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# Ultimate Boundedness Sets for Continuous-time Linear Systems with Deadzone Feedback Controls

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*Abstract*— **This paper is concerned with the construction of positively invariant convex polyhedral uniform ultimate boundedness sets for linear continuous-time systems with stabilizing deadzone feedback control laws. The objective is delimitation and region of attraction estimation of possible limit cycles around origin of open-loop unstable systems. Limit cycle delimitation is performed via construction of a positively invariant convex compact polyhedral estimate of the minimal positively invariant set containing an arbitrarily small neighborhood of origin. Region of attraction estimation is performed via construction of a piecewise-affine Lyapunov function assuring uniform ultimate boundedness in the above mentioned convex positively invariant polyhedral set.**

## I. Introduction

Deadzone nonlinearities may occur in various components of control systems, including sensors, amplifiers and actuators. Possible undesirable effects of deadzones in control systems are decrease of static output accuracy and system instability [1]. The stability of linear time-invariant systems in feedback interconection with deadzone nonlinearities have been widely studied in literature, considering mostly single-input single-output systems and frequency-domain techniques [1], [2], [3], [4]. The feedback interconnection of linear time-invariant systems and deadzone nonlinearities results in piecewise-linear closed-loop systems with possibly multiple equilibrium points and/or limit cycles. For stable closed-loop systems which are open-loop unstable, their trajectories will not converge asymptotically to origin inside deadzone but they will converge to nearby stable equilibrium points or limit cycles. The affine structure of the closed-loop system makes the determination and stability verification of the equilibrium points a simple task. Conversely, the determination and stability analysis of the limit cycles cannot be easily performed.

This paper is concerned with the construction of positively invariant convex polyhedral uniform ultimate boundedness sets [5] for linear continuous-time systems with stabilizing deadzone feedback control laws. The objective is delimitation and region of attraction estimation of possible limit cycles around origin of open-loop unstable systems. The results presented here are a continuous-time extension of discrete-time ones summarily presented in [6].

## II. Preliminaries

Throughout this paper: for two $n \times m$ real matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A \leq B$ is equivalent to $a_{ij} \leq b_{ij}$ for all $i$, $j$ such that $1 \leq i \leq n$ and $1 \leq j \leq m$. For any real $\epsilon \geq 0$, the set $\epsilon\Omega$ is defined as $\{x = \epsilon y, y \in \Omega\}$.

Consider the continuous-time system

$$\dot{x}(t) = f(x(t)) \tag{1}$$

where function $f : \Re^n \rightarrow \Re^n$ is possibly nonlinear and $f(0) = 0$.

**Definition 2.1:** [7] The upper Dini derivative of a locally Lipschitz function $\Psi : \Re^n \rightarrow \Re$ [8] w.r.t. system (1) is given by:

$$D^+\Psi(x) = \limsup_{\tau \rightarrow 0^+} \frac{\Psi(x + \tau f(x)) - \Psi(x)}{\tau} \tag{2}$$

**Definition 2.2:** A convex compact set $\Omega$, containing the origin in its interior, defines the Minkowski function $\Psi_\Omega : \Re^n \rightarrow \Re$:

$$\Psi_\Omega(x) = inf\{\mu \geq 0 : x \in \mu\Omega\} \tag{3}$$

The Minkowski function is a locally Lipschitz convex function, positively homogeneous of order one [9]:

$$\Psi_\Omega(\xi x) = \xi \Psi_\Omega(x) \ , \ \xi \geq 0 \tag{4}$$

**Definition 2.3:** A set $\Omega$ is positively invariant w.r.t. system (1) if $x(t) \in \Omega$ for all $x(0) \in \Omega$ and $t > 0$. A convex compact set $\Omega$, containing the origin in its interior, has contraction index $\beta \geq 0$ if the upper Dini derivative (2) of its Minkowski function $\Psi_\Omega(x)$ (3) satisfies:

$$D^+\Psi_\Omega(x) \leq -\beta \ \forall \ x \in \partial\Omega \tag{5}$$

Consider the discrete-time Euler Approximating Systems (EAS) of system (1):

$$x(t+1) = x(t) + \tau f(x(t)) \quad , \quad \tau > 0. \tag{6}$$

where $\tau$ is the step size.

**Lemma 2.1:** A locally Lipschitz convex function $\Psi : \Re^n \rightarrow \Re$ has the following properties:

**(i)** The upper Dini derivative w.r.t. system (1) satisfies:

$$D^+\Psi(x) \leq \frac{\Psi(x + \tau f(x)) - \Psi(x)}{\tau} \ , \ \tau > 0$$

**(ii)** If $\Omega \in \Re^n$ is convex compact, $\Psi(x)$ is positive definite and system (1) is affine ($f(x)$ affine), then $\tilde{\Psi}(x) = \Psi(x + \tau f(x))$ is convex and hence:

$$\max_{x \in \Omega} \tilde{\Psi}(x) = \max_{x \in \partial\Omega} \tilde{\Psi}(x)$$

