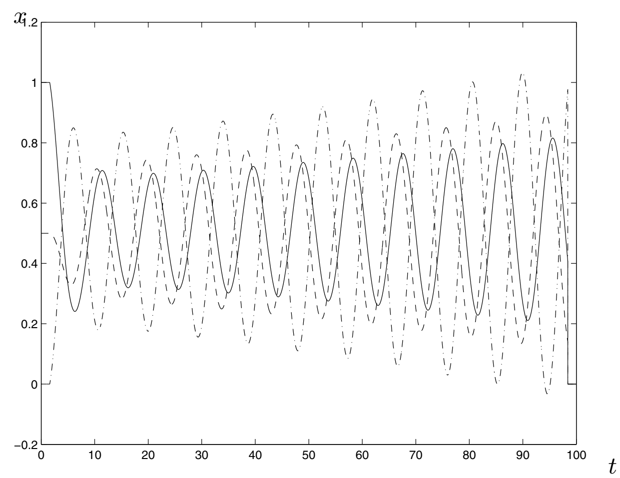

Fig. 3. Plots of x_1 (solid) and x_2 (dashed) and x_3 (dashed-dotted) as functions of time for $\tau = 1.6$

IV. INDIRECT STABILITY DETERMINATION: NYQUIST CRITERION

It is shown in [2] how the stability of a formation may be determined indirectly after a Schur transformation of the original linear dynamic system. In the case where there are delays in the information transfer this cannot be done directly. However a similar method can be used on the Laplace-transformed system. This approach is the subject of this section.

A Laplace transform of (8) gives

$$sIX = (\widehat{A} + B\widehat{K_1C_1})X + \widehat{BH}V + BK_2C_2LX \quad (33)$$
$$sIV = \widehat{G_1C_1}X + \widehat{F}V + \widehat{G_2C_2}LX \quad (34)$$

where LX is the Laplace transform of $\bar{L}x$.

To further analyze (34) we define the "delay-sensitive" adjacency matrix $A^{(\tau)}$ such that $A^{(\tau)}_{ij} = e^{\tau_{ij}}$ if there is an arc between nodes i and j and and τ_{ij} is the delay in the information transfer along that arc, and $A^{(\tau)}_{ij} = 0$ otherwise. Now let D be the matrix of the in-degree of each vertex along the diagonal. The delay-sensitive Laplacian can be defined through

$$L^{(\tau)} = I - D^{-1}A^{(\tau)} \quad (35)$$

Note that with the above definition of the delay sensitive Laplacian, the matrix L in (34) is a dimensional extension of $L^{(\tau)}$, i.e. $L = L^{(\tau)}_{(n)}$. The dimensional extension of an $N \times N$ matrix A denoted by $A_{(n)}$ is defined such that $A_{(n)}$ is an $n \times n$ matrix of $N \times N$ diagonal matrices. The diagonal elements of component matrix ij of A_n are equal to A_{ij}.

Next, denoting the Schur transformation of of $L^{(\tau)}$ by T, i.e.

$$L^{(\tau)} = TUT^{-1} \quad (36)$$

where U is upper triangular with the eigenvalues of $L^{(\tau)}$ along the diagonal, and using *Lemma* 2 in [1], it follows that

$$\begin{bmatrix} sI\tilde{X} \\ sI\tilde{V} \end{bmatrix} = \begin{bmatrix} \widehat{A} + B\widehat{K_1}C_1 + B\widehat{K_2}C_2U_{(n)} & \widehat{BH} \\ \widehat{G_1C_1} + \widehat{G_2C_2}U_{(n)} & \widehat{F} \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{V} \end{bmatrix} \quad (37)$$

where $\tilde{X} = T^{-1}_{(n)}X$ and $\tilde{V} = T^{-1}_{(m)}V$ All the terms in the above coefficient matrix are either block diagonal or upper triangular. Stability of the system therefore only depends only on the diagonal elements of U, i.e. the eigenvalues of L. Introducing the diagonal matrix λ consisting of the eigenvalues of $L^{(\tau)}$ one can conclude that the stability of (37) is equivalent to the stability of the system

$$\begin{bmatrix} sI\tilde{X} \\ sI\tilde{V} \end{bmatrix} = \begin{bmatrix} \widehat{A} + B\widehat{K_1C_1} + B\widehat{K_2C_2}\lambda & \widehat{BH} \\ \widehat{G_1C_1} + \widehat{G_2C_2}\lambda & \widehat{F} \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{V} \end{bmatrix} \quad (38)$$

Note that this last system is stable if for each i, the system

$$\begin{bmatrix} sI\tilde{X}_i \\ sI\tilde{V}_i \end{bmatrix} = \begin{bmatrix} \widehat{A} + B\widehat{J_1C_1} + B\widehat{J_2C_2}\lambda_i & \widehat{BH} \\ \widehat{G_1C_1} + \widehat{G_2C_2}\lambda_i & \widehat{F} \end{bmatrix} \begin{bmatrix} \tilde{X}_i \\ \tilde{V}_i \end{bmatrix} \quad (39)$$

is stable.

