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Design of an Active Fault Tolerant Control and Polytopic Unknown Input Observer for Systems described by a Multi-Model Representation

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Abstract— **In this paper, an active Fault Tolerant Control (FTC) strategy is developed to systems described by multiple linear models to prevent the system deterioration by the synthesis of adapted controllers. First, a Polytopic Unknown Input Observer is synthesized for providing actuator fault estimation. The actuator fault estimation is used in a FTC scheme which schedules some predefined state feedback gains. These gains are performed through $\mathcal{LMI}$ both in fault-free and faulty cases in order to preserve the system performances over a wide operating range. For each separate actuator, a pole placement is designed by pole clustering. The effectiveness and performances of the method have been illustrated in simulation considering a hydraulic system: a three-tank system.**

I. INTRODUCTION

The objective of Fault Tolerant Control system (FTC) is to maintain current performances closed to desirable performances and preserve stability conditions in the presence of component and/or instrument faults. Accommodation capability of a control system depends on many factors such as the severity of the failure, the robustness of the nominal system, and the actuators redundancy. Various approaches for FTC have been suggested in the literature [1], [2] and [3] but often deal with linear systems. For nonlinear systems, the design of Fault Tolerant controller is far more complicated. Nonlinear systems based on multiple linear models, represents an attractive solution to deal with the control of nonlinear systems [4], [5] or Fault Detection and Isolation (FDI) methods as in the chapter nine of [6] where nonlinear dynamic systems are described by a number of locally linearized models based on the idea of Tagaki-Sugeno fuzzy models or as interpolated multiple linear models [7]. A great number of gain scheduling strategies are developed in fault-free case [4] and we proposed to develop one of them in faulty-case. Various recent FDI/FTC studies, based on a multiple model method have been developed in order to detect, isolate and estimate an accurate state of a system in presence of faults/failures around an unique operating point [8] and in chapter 7 of [2]. Multiple Model Adaptive Estimation (MMAE) [8] or Interacting Multiple Model (IMM) introduce a multi-model approach for FDI, but these techniques are developed around an unique operating point ($\mathcal{OP}$). Indeed, these methods consider a particular multiple or multi-model approach where each model is dedicated to a specified fault.

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A polytopic representation is sometimes used in multi-model or piecewise linear models [9].
This paper addresses a more general method that could allow to detect actuator fault in a nonlinear system. Compared to multi-model based reconfigurable control method presented by [10], this paper does not consider some redundant hardware which is very useful when failures are supposed to occur on the system. In this paper, an active fault tolerant control strategy is developed so as to avoid actuator fault effect on nonlinear system where faults are assumed to be incipient, abrupt but not generate a total actuator fault i.e. a failure. The developed method preserves the system performances through an appropriate gain scheduling synthesis in faulty case with a FDI module dedicated to actuator fault estimation in multi-model framework. Compared to recent work applied to similar nonlinear system [11], where a multi-model representation is considered, the proposed FTC strategy is not based on an additional control law but on the redesign of appropriate gain in faulty case allowing stability and performances of the system.

The paper is organized as follows. In section II, we introduce a state space representation for nonlinear systems with additive and multiplicative fault around predefined operating points. A Polytopic Unknown Input Observer is synthesized for estimating actuator fault in systems represented by multi-models. In section III, we introduce a pole placement by $\mathcal{LMI}$ region and then a gain synthesis for each actuator generates an active global state feedback synthesis. A simulation example is given in section IV to illustrate the proposed method. Finally, concluding remarks are given in the last section.

II. NONLINEAR REPRESENTATION AND FDI FOR POLYTOPIC SYSTEMS

A. Nonlinear representation

Consider a discrete-time nonlinear dynamical system in fault-free case described by:

$$\left\{ \begin{array}{ccc} x_{k+1} & = & g(x_k, u_k) \\ y_k & = & h(x_k, u_k) \end{array} \right. \quad (1)$$

where $x_k \in \mathcal{X} \in \mathbb{R}^n$ represents the state vector, $u_k \in \mathcal{U} \in \mathbb{R}^p$ is the input vector and $y \in \mathbb{R}^m$ is the output vector. Functions $g(x_k, u_k)$ and $h(x_k, u_k)$ are continuously differentiable.