**Proof:** Can be performed using known properties of convex functions and convex programming [10].□

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**Definition 2.4:** The one-step set ahead from set $\Omega$ w.r.t. EAS (6) is given by:

$$\mathcal{T}(\Omega,\tau)=\{x+\tau f(x)\in\Re^n : x\in\Omega\} \tag{7}$$

**Proposition 2.1:** A convex compact set $\Omega$ containing the origin in its interior is contractive w.r.t. an affine system (1), with contraction index $\beta\geq 0$, iff there is $\tau>0$ such that:

$$\mathcal{T}(\Omega,\tau)\subset(1-\tau\beta)\Omega \tag{8}$$

**Proof:** Definition 2.3 gives:

$$D^+\Psi_\Omega(x)\leq-\beta\ \forall\ x\in\partial\Omega \tag{9}$$

Noting that $\Psi_\Omega(x)=1$ for $x\in\partial\Omega$ and applying Lemma 2.1 (i), it can be verified from (9) that

$$\Psi_\Omega(x+\tau f(x))\leq 1-\tau\beta\ ,\ \forall\ x\in\partial\Omega \tag{10}$$

must hold for some $\tau>0$. Using Lemma 2.1 (ii), (10) can be equivalently stated as:

$$\Psi_\Omega(x+\tau f(x))\leq 1-\tau\beta\ ,\ \forall\ x\in\Omega \tag{11}$$

Using Definition 2.2 and the positive homogeneity property of $\Psi_\Omega(x)$ (4), it can be verified that (11) corresponds to:

$$x+\tau f(x)\in(1-\tau\beta)\Omega\ ,\ \forall x\in\Omega$$

which, from Definition 2.4, is equivalent to (8), concluding the proof. □

Let $\mathcal{S}_I$ be a non empty family of all positively invariant sets $\Omega$ w.r.t. affine system (1), with a contraction rate $\beta$, containing a closed set $\Gamma$.

**Proposition 2.2:** The family $\mathcal{S}_I$ has a minimum element given by the limit solution of recurrence:

$$\Omega_{i+1}=\Omega_i\cup(1-\tau\beta)^{-1}\mathcal{T}(\Omega_i,\tau)\ \ ;\ \ \Omega_0=\Gamma \tag{12}$$

where $\tau>0$ must be chosen sufficiently small.

**Proof:** Applying Definition 2.3, it can be verified that $\mathcal{S}_I$ is closed under the operation of set intersection. Hence, it must have a minimum element given by the intersection of all family members. It can be verified that the limit solution of (12) is an element of $\mathcal{S}_I$: it contains $\Gamma$ and Proposition 2.1 assures that it is positively invariant because it is obtained when $\Omega_i=\Omega_i\cup(1-\tau\beta)^{-1}\mathcal{T}(\Omega_i,\tau)$, which corresponds to $\mathcal{T}(\Omega_i,\tau)\subset(1-\tau\beta)\Omega_i$. Given any positively invariant set $\Omega\supset\Gamma$, it can be verified that: $\Omega_1\subset\Omega$ and if $\Omega_i\subset\Omega$ then $\Omega_{i+1}\subset\Omega$. So, by induction, it can be concluded that $\Omega_i\subset\Omega$ holds for all $i$ and consequently the limit of (12) is contained in any $\Omega\in\mathcal{S}_I$, being, hence, its minimum element. □

**Definition 2.5:** A closed set $\Omega\subset\Omega_0$ defined in state space of system (1) is a local uniform ultimate boundedness set in $\Omega_0$ w.r.t. system (1) if for every initial condition $x(0)\in\Omega_0$, there is a $t_0$ such that $x(t)\in\Omega$ for all $t\geq t_0$. Set $\Omega_0$ is a region of attraction of $\Omega$. If $\Omega_0$ is the whole state space of (1), one has global uniform ultimate boundedness.

Let $\mathcal{S}_U$ be the family of all local uniform ultimate boundedness sets in $\Omega_0$ w.r.t. system (1). If $\mathcal{S}_U$ is non empty, it can be verified from Definition 2.5 that it has a minimum element given by the intersection of all family members.

For a locally Lipschitz function $\Psi:\Re^n\to\Re$ consider the possibly empty annular set

$$\mathcal{B}(\Psi,\check{b},\hat{b})=\{x\in\Re^n : \check{b}\leq\Psi(x)\leq\hat{b}\} \tag{13}$$

where $0\leq\check{b}\leq\hat{b}$. In the sequel we use also the same notation with a single argument for the ball set

$$\mathcal{B}(\Psi,\hat{b})=\mathcal{B}(\Psi,0,\hat{b})=\{x\in\Re^n : \Psi(x)\leq\hat{b}\} \tag{14}$$

Consider also the $\epsilon$ level set of the function $\Psi:\Re^n\to\Re$ given by

$$\partial\mathcal{B}(\Psi,\epsilon)=\{x\in\Re^n : \Psi(x)=\epsilon\}, \tag{15}$$