A. Example

The Schur transformation of L in (34) generally leads to a complicated set of equations that is not readily solved. For that reason we study, as an example, the stability of a one-dimensional formation of two vehicles with dynamics given by

$$\dot{x} = Ax + Bx(t - \tau) \quad (40)$$

where the right hand side of (40) corresponds to the term $\widehat{BK_2C_2}\bar{L}$ in (8), i.e. all other terms in that equation are zero. We assume that the matrices A and B have the form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_1 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & k_2 & d_2 \end{bmatrix} \quad (41)$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -k_1 & -d_1 \\ 0 & 0 & 1 & 0 \\ -k_2 & -d_2 & 0 & 0 \end{bmatrix} \quad (42)$$

Note that in this case the procedure described above involving the Schur transformation of L is equivalent to the diagonalization of the coefficient matrix of X on the right hand side of the Laplace transformed system. The diagonal elements can subsequently be studied from the point of view of the Nyquist Stability Criterion. The four Nyquist diagrams for the system are shown in Figure 4. The value of τ used is 1.95 i.e. just above the bifurcation value which is reflected by the fact that one branch of the diagram just encircles $s = -1$.

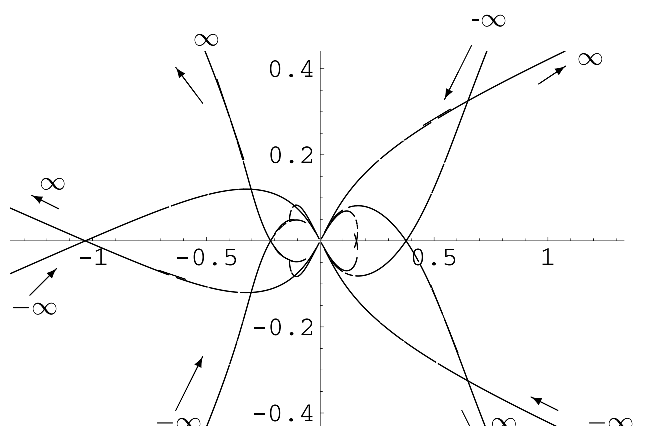

Fig. 4. Nyquist plots of the four "eigensystems" for $\tau = 1.95$. (The arrows show direction of motion with increasing ω.)

We also use the directed method described in section 2 to determine the stability region for this problem. The characteristic equation is given by

$$D(s, \tau) = Det[sI - A - Be^{-s\tau}] \tag{43}$$

An evaluation of the determinant and its real and imaginary parts and the use of (22) now allows us to obtain the contour plot shown in Figure 5. From Figure 5 we conclude

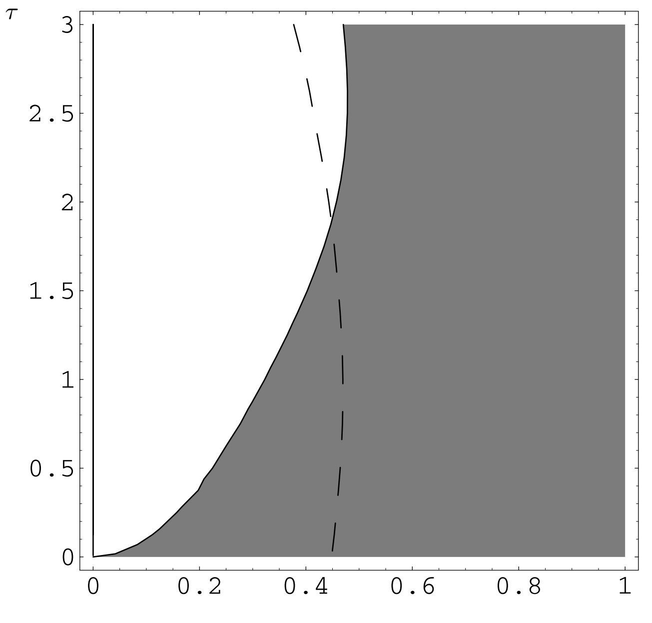

Fig. 5. Contour plots of $R(\omega) = 0$ (dashed) and $S(\omega) = 0$ (solid). $S(\omega)$¡0 in the dashed region.

that the system is stable for $\tau < 1.95$.

V. CONCLUSIONS

The amount of information flow between the vehicles in a formation, i.e. for example how many neighbors a given vehicle has, how often the information is updated, whether there are delays in the information, etc. is crucial in determining the stability of a formation. On the other hand, the design of the information flow itself may need to obey certain constraints, e.g. the maximum number of vehicles a given vehicle may communicate with, the rate of update of information etc. This may be due to band-width constraints, power constraints, or even for the sake of minimizing the risk of interception of communication. For these reasons it is important to study the interplay between the information flow, the dynamics of the vehicles, and the stability of the formation. A good understanding of this interplay allows for the design of efficient communication channels that furnish just the right amount of information to each vehicle.

The results in this paper, regarding the determination of the maximum amount of allowable delay that does not lead to instability, can be used in the design of the information flow architecture of a formation.

It is apparent, however, that the calculations involved to get the allowable delay as presented in this paper (though straightforward in principle) are very complicated in implementation. Additional work to study other possible approaches for the determination of the allowable delay would therefore be of interest.

Another issue that has not been addressed in this paper is that the delay in information exchange between vehicles may not be uniform, i.e. different pairs of vehicles may experience different amounts of delay in their respective communications. This is also a subject that requires further study.

VI. ACKNOWLEDGEMENTS

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