It is assumed that dynamic behaviour of the system operating at different operating points can be approximated by a set of N linear time invariant models as presented in [12], [13]. Consider the following state space representation of a

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nonlinear system around j-th operating point, $\forall j \in [1 \dots N]$, with additive actuator faults:

$$\begin{array}{rcl} x_{k+1} - x_e^j & = & A_j(x_k - x_e^j) + B_j(u_k - u_e^j) + F_j f_k \\ y_k - y_e^j & = & C_j(x_k - x_e^j) + D_j(u_k - u_e^j) \end{array} \quad (2)$$

Matrices (A_j, B_j, C_j, D_j) are invariant matrices defined around the j^{th} operating point $(\mathcal{OP}_j)$ generally obtained from a nonlinear system using a first-order Taylor expansion around (x_e^j, u_e^j) with $y_e^j = C_j x_e^j + D_j u_e^j$ or by an identification procedure of a nonlinear system around predefined operating points as for example in chemical processes in [9]. Fault distribution matrix is represented by $F_j \in \mathbb{R}^{n \times p}$. $f_k \in \mathbb{R}^p$ represents the actuator fault magnitude vector which in fault-free case it is obviously equal to zero. In the following, we consider that $D_j = 0$. This linear system (2) can be specified by the set of system matrices:

$$M_j = \begin{bmatrix} A_j & B_j & F_j \\ C_j & & \end{bmatrix}, \forall j = [1, 2, ..., N] \quad (3)$$

Let M_k be a matrix sequence varying within a convex set, defined as:

$$S_k := \left\{ \sum_{j=1}^N \rho_k^j M_j : \rho_k^j \geq 0, \sum_{j=1}^N \rho_k^j = 1 \right\} \quad (4)$$

In the multi-model framework, M_k characterizes at each sample the nonlinear system and consequently, the dynamic behavior of nonlinear system can be defined by a convex set of a multiple LTI models through interpolation or activation functions ρ_k^j. These activation functions $\rho_k^j \quad \forall j \in [1, 2, .., N]$ lie in a convex set $\Omega = \{\rho_k^j \in \mathbb{R}^N, \rho_k = [\rho_k^1 \quad .... \rho_k^N]^T, \quad \rho_k^j \geq 0 \quad \forall j$ and $\sum_{j=1}^N \rho_k^j = 1\}$ and these functions are directly generated via works of [14] which permit to generate insensitive residual to faults and some uncertainties. We will use these works for providing the activation functions as scheduling variable. So, activation functions are robust against faults and the dynamic system behaviour is well represented. based on (2) and around an operating point, the nonlinear system can be represented by the following way as:

$$\begin{array}{rcl} x_{k+1} & = & A_j x_k + B_j u_k + F_j f_k + \Delta x_j \\ y_k & = & C_j x_k \end{array} \quad (5)$$

with the term Δx_i, around an jth operating point, equals to:

$$\Delta x_j \quad = \quad -A_j x_e^j - B_j u_e^j + x_e^j \quad (6)$$

As referred in (3), a new set of matrices is designed with the additional term Δx_j such that:

$$S_j = \begin{bmatrix} A_j & B_j & F_j & \Delta x_j \\ C_j & & & \end{bmatrix}, \forall j \in [1 \dots N] \quad (7)$$

In this paper, a diagnosis procedure is developed to detect and isolate a particular fault among several others. In the following a unique matrix B is considered: some plants can be modelled in such way as presented in the last section. While a single residual is sufficient to detect a fault, a set of structured residuals is required for fault isolation. Several approaches have been proposed by [6] to generate structured residuals. Here, a residual generation using unknown input observer scheme is considered in order to be sensitive to fault vector f_k^* and insensitive to f_k^d. We consider that only a single actuator fault may occur at a given time, simultaneous faults are not considered. Hence, vector f_k^d is a scalar and it is considered as an unknown input. In case of an i-th actuator fault, the system can be represented by:

$$\begin{array}{rcl} x_{k+1} & = & \sum_{j=1}^N \rho_k^j \left[A_j x_k + B u_k + F_d f_k^d + F_x^* f_k^* + \Delta x_j \right] \\ y_k & = & C x_k \end{array} \quad (8)$$

with matrix F_d equals to B^i which is the ith column of B, and matrix F_x^* equals to $\bar{B}^i$ which is the matrix B without the ith column. In the following, we will consider only one output matrix C for all $\mathcal{OP}_j$.

B. Polytopic $\mathcal{UIO}$ synthesis for actuator fault estimation

In a general way, the $\mathcal{UIO}$ allows to vanish undesirable information by decoupling unknown input. In this FDI scheme, the $\mathcal{UIO}$ is used to isolate an actuator fault.