**Definition 2.6:**[9] A positive definite locally Lipschitz function $\Psi:\Re^n\to\Re$ is a Lyapunov function in strong sense, for time-invariant continuous-time system (1) in $\mathcal{B}[\Psi,\check{b},\hat{b}]$ (13), if for all $x\in\mathcal{B}[\Psi,\check{b},\hat{b}]$, the upper Dini derivative (2) satisfies:

$$D^+\Psi(x)\leq-\beta\Psi(x), \tag{16}$$

for some $\beta>0$. The following remarks on Definition 2.6 are opportune:

- the balls $\mathcal{B}(\Psi,\epsilon)$, $\check{b}\leq\epsilon\leq\hat{b}$ are contractive sets with contraction index $\beta>0$;
- for $\check{b}=0$, $\mathcal{B}[\Psi,\hat{b}]$ is a region of asymptotic stability of the origin with convergence rate $\beta$;
- for $\check{b}>0$, it is assured uniform ultimate boundedness of (1) in the ball $\mathcal{B}[\Psi,\check{b}]$, with convergence rate $\beta$ and a region of attraction $\mathcal{B}[\Psi,\hat{b}]$;
- the asymptotic stability of origin and uniform ultimate boundedness are local if $\hat{b}$ is finite and they are global if $\hat{b}\to\infty$ and $\Psi(x)$ is radially unbounded [3].

**Proposition 2.3:** A positive definite locally Lipschitz function $\Psi(x)$ is a Lyapunov function in strong sense, for system (1) in $\mathcal{B}[\Psi,\check{b},\hat{b}]$ (13), iff there is a $\beta>0$ such that:

$$D^+\Psi(x)\leq-\beta\epsilon\ \ \forall x\in\partial\mathcal{B}(\Psi,\epsilon)\ ,\ \check{b}\leq\epsilon\leq\hat{b}. \tag{17}$$

**Proof:** Noting that $\Psi(x)=\epsilon$ for $x\in\partial\mathcal{B}(\Psi,\epsilon)$ and that $\forall x\in\partial\mathcal{B}(\Psi,\epsilon)\ ,\ \check{b}\leq\epsilon\leq\hat{b}$ is equivalent to $\forall x\in\mathcal{B}[\Psi,\check{b},\hat{b}]$, the proof is immediate from Definition 2.6. □

**Definition 2.7:** The one-step admissible set to $\mathcal{B}(\Psi,\epsilon)$ w.r.t. EAS (6) is given by:

$$\mathcal{B}(\Psi_f,\epsilon,\tau)=\{x:\ \ \Psi(x+\tau f(x))\leq\epsilon\} \tag{18}$$

**Proposition 2.4:** A convex positive definite locally Lipschitz function $\Psi(x)$ is a Lyapunov function in strong sense, for an affine system (1) in $\mathcal{B}[\Psi,\check{b},\hat{b}]$ (13), iff there are $\beta>0$ and $\tau>0$ such that:

$$\mathcal{B}(\Psi,\epsilon)\subset\mathcal{B}(\Psi_f,(1-\tau\beta)\epsilon,\tau)\ \ \forall\ \ \check{b}\leq\epsilon\leq\hat{b} \tag{19}$$

**Proof:** Definition 2.6 gives:

$$D^+\Psi(x)\leq-\beta\Psi(x)\ \ \forall x\in\ \mathcal{B}[\Psi,\check{b},\hat{b}] \tag{20}$$

Applying Lemma 2.1 (i) in (20), gives:

$$\Psi(x+\tau f(x))\leq(1-\tau\beta)\Psi(x)\ \ \forall x\in\ \mathcal{B}[\Psi,\check{b},\hat{b}] \tag{21}$$

Using Lemma 2.1 (ii), from (14) and Definition 2.7, it can be verified that (21) is equivalent to (19). □

## III. DEADZONE FEEDBACK CONTROL MODEL

Consider the continuous-time linear system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{22}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and control variables, respectively, $A$, $B$, are real and of appropriate dimensions. Assume the deadzone nonlinear feedback control law

$$u(t) = dz(Fx(t)) \tag{23}$$

where $F \in \Re^{m\times n}$ is constant. The components of $dz(Fx)$ are given by:

$$dz(Fx)_i = \begin{cases} f_i x + \check{u}_i & if \quad f_i x < -\check{u}_i \\ 0 & if \quad -\check{u}_i \le f_i x \le \hat{u}_i \\ f_i x - \hat{u}_i & if \quad f_i x > \hat{u}_i \end{cases} \tag{24}$$

where: $f_i$ denotes the ith row of matrix $F$ and $\check{u}_i$, $\hat{u}_i$, denote ith rows of deadzone function lower and upper bound vectors $\check{u} \ge 0$, $\hat{u} \ge 0$, respectively.