Definition 1: (Polytopic Unknown Input Observer)[15] A polytopic observer is defined as a polytopic unknown input observer ($\mathcal{PUIO}$) for the system described by (8) without fault ($f_k^d, f_k^* = 0$), if the estimation error tends asymptotically to zero despite unknown disturbances on the system.□ This polytopic unknown input observer is defined such that:

$$\begin{array}{l} z_{k+1} = \sum_{j=1}^N \rho_k^j \left[S_j z_k + T B u_k + K_j y_k + \Delta z_j \right] \\ \widehat{x}_{k+1} = z_{k+1} + H y_{k+1} \end{array} \quad (9)$$

where $\widehat{x}$ is the state space estimation of x. The estimation error is equivalent to

$$e_{k+1} = x_{k+1} - \widehat{x}_{k+1} \quad (10)$$

where, with notation $(\centerdot)(\rho)$ stands for $\sum_{j=1}^N \rho_k^j (\centerdot)_j$, $K(\rho) = K^1(\rho) + \Pi(\rho)$. $S(\rho), T, K(\rho)$ and H are designed so as to ensure the stability and the convergence of the estimation error $e_k = x_k - \widehat{x}_k$ without fault on the system ($f_k = 0$). To obtain an exact decoupling, the following conditions should be satisfied:

$$\begin{array}{lll} S(\rho) = TA(\rho) - K^1(\rho)C & \Pi(\rho) = S(\rho)H & \\ (I - HC)F_d = 0 & T = I - HC & T\Delta x(\rho) = \Delta z(\rho) \end{array} \quad (11)$$

The robust polytopic $\mathcal{UIO}$ design is realized when relations (11) hold true and $S(\rho)$ is stable. So, estimation error without fault occurrence, denoted $\bar{e}_k$, tends asymptotically to zero if all these conditions (11) are satisfied. The necessary and sufficient conditions for the existence of a $\mathcal{PUIO}$ are:
(i) $Rank(CF_d) = Rank(F_d)$
(ii) (TA_j, C) are detectable pairs, $\forall j \in [1, 2, .., N]$. These conditions are related in linear case [6]. If condition (i) is true, the synthesis of matrix H, which can permit to avoid unknown input effects, is performed by:

$$H \quad = \quad F_d(CF_d)^+ \quad (12)$$

It should be noted that the matrix F_d is a constant matrix $\forall j \in [1, \dots, N]$. If conditions (11) hold true, the estimation error e_k and the residual r_k are described as:

$$\begin{array}{rcl} e_{k+1} & = & S(\rho)e_k + TF_x^* f_k^* \\ r_k & = & Ce_k \end{array} \quad (13)$$

If conditions (11-12) are fulfilled, an unknown input observer provides an estimation of the state vector , used to generate a residual vector $r^l_k = y_k - C\widehat{x}_k$ $(l \in [1,\ldots,p])$, independent from f^d_k. This means that $r_k = 0$ if $f^*_k = 0$ and $r_k \neq 0$ if $f^*_k \neq 0$ whatever u_k and f^d_k (see (13)). The fault isolation is realized by a common bank of p polytopic unknown input observers. Each residual vector $r^l_k = y_k - C\widehat{x}_k$, produced by the l-th polytopic unknown input observer, may be used to detect a fault according to a statistical test: Page-Hinkley test, limit checking test, generalized likelihood ratio test. The reader could see [16] for more details about residual generation and isolation purposes. Some relations have to be verified for detection purposes, ensuring that such synthesis do not affect fault detection:

$$Rank[TF^*_x] = rank[F^*_x] \tag{14}$$

This condition allows to verify that de-coupling do not affect fault detection by the estimation error [15]. This condition can be translated in a geometric way such as:

$$Im(F^*_x) \subseteq Im(T^T) \tag{15}$$

If the union image F^*_x is included in the image of de-coupling matrix T^T, then fault detection is possible over all the operating range. It is equivalent to:

$$Im(F^*_x) \bigcap Ker(T) = \{0\} \tag{16}$$

If condition (16)(which can be checked off-line) is fulfilled, it ensures that de-coupling do not affect fault detection. It could be noticed that F^*_x is supposed to be full column rank in order to avoid fault compensation of non-independent columns.

According to the fault isolation, the fault magnitude estimation of the corrupted element is extracted directly from the l-th polytopic unknown input observer which is built to be insensitive to the l-th fault ($f^*_k = 0$).

Proposition 1: If (9) is a Polytopic Unknown Input Observer, then fault estimation can be done by:

$$\widehat{f}^d_k = F^+_d\big(\widehat{x}_{k+1} - A(\rho)\widehat{x}_k - Bu_k - \Delta x(\rho)\big) \tag{17}$$

■

Proof:
Then, if (9) is a polytopic unknown input observer, $\widehat{x}_k$ will coincide either asymptotically or in finite time with x_k when $f^*_k = 0$. Thus, substituting x_k by $\widehat{x}_k$ in (8), multiplying left by F^+_d leads to (17). □

Note that in the presence of an actuator fault, F_d is a matrix of full column rank. The stability condition of the polytopic unknown input observer can be performed through the use of $\mathcal{LMI}$.

C. Pole placement of the polytopic $\mathcal{UIO}$

According to (13), the estimation error $\bar{e}_k$, without fault, is expressed as

$$\begin{array}{rcl} \bar{e}_{k+1} & = & S(\rho)\bar{e}_k = \big(TA(\rho) - K^1(\rho)C\big)\bar{e}_k \\ & = & \big(\bar{A}(\rho) - K^1(\rho)C\big)\bar{e}_k \end{array} \tag{18}$$

Proposition 2: The estimation error (18) is called quadratically stabilizable if there exists matrices R_j and if there exists a positive symmetric matrix $X > 0$ such that $\forall j \in [1,\ldots,N]$:

$$\begin{pmatrix} X & \bar{A}^T_j - C^T R^T_j \\ X\bar{A}_j - R_j C & X \end{pmatrix} > 0 \tag{19}$$

with gain matrices K^1_j equal to $X^{-1}R_j$. ■

Proof: By weighting each $\mathcal{LMI}$ defined in (19) by interpolation functions ρ^j defined in the previous section, using notation $R_j = XK^1_j$ and summing all of them with $\sum_{j=1}^N \rho^j = 1$, (19) becomes $\forall j \in [1,\ldots,N]$ equivalent to

$$\begin{pmatrix} X & \sum_{j=1}^N \rho^j(\bar{A}_j - K^1_jC)^T X \\ X\sum_{j=1}^N \rho^j(\bar{A}_j - K^1_jC) & X \end{pmatrix} > 0 \tag{20}$$

With a Schür Complement, $\mathcal{LMI}$ in (20) expresses quadratic stability and we find after some computations:
$(\bar{A}_j - K^1_jC)^T P(\bar{A}_j - K^1_jC) - P < 0$ and $X^{-1} = P > 0$ that is the quadratic stability in discrete case of the estimation error. □

III. Fault Tolerant Control synthesis

The equation (8) considers additive fault representation but there exists multiplicative representation for specific actuator fault. So, a local multiplicative actuator fault representation is defined as:

$$\left\{\begin{array}{rcl} x_{k+1} & = & A_jx_k + B_j(I - \gamma^a)u_k + \Delta x_j \\ y_k & = & C_jx_k \end{array}\right. \tag{21}$$

with $\gamma^a \triangleq diag[\gamma^a_1,\ldots,\gamma^a_p]$, $\gamma^a_1 \in \mathbb{R}$, such that $\gamma^a_i = 1$ represents a failure of i-th actuator and $\gamma^a_i = 0$ implies that i-th actuator operates normally. The relationship between state space representations (2) and (21) is equivalent to

$$F_jf_k = -B_j\gamma^a u_k \tag{22}$$

where the faulty matrix distribution F_j is equal to matrix B_j in an actuator fault case. The estimation of $\widehat{\gamma}^a$ is obtained from fault estimation $\widehat{f}_k$ provided by the $\mathcal{PUIO}$ and availability of nominal control inputs u^{nom}_k(in fault-free case). We attract attention that the system has to be observable in all around each operating points and we will now only consider linear models and not affine models, i.e. $\Delta x_j = 0$.