From (22), (23), the closed-loop system is given by nonlinear model:

$$\dot{x}(t) = Ax(t) + Bdz(Fx(t)) \tag{25}$$

Considering all $x \in \Re^n$, each one of the $m$ components of the deadzone law (24) has 3 possible states: below deadzone, inside deadzone and above deadzone. Consequently, $\Re^n$ can be decomposed into $j = 1 : 3^m$ regions $S(R_j, d_j) \subset \Re^n$, denoted as deadzone regions given by polyhedra of the form:

$$S(R_j, d_j) = \{x \in \Re^n; R_j x \le d_j\} \tag{26}$$

$$R_j = \begin{bmatrix} F_{iz}^j \\ -F_{iz}^j \\ -F_{az}^j \\ F_{bz}^j \end{bmatrix} \;;\; d_j = \begin{bmatrix} \hat{u}_{iz}^j \\ \check{u}_{iz}^j \\ -\hat{u}_{az}^j \\ -\check{u}_{bz}^j \end{bmatrix} \tag{27}$$

where $F_{iz}^j$, $\hat{u}_{iz}^j$, $\check{u}_{iz}^j$, $F_{az}^j$, $\hat{u}_{az}^j$, $F_{bz}^j$, $\check{u}_{bz}^j$, denote matrices and vectors appropriately formed by the rows of $F$, $\hat{u}$, $\check{u}$, related, respectively, to the components inside deadzone, above deadzone and below deadzone, which characterize jth region.

Within each deadzone region $S(R_j, d_j)$, closed-loop system (25) can be represented by an affine model:

$$\begin{aligned} \dot{x}(t) &= A_j x(t) + p_j \\ A_j &= [A + B_{az}^j F_{az}^j + B_{bz}^j F_{bz}^j] \\ p_j &= -B_{az}^j \hat{u}_{az}^j + B_{bz}^j \check{u}_{bz}^j \end{aligned} \tag{28}$$

where $B_{az}^j$ and $B_{bz}^j$ denote matrices appropriately formed by the columns of $B$ related to $F_{az}^j$ and $F_{bz}^j$, respectively. Throughout the paper, it will be assigned $j = 1$ for the region inside deadzone, described by:

$$R_1 = \begin{bmatrix} F \\ -F \end{bmatrix} \;;\; d_1 = \begin{bmatrix} \hat{u} \\ \check{u} \end{bmatrix} \;;\; A_1 = A \;;\; p_1 = 0 \tag{29}$$

## IV. POLYHEDRAL POSITIVELY INVARIANT SETS

Consider the polyhedron

$$\Omega(G, w) = \{x \in \Re^n : Gx \le w\} \tag{30}$$

where $G \in \Re^{r\times n}$ and $w > 0 \in \Re^r$.

According to Definition 2.2, $\Omega(G, w)$ (30) defines the Minkowski function

$$\Psi_\Omega(x) = inf\{\epsilon \ge 0 : x \in Gx \le \epsilon w\} \tag{31}$$

which can be also equivalently stated as:

$$\Psi_\Omega(x) = \max_{1\le i\le r} w_i^{-1} g_i x, \tag{32}$$

**Proposition 4.1:** A bounded polyhedron $\Omega(G, w)$ (30) is positively invariant w.r.t. closed-loop system (25) with contraction index $\beta > 0$ iff for the $i = 1 : r$ facets of $\Omega(G, w)$ and for the $j = 1 : 3^m$ deadzone regions (26), (27), (28):

$$g_i A_j x \le -\beta w_i \tag{33}$$

holds for any $x$ satisfying:

$$\begin{aligned} & g_i x = w_i \\ & g_l x \le w_l \;,\; l = 1 : r \;,\; l \ne i \\ & R_j x \le d_j \end{aligned} \tag{34}$$

**Proof:** From Definition 2.3,

$$D^+\Psi_\Omega(x) \le -\beta \tag{35}$$

must hold $\forall x \in \partial\Omega(G, w)$. From (30) and considering the $j = 1 : 3^m$ deadzone regions $S(R_j, d_j)$ (26), (27), it can be verified that $\partial\Omega(G, w)$ is given by:

$$\partial\Omega(G, w) = \bigcup_{i=1}^{i=r} \bigcup_{j=1}^{j=3^m} \partial_i^j \Omega(G, w) \tag{36}$$

where $\partial_i^j \Omega(G, w)$ is given by:

$$\begin{aligned} & g_i x = w_i \\ & g_l x \le w_l \;,\; l = 1 : r \;,\; l \ne i \\ & R_j x \le d_j \end{aligned} \tag{37}$$

From (2), (28), (32), (37), for $x \in \partial_i^j \Omega(G, w)$, (35) is given by:

$$g_i A_j x \le \beta w_i \tag{38}$$

From (36), (37), it can be verified that (35), (38) hold $\forall x \in \partial\Omega(G, w)$ iff (38) hold for all $x$ satisfying (37) for $i = 1 : r$ and $j = 1 : 3^m$, concluding the proof. □

A primal linear programming (LP) formulation to Proposition 4.1 is given by:

$$\max_{i,j}\{\sigma_j^i\} \le 0 \;;\; 1 \le i \le r \;;\; 1 \le j \le 3^m \tag{39}$$

where $\sigma_j^i$ are obtained solving the following independent feasible linear programs:

$$\sigma_j^i = \max_x g_i A_j x + \beta w_i \;\; s.t. \; (34)$$

Consider the discrete-time Euler Approximating Systems (EAS) of closed-loop system (25):

$$x(t+1) = (I+\tau A)x(t) + \tau B dz(Fx(t)) \quad ; \quad \tau > 0 \quad (40)$$

Let $T(G,w,\tau)$ be the one step set ahead from $\Omega(G,w)$ (30) w.r.t. system (40):

$$T(G,w,\tau) = \{(I+\tau A)x + \tau B dz(Fx) \in \Re^n; \\ x \in \Omega(G,w)\} \quad (41)$$

Considering the $j = 1 : 3^m$ deadzone regions (26), (27), (28), it can be verified that set $T(G,w,\tau)$ (41) is given by:

$$T(G,w,\tau) = \bigcup_{j=1}^{j=3^m} T_j(G,w,\tau) \quad (42)$$

where $T_j(G,w,\tau) \subset \Re^n$ is the one step set ahead from $\Omega(G,w) \cap S(R_j,dj)$ given by polyhedron:

$$\begin{bmatrix} G \\ R_j \end{bmatrix} (I+\tau A_j)^{-1} x \leq \begin{bmatrix} w \\ d_j \end{bmatrix} + \begin{bmatrix} G \\ R_j \end{bmatrix} (I+\tau A_j)^{-1} \tau p_j \quad (43)$$

where, without loss of generality, matrices $I + \tau A_j$ are assumed non singular.

For a system (25), with unstable origin $x = 0$ but stable overall system ($A + BF$ Hurwitz), the trajectories starting inside deadzone region will converge to limit cycles or stable equilibrium points in nearby deadzone regions. In this case, Proposition 2.2 assures the existence of a minimal positively invariant set containing an arbitrarily small compact neighborhood of origin. If this minimal positively invariant set do not contain any stable equilibrium point of (25), it can be concluded that it contains a limit cycle. In other words, it delimits a limit cycle. Based on Propositions 2.2 and 4.1, the following procedure gives a close convex external estimate of the minimal convex positively invariant polyhedron w.r.t. system (25).

**Procedure 4.1:** Estimation of the minimal positively invariant polyhedron w.r.t. system (25) containing an arbitrarily small neighborhood of origin.

1) Define data:
   $\beta$ : contraction index , $\tau$: step size
   $\Omega(G,w)$ : initial arbitrarily small bounded closed polyhedron containing origin $(w > 0)$
   itmax : maximum number of iterations , Set $it = 0$
2) Iterate:
   a) Compute one-step ahead sets $T_j(G,w,\tau)$ (42), (43).
   b) Check if $\Omega(G,w)$ is positively invariant with contraction index $\beta$ using Proposition 4.1:
      - Yes: **Stop** : $\Omega(G,w)$ is desired polyhedron
      - No: Set $it = it + 1$
        if $it > itmax$: **Stop** - convergence failed
   c) Compute convex polyhedral approximation $Cvp$:

   $$Cvp \approx \left[ \bigcup_{j=1}^{j=3^m} T_j(G,w,\tau) \right] \bigcup \Omega(G,w)$$

   d) Set $\Omega(G,w) = Cvp$ ; Return to step 2a.

The computational efficiency and precision of Procedure 4.1 are strongly dependent on the step size $\tau$ specified in step 1 and on the polyhedral approximation $Cvp$ used in step 2c. The step size $\tau$ should be sufficiently small to assure that EAS (40) is a good discrete-time approximate of system $\dot{x} = (A + BF)x$. The ideal $Cvp$ is the convex hull of the union of polyhedra in step 2c, which is known to be a very hard NP computational problem. A fairly good compromise solution between computational efficiency and precision can be obtained taking $Cvp$ as the minimal polyhedron $\Omega(G,\rho)$ such that:

$$\Omega(G,\rho) \supset \left[ \bigcup_{j=1}^{j=3^m} T_j(G,w,\tau) \right] \bigcup \Omega(G,w) \quad (44)$$

In this case, Procedure 4.1 progressively expands the initial polyhedron $\Omega(G,w)$ changing only its right hand vector $w$. The precision of $Cvp$ increases with the number of facets allowed to $\Omega(G,w)$.

## V. Piecewise-affine Lyapunov functions

Consider the piecewise-affine (PWA) function:

$$\Psi(x) = \max_{1 \leq i \leq r} w_i^{-1} \{g_i x + c_i\} \quad (45)$$

where $x \in \Re^n$ and $w_i$, $g_i$, $c_i$ are ith rows of $w > 0 \in \Re^r$, $G \in \Re^{r \times n}$, $c \leq 0 \in \Re^r$, respectively. Function $\Psi(x)$ (45) can also be defined as:

$$\Psi(x) = \min_{\epsilon \in \Re} \epsilon \quad s.t. \quad Gx + c \leq w\epsilon \quad (46)$$

It can be verified that PWA function $\Psi(x)$ (45), (46) is locally Lipschitz, convex and, under mild conditions, positive definite and radially unbounded [10], [11].