A. Nominal case

Let consider the state space representation (21) of non-linear system defined around the Equilibrium Points ($\mathcal{OP}_j$), $\forall j = 1,\ldots,N$

$$\begin{aligned} x_{k+1} &= \sum_{j=1}^N \rho^j_k\big[A_jx_k + \sum_{i\in\mathcal{I}} B^i_j(I - \gamma^a)u_k\big] \\ y_k &= \sum_{j=1}^N \rho^j_k[C_jx_k] \end{aligned} \tag{23}$$

with $\mathcal{I} : i = 1,\ldots,p$ the actuators for each $\mathcal{OP}_j$ and matrices $(A_j, B_j, C_j) \in S_k$ defined in (4). Consider the matrix representing total faults in all actuators but the i-th:

$$B^i_j = [0,\ldots,0,b^i_j,0,\ldots,0] \tag{24}$$

and $B_j = [b^1_j, b^2_j,\ldots,b^p_j,]$ with $b^i_j \in \mathbb{R}^{n\times 1}$. It is assumed that each column of B_j is full column rank whatever the $\mathcal{OP}_j$. The pairs $(A_j, b^i_j), \forall i = 1,\ldots,p$ are assumed to be controllable for all $\forall j = 1,\ldots,N$. Let $\mathcal{D}$, a $\mathcal{LMI}$ region defining a disk with a center $(-q,0)$, and a radius r with

$(q+r) < 1$ for defining pole assignment in the unit circle [17]. Assume that for each B_j^i, there exist matrices $X_i = X_i^T > 0$ and Y_i, $\forall j = 1,\dots,N, \forall i = 1,\dots,p$ such as:

$$\begin{pmatrix} -rX_i & qX_i + (A_jX_i - B_j^iY_i)^T \\ qX_i + A_jX_i - B_j^iY_i & -rX_i \end{pmatrix} < 0 \tag{25}$$

It can be noticed that if $q = 0$ and $r = 1$, the previous equation (25) is equivalent to solve a classical quadratic stability problem. Due to space limitation basic principles in $\mathcal{LMI}$ are omitted. Thus, for more details in $\mathcal{LMI}$ pole placement, the reader could refer to [17]. Based on the assumptions that for each $\mathcal{OP}_j$ each pairs (A_j, b_j^i) are controllable, it is possible to find a Lyapunov matrices $X_i > 0$ and state-feedback K_i with $Y_i = K_iX_i$.

Theorem 1: Consider the system (23) in fault-free case ($\gamma^a = 0$) defined for all $\mathcal{OP}_j$, $j = 1,\dots,N$: it is possible to develop a mixing of pre-designed state-feedback gains matrices $K_i = Y_iX_i^{-1}$ for each actuator i with $i = 1,\dots,p$ such that (25) holds for all $j = 1,\dots,N$. The state feedback control for each operating point is given by:

$$\begin{aligned} u_{nom}^j &= -(\sum_{i=1}^{p} G_iY_i)(\sum_{i=1}^{p} X_i)^{-1}x_k \\ &= -YX^{-1}x_k = -K_{nom}x_k \end{aligned} \tag{26}$$

with $\sum_{i=1}^{p} G_iY_i = Y$, $X = \sum_{i=1}^{p} X_i$ and $G_i = B_j^{i+}B_j^i$ is matrix that has zeros everywhere except in entry (i,i) where it has a one. The general control law for all $\mathcal{OP}_j$ could be defined as:

$$u_k = \sum_{j=1}^{N} \rho_k^j u_{nom}^j = -\sum_{j=1}^{N} \rho_k^j K_{nom}x_k = -K_{nom}x_k \tag{27}$$

■

Proof: Summation of (25) for $i = 1,\dots,p$ gives for one equilibrium point j

$$\sum_{i=1}^{p} \begin{pmatrix} -rX_i & qX_i + (A_jX_i - B_j^iY_i)^T \\ qX_i + (A_jX_i - B_j^iY_i) & -rX_i \end{pmatrix} < 0 \tag{28}$$

related to the quadratic $\mathcal{D}$-stability in a prescribed $\mathcal{LMI}$ region. Next, denote $X = \sum_{i=1}^{p} X_i$ (with $X = X^T > 0$) to obtain

$$\begin{pmatrix} -rX & qX + (A_jX - \sum_{i=1}^{p} B_j^iY_i)^T \\ qX + (A_jX - \sum_{i=1}^{p} B_j^iY_i) & -rX \end{pmatrix} < 0 \tag{29}$$

Now, denote the l-th row of the matrix Y_i as Y_i^l, $i = 1,\dots,p$ and $l = 1,\dots,p$, i.e.

$$Y_i^l = G_lY_i \tag{30}$$

Therefore,

$$\sum_{i=1}^{p} B_j^iY_i = \sum_{i=1}^{p} [0,\dots,0,b_j^i,0,\dots,0]Y_i^i = B_j\sum_{i=1}^{p} Y_i^i \tag{31}$$

leading to

$$\sum_{i=1}^{p} B_j^iY_i \quad = \quad B_j(\sum_{i=1}^{p} G_iY_i) \tag{32}$$

Thus, taking $Y = \sum_{i=1}^{p} G_iY_i$, equation (32) becomes

$$\sum_{i=1}^{p} B_j^iY_i = \sum_{i\in\mathcal{I}} B_j^iY_i = B_jY \tag{33}$$

which, substituted into $\mathcal{LMI}$ (29), finally makes

$$\begin{pmatrix} -rX & qX + (A_jX - B_jY)^T \\ qX + (A_jX - B_jY) & -rX \end{pmatrix} < 0 \tag{34}$$

for all $\mathcal{OP}_j$, $j = 1,\dots,N$. By multiplying each $\mathcal{LMI}$ (34) by ρ_k^j and summing all of them, we obtain

$$\begin{pmatrix} -rX & qX + \sum_{j=1}^{N} \rho_k^j(A_jX - B_jY)^T \\ qX + \sum_{j=1}^{N} \rho_k^j(A_jX - B_jY) & -rX \end{pmatrix} < 0 \tag{35}$$

it is equivalent to

$$\begin{pmatrix} -rX & qX + (A(\rho)X - B(\rho)Y)^T \\ qX + (A(\rho)X - B(\rho)Y) & -rX \end{pmatrix} < 0 \tag{36}$$

with $A(\rho) = \sum_{j=1}^{N} \rho_k^jA_j \in S_k$ and $B(\rho) = \sum_{j=1}^{N} \rho_k^jB_j \in S_k$. Hence quadratic $\mathcal{D}$-stability is ensured by solving (35) and $Y = K_{nom}X$ quadratically stabilizes the system (23) under the set S_k by solving (36) with a state feedback law $u_k = \sum_{j=1}^{N} \rho_k^j u_{nom}^j = -YX^{-1}x_k$. □

Remark 1: If all local models have the same B_j matrices, i.e. $B_j = B$ for all $j = 1,\dots,N$, it is also possible to use the following parameter-dependent state feedback law instead of (4):

$$u_k = \sum_{j=1}^{N} \rho_k^j u_{nom}^j = -\sum_{j=1}^{N} \rho_k^j K_jx_k \tag{37}$$

with $K_j = Y_jQ^{-1}$. The resulting matrices remain the same except that K_j replaces K, B replaces B_j and Y_j replaces Y.

B. Active Fault Tolerant Control design

By considering the system (23) and based on the previous synthesis control law, the FTC method can be developed in this section under assumptions that fault occurrence and fault magnitude γ^a are known, provided by the $\mathcal{PUIO}$.

Theorem 2: Consider the system (23) in faulty case ($\gamma^a \neq 0$) coupled with regulators with gains $K_i = Y_iX_i^{-1}$ for all equilibrium point $j = 1,\dots,N$ and for each actuator i with $i = 1,\dots,p$. Let introduce the set of indexes of all actuators that are not completely lost, i.e.

$$\Theta \triangleq \{i : i \in (1,\dots,p), \gamma_i^a \neq 1\}$$

The control action is

$$u_{FTC}^j = -(I-\gamma^a)^+\big(\sum_{i\in\Theta} G_iY_i\big(\sum_{i\in\Theta} X_i\big)^{-1}\big)x_k \tag{38}$$

where $G_i = B_j^{i+}B_j^i$, applied to the faulty system allows to constrain pole placement in prescribed $\mathcal{LMI}$ region. ■

Proof: Applying the new control law (38) to the faulty system (21), leads to the following equation

$$B_j(I-\gamma^a)u_{FTC}^j = B_j\Gamma^a(\sum_{i\in\Theta} G_iY_i)(\sum_{i\in\Theta} X_i)^{-1}x_k$$

$$\text{with} \quad \Gamma^a = \begin{pmatrix} I_{p-h} & 0 \\ 0 & O_h \end{pmatrix}$$

Γ^a is a diagonal matrix which contains only entries zero (representing total faults) and one (no fault). Due to the fact that only the set Θ is considered and since $B_j\Gamma^a = \sum\limits_{i\in\Theta} B_j^i$ models only the actuators that are not completely lost, then performing the summations in the proof of Theorem (1) over the elements of Θ shows that $(\sum\limits_{i\in\Theta} G_iY_i)(\sum\limits_{i\in\Theta} X_i)^{-1}$ is the state-feedback gain matrix for the faulty system $(A_j, \sum\limits_{i\in\Theta} B_j^i, C_j)$. $\square$