It can be verified that the annular set (13), the $\epsilon$ ball set (14) and the $\epsilon$ level set (15) of $\Psi(x)$ (45), (46) are, respectively, given by:

$$\mathcal{B}(\Psi, \check{b}, \hat{b}) = \{x \in \Re^n : \quad w\check{b} \leq Gx + c \leq w\hat{b}\} \quad (47)$$

$$\mathcal{B}(\Psi, \epsilon) = \{x \in \Re^n : \quad Gx + c \leq w\epsilon\} \quad (48)$$

$$\partial\mathcal{B}(\Psi, \epsilon) = \bigcup_{i=1}^{i=r} \partial_i \mathcal{B}_\Psi(\epsilon) \quad (49)$$

where

$$\partial_i \mathcal{B}(\Psi, \epsilon) = \{x \in \Re^n : \quad g_i x + c_i = w_i \epsilon \ ; \\ g_l x + c_l \leq w_l \epsilon \ , \ l = 1 : r \ , \ l \neq i\}$$

From Definition 2.7, the one-step admissible set to (48) w.r.t. EAS (40) is given by:

$$\mathcal{B}(\Psi_f, \epsilon, \tau) = \bigcup_{j=1}^{j=3^m} \mathcal{B}_j(\Psi_f, \epsilon, \tau) \quad (50)$$

where

$$\mathcal{B}_j(\Psi_f, \epsilon, \tau) = \{x \in S(R_j, d_j) : \\ G(I + \tau A_j)x \leq w\epsilon - G\tau p_j - c\} \quad (51)$$

**Proposition 5.1:** A positive definite PWA function $\Psi(x)$ (45), (46) is a Lyapunov function in strong sense, for closed-loop system (25) in $\mathcal{B}(\Psi, \check{b}, \hat{b})$ (47), iff for the $i = 1 : r$ components of $\Psi(x)$ and the $j = 1 : 3^m$ deadzone regions (26), (27), (28), there is a positive $\beta$ such that:

$$g_i A_j x + g_i p_j \le -\beta w_i \epsilon \quad (52)$$

holds for any $(\epsilon, x)$ satisfying:

$$\begin{array}{l} g_i x + c_i = w_i \epsilon \\ g_l x + c_l \le w_l \epsilon \ , \ l = 1 : r \ , \ l \ne i \\ R_j x \le d_j \\ \check{b} \le \epsilon \le \hat{b} \end{array} \quad (53)$$

**Proof:** From Proposition 2.3,

$$D^+ \Psi(x) \le -\beta \epsilon \quad (54)$$

must hold $\forall x \in \partial \mathcal{B}(\Psi, \epsilon)$ , $\check{b} \le \epsilon \le \hat{b}$. Considering the $j = 1 : 3^m$ deadzone regions $S(R_j, d_j)$ (26), from (49) it can be verified that $\partial \mathcal{B}(\Psi, \epsilon)$ is given by:

$$\partial \mathcal{B}(\Psi, \epsilon) = \bigcup_{i=1}^{i=r} \bigcup_{j=1}^{j=3^m} \partial_i^j \mathcal{B}(\Psi, \epsilon) \quad (55)$$

where $\partial_i^j \mathcal{B}(\Psi, \epsilon)$ is given by:

$$\begin{array}{l} g_i x + c_i = w_i \epsilon \\ g_l x + c_l \le w_l \epsilon \ , \ l = 1 : r \ , \ l \ne i \\ R_j x \le d_j \end{array} \quad (56)$$

From (2), (28), (45), (56) and for $x \in \partial_i^j \mathcal{B}_\Psi(\epsilon)$, it can be verified that (54) is given by:

$$g_i A_j x + g_i p_j \le -\beta w_i \epsilon. \quad (57)$$

From (55), (56), it can be verified that (54), (57) hold $\forall x \in \partial \mathcal{B}_\Psi(\epsilon)$ , $\check{b} \le \epsilon \le \hat{b}$ iff (57) hold for all $(x, \epsilon)$ satisfying (56) for $i = 1 : r$, $j = 1 : 3^m$, $\check{b} \le \epsilon \le \hat{b}$, concluding the proof. □

A primal LP formulation to Proposition 5.1 is given by:

$$\max_{i,j} \{\sigma_j^i\} \le 0, \quad 1 \le i \le r, \quad 1 \le j \le 3^m \quad (58)$$

where $\sigma_j^i$ are obtained solving the following independent linear programs:

$$\sigma_j^i = \max_{x, \epsilon} g_i A_j x + g_i p_j + \beta w_i \epsilon \ s.t. \ (53) \quad (59)$$

Furthermore, let $(x_j^i, \epsilon_j^i)$ be an optimal solution related to a $\sigma_j^i > 0$. This indicates that $(x_j^i, \epsilon_j^i) \in \partial_i^j \mathcal{B}(\Psi, \epsilon)$, $\check{b} \le \epsilon \le \hat{b}$ (56) is outside the ith half-space defining $\mathcal{B}_j(\Psi_f, (1 - \tau\beta)\epsilon)$ (51) at jth deadzone region:

$$\begin{array}{c} \mathcal{B}_j^i = \{(x, \epsilon) \in \Re^{n+1} \ : \\ g_i (I + \tau A_j) x - (1 - \tau\beta) w_i \epsilon \le -\tau g_i p_j - c_i\} \end{array} \quad (60)$$