The control law in equation (38) implies that

$$\begin{aligned} u_{FTC}^j &= -K_{FTC}x_k \\ \text{with} \quad K_{FTC} &= (I-\gamma^a)^+\sum_{i\in\Theta} G_iY_i(\sum_{i\in\Theta} X_i)^{-1} \end{aligned}$$

The global control law U_{FTC} of the system is realized as:

$$\begin{aligned} u_k &= \sum_{j=1}^N \rho_k^j u_{FTC}^j \\ &= -\sum_{j=1}^N \rho_k^j K_{FTC}x_k = -K_{FTC}x_k \end{aligned} \tag{39}$$

IV. Application

A. Process description

The approach presented in this paper has been applied to the well known three tanks system as in [11]. As all the three liquid levels are measured by level sensors, the output vector Y is $[l_1 \quad l_2 \quad l_3]^T$. The control input vector is $U = [q_1 \quad q_2]^T$. The goal is to control the system around three equilibrium points. Thus, 3 linear models have been identified around each of these equilibrium points and the operating conditions are given in Table (I). The linearized

TABLE I

OPERATING POINT DEFINITION

Operating point	n°1	n°2	n°3
$y_e^j(m)$	0.20; 0.15; 0.175	0.50; 0.15; 0.325	0.50; 0.405; 0.45
u_e^j $(m^3/s)\times 10^{-5}$	1.7509; 4.0390	4.6324; 1.1574	2.4761; 6.9787

system is described by a discrete state space representation with a sampling period $Ts = 1s$. For each $\mathcal{OP}$, each control matrix pair (A_j, b_j^i) is controllable. Controllers have been designed for levels l_1 and l_2 to track reference input vector $Y^r \in \mathbb{R}^2$. Nominal controllers have been designed through Theorem (1), leading to two state feedback gain matrices K_1, K_2 (due to 2 actuators) for the three $\mathcal{OP}$ in order to achieve satisfying tracking performances. The simulation of actuator faults on the system does not affect the controllability and observability of the system such as assumption $l_1 > l_3 > l_2$ is assumed to be always true.

B. Results and comments

Simulations have been performed such as the 3 operating conditions described in Table (I) are reached and weighting functions for each local model are presented in figure (1(b)) always close to the dynamic behaviour of the nonlinear system according to the considered operating regimes [14].

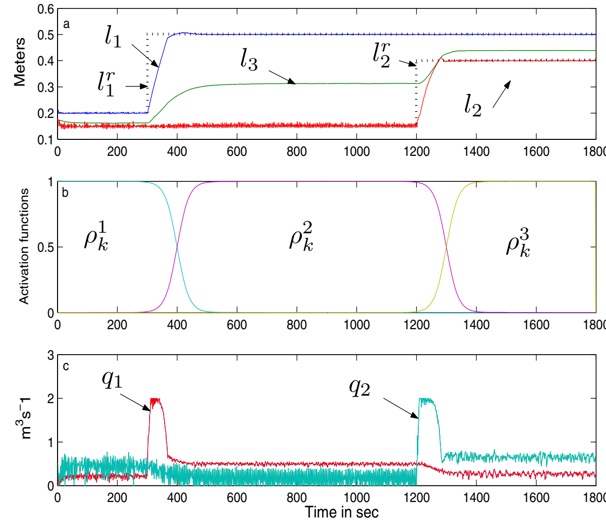

Fig. 1. System dynamic behaviour in fault-free case :(a) system outputs, (b) activation functions ρ_k^j, (c) system inputs

Figure (1) shows the outputs with respect to set-point changes occur at time instant $300s$ and after at time instant $1200s$. In the simulation, gaussian noises $(N(0, 1e^{-4^2}))$ are added to each output signal. The reference inputs correspond to step changes for l_1 and l_2. The consequence of an actuator fault is illustrated in figure (2). A gain degradation of pump 1 (clogged or rusty pump) equivalent to 80% loss of effectiveness is supposed to occur at time instant 600s. Consequently, the dynamic behaviour of the other levels is also affected by this fault and control system tries to cancel the static error created by the corrupted input. Consequently, the real output is different from the reference input and the control law is different from its nominal value. The controller tries to cancel the fault effect with the signal q_{s1} on figure (2(b) but the real available control signal is described by variable q_1. Since an actuator fault acts on the system as a perturbation, and in spite of the presence of an integral controller, the system outputs can not reach again their nominal values. The presence of an integrator in the control law allows the system to reach the nominal values as illustrated in the third operating point (after time instant 1200s) in spite of fault persistence as illustrated in figure (4).