Or, equivalently, $x_j^i$ is outside the $\epsilon_j^i$ ball of elementary PWA function given by:

$$\Psi^{ij}(x) = (1 - \tau\beta)^{-1} w_i^{-1} \{g_i (I + \tau A_j) x + \tau g_i p_j + c_i\}. \quad (61)$$

Consider a positive definite PWA function $\Psi(x)$ (45), its $\epsilon$ ball $\mathcal{B}(\Psi, \epsilon)$ (48) and its one-step admissible set $\mathcal{B}(\Psi_f, \epsilon, \tau)$ (50), (51) w.r.t. EAS (40) of closed-loop system (25). According to Proposition 2.4, $\Psi(x)$ is a Lyapunov function with convergence rate $\beta > 0$ iff there is $\tau > 0$ such that

$$\mathcal{B}(\Psi, \epsilon) \subset \mathcal{B}(\Psi_f, (1 - \tau\beta)\epsilon, \tau) \ \ \forall \ \ \check{b} \le \epsilon \le \hat{b} \quad (62)$$

which is equivalent to:

$$\mathcal{B}(\Psi_f, (1 - \tau\beta)\epsilon, \tau) \cap \mathcal{B}(\Psi, \epsilon) = \mathcal{B}(\Psi, \epsilon) \ \ \forall \ \ \check{b} \le \epsilon \le \hat{b} \quad (63)$$

If (63) is not satisfied, the next natural candidate should be a PWA function with $\epsilon$ ball given by $\mathcal{B}(\Psi_f, (1 - \tau\beta)\epsilon, \tau) \cap \mathcal{B}(\Psi, \epsilon)$. However, $\mathcal{B}(\Psi_f, (1 - \tau\beta)\epsilon, \tau)$ is possibly non convex, being, hence, not possible to assure the convexity of the resulting intersection set and its related PWA function. A next satisfactory convex PWA Lyapunov function candidate can be often obtained taking the PWA function with $\epsilon$ ball given by the intersection $\mathcal{B}(\Psi, \epsilon) \cap \mathcal{B}(\Psi^{ij}, \epsilon)$ where $\Psi^{ij}$ is conveniently selected among elementary PWA functions (61) given by the primal LP formulation of Proposition 5.1. From (45), (61), it can be verified that this procedure corresponds to recurrence:

$$\Psi_{k+1}(x) = max\{\Psi_k(x), \Psi_k^{ij}(x)\} \ , \ \Psi_0(x) = \Psi(x) \quad (64)$$

The following procedure uses recurrence (64) and Proposition 5.1 for construction of a PWA Lyapunov function assuring uniform ultimate boundedness of system (25) in a given positively invariant compact polyhedron. In what follows, PWA function $\Psi(x)$ (45) and $\epsilon$ ball $\mathcal{B}(\Psi, \epsilon)$ (48) will be denoted in compact form as $\Psi[G, c, w]$ and $\mathcal{B}_\Psi[G, c, \epsilon w]$, respectively.

**Procedure 5.1:** Construction of PWA Lyapunov function assuring uniform ultimate boundedness of (25) in a positively invariant set $Gx \le w$.

1) Define $\Psi[G, c = 0, w]$ - initial PWA Lyapunov function candidate in $\mathcal{B}[\Psi, \check{b} = 1, \hat{b} = \infty]$, corresponding to $Gx \le w$, a positively invariant polyhedron containing origin ($w > 0$) with contraction index $\beta$.
2) Check if initial $\Psi[G, c, w]$ is a Lyapunov function using primal LP formulation of Proposition 5.1:
   - Yes : **Stop** :
     $\Psi[G, c = 0, w]$ is a PWA Lyapunov function in $\mathcal{B}[\Psi, \check{b} = 1, \hat{b} = \infty]$.
   - No: Identify $\Psi^*$, the elementary PWA function (61) with minimum $\epsilon_j^i$ not satisfying (58) in the primal LP formulation of Proposition 5.1:

     $$\begin{array}{c} \Psi^* = \Psi[g_{i^*}(I + \tau A_{j^*}), \\ \tau g_{i^*} p_{j^*} + c_{i^*}, (1 - \tau\beta) w_{i^*}] \end{array}$$

     Set $\hat{b}_a = \epsilon_j^i$ , Update PWA:

     $$\Psi[G, c, w] = \max\{\Psi[G, c, w] \ , \ \Psi^*\}$$

3) Iterate:
   a) Check if $\Psi[G, c, w]$ is a Lyapunov function using primal LP formulation of Proposition 5.1:
      - Yes: **Stop** :

$\Psi[G, c, w]$ is a PWA Lyapunov function in $\mathcal{B}[\Psi, \check{b} = 1, \hat{b} = \infty]$.

- No: Identify $\Psi^*$, the elementary PWA function (61) with minimum $\epsilon_j^i$ not satisfying (58) in the primal LP formulation of Proposition 5.1:

$$\Psi^* = \Psi[g_{i^*}(I + \tau A_{j^*}), \\ \tau g_{i^*} p_{j^*} + c_{i^*}, (1 - \tau\beta) w_{i^*}]$$

b) Check if $\epsilon_j^i < \hat{b}_a$:

- Yes : **Stop** : $\Psi[G, c, w]$ is is a PWA Lyapunov function in $\mathcal{B}[\Psi, \check{b} = 1, \hat{b} = \hat{b}_a]$.
- No : Set $\hat{b}_a = \epsilon_j^i$ , Update PWA:

$$\Psi[G, c, w] = \max\{\Psi[G, c, w] \ , \ \Psi^*\}$$

Return to step 3a.

The main result of Procedure 5.1 is the region of attraction $\mathcal{B}[\Psi, 0, \hat{b}]$, with convergence rate $\beta$, of the positively invariant uniform ultimate boundedness set $Gx \leq w$ given in step 1. If $Gx \leq w$ is an estimate of the minimal positively invariant polyhedron given by Procedure 4.1, one has a close estimate of the region of attraction of a possible limit cycle inside it.

## VI. Numerical Example

Consider the continuous-time linear system with deadzone feedback control law :

$$\dot{x}(t) = Ax(t) + Bu(t) \ ; \ u(t) = dz(Fx(t)) \qquad (65)$$

$$A = \begin{bmatrix} 0 & 0.5 \\ -2.5 & 1.0 \end{bmatrix} \ ; \ B = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

$$\check{u} = \hat{u} = 2.0 \ ; \ F = \begin{bmatrix} -1.00 & -6.00 \end{bmatrix}$$

It can be verified that system (65) has: unstable equilibrium point $(0, 0)$ inside deadzone region, no equilibrium points in other regions and matrix $A + BF$ is Hurwitz. In this case, the trajectories of system (65) starting in $\{\Re^2 - (0, 0)\}$ will converge to a limit cycle. The minimal global uniform ultimate boundedness set, according to Definition 2.5, is the set formed by the union of $(0, 0)$ and the limit cycle. Using convex polyhedral approximation (44), contraction index $\beta = .02$ and step size $\tau = 0.1$, Procedure 4.1 gives the minimal positively invariant polyhedron $\Omega(G, w)$ in Figure 6.1. The PWA Lyapunov function given by Procedure 5.1 assures global uniform ultimate boundedness of system (65) in $\Omega(G, w)$. Consequently, the region of attraction of the limit cycle contained in $\Omega(G, w)$ is correctly estimated as $\{\Re^2 - (0, 0)\}$. The phase portrait of some trajectories of system (65), doted lines in Figure 6.1, show that $\Omega(G, w)$ gives a close delimitation of the limit cycle, being also a close convex polyhedral outer approximation of the minimal global uniform ultimate boundedness set.

## VII. Conclusion

This paper dealt with the construction of positively invariant convex polyhedral uniform ultimate boundedness sets for linear continuous-time systems with stabilizing deadzone feedback control laws. The objective was delimitation and region of attraction estimation of possible limit cycles around origin of open-loop unstable multivariable systems. Using a piecewise-linear model of the closed-loop system, the following results were presented. Limit cycle delimitation: a procedure was proposed for construction of a positively invariant convex compact polyhedral estimate of the minimal positively invariant set containing an arbitrarily small compact polyhedral neighborhood of origin; this convex positively invariant set contains a limit cycle if it does not contain any asymptotically stable equilibrium point. Region of attraction estimation: new necessary and sufficient conditions for a piecewise-affine function be Lyapunov function were derived; based on linear programming formulation of these necessary and sufficient conditions, a procedure was proposed for construction of a piecewise-affine Lyapunov function assuring uniform ultimate boundedness in the above mentioned convex positively invariant polyhedral set.

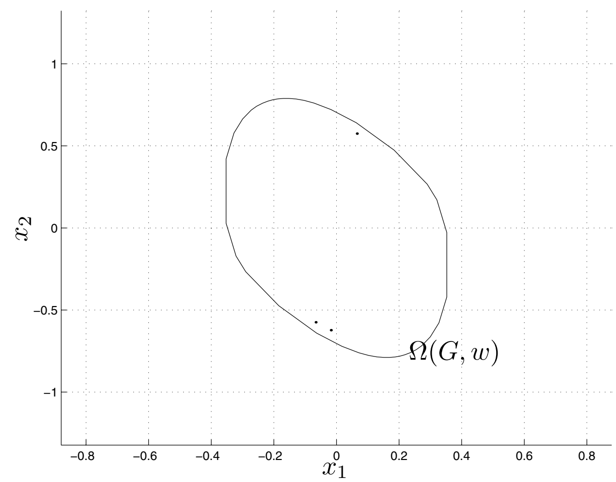


Figure 6.1: Global ultimate boundedness set $\Omega(G, w)$

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