The $\mathcal{PUIO}$ is synthesized as in first section. The fault is detected, estimated and depicted on figure (4) where we can see its effectiveness. In the same way, the actuator Fault Tolerant Control method's ability to compensate faults is illustrated in the presence of the same fault. Once the fault is isolated and simultaneously estimated, a new control law (39) is computed in order to reduce the fault effect on the system. Indeed, since the effect of an actuator fault is quite similar to the effect of a perturbation, the system outputs reach again their nominal values, as illustrated in figure (3). There exists a little time delay between fault occurrence and fault compensation due to FDI module. Computation of the tracking error norm in fault-free case, in faulty case without

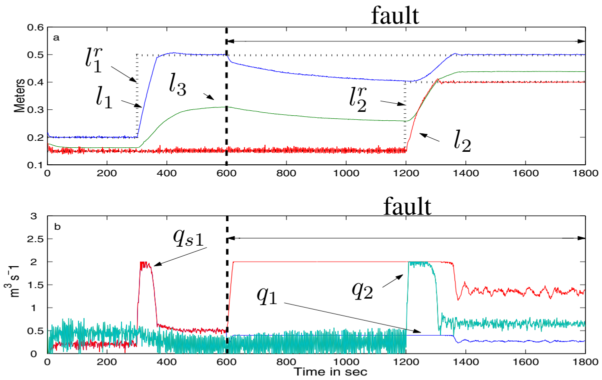

Fig. 2. System dynamic behaviour in faulty-case without FTC :(a) system outputs, (b) system inputs

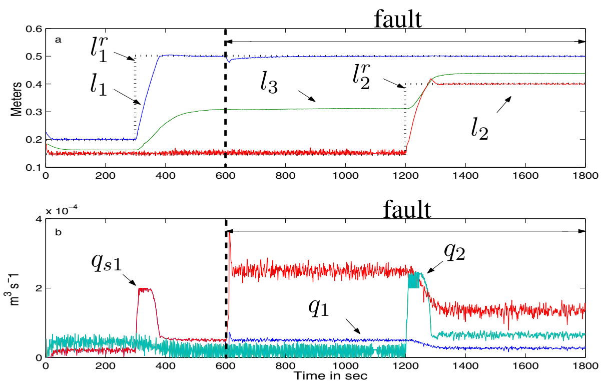

Fig. 3. System dynamic behaviour in faulty-case with FTC :(a) system outputs, (b) system inputs

and with FTC underlines the performances of this approach as seen in Table (II). With FTC, the tracking error norm for output l_1 is a bit larger than with the fault-free case but it still widely smaller than the one without FTC. The actuator Fault Tolerant Control is able to maintain performances as close as possible to nominal ones and to ensure the closed-loop stability despite the presence of instruments malfunction.

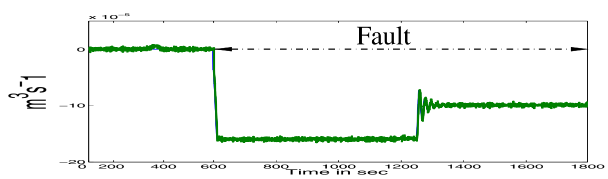

Fig. 4. Actuator fault magnitude and its estimation

TABLE II

ERROR NORM COMPARAISON

Error norm	Fault-free case	Actuator fault without FTC	Actuator fault with FTC
e_{l1}	1.1063	3.3365	1.1176

Remark: This proposed scheme could have an interesting development in aeronautic, for example, where redundant actuators are available and where a wide operating range is considered.

V. CONCLUSION

The method developed in this paper emphasises the importance of the active Fault Tolerant Control for systems based on multi-model representation. This method is suitable for partial actuator faults on a wide operating range of the system. Actuator faults are estimated with a Polytopic Unknown Input Observer. A robust controller is designed for each separate actuator through an $\mathcal{LMI}$ pole placement in fault-free case and faulty case. It allows the system to continue operating safely, to avoid stopping it immediately and to ensure stability. The synthesis of this active state feedback control takes into account the information provided on-line by the $\mathcal{PUIO}$. The performances and the effectiveness of this active Fault Tolerant Control based on a multiple model approach have been illustrated through an hydraulic system.

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