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Modular Modelling of Flexible Thin Beams in Multibody Systems

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Abstract— **In this paper the problem of modelling flexible thin beams in multibody systems is tackled. The proposed model, implemented with the object-oriented physical systems modelling language *Modelica*, is completely modular, allowing the realization of complex systems by simple aggregation of basic components. The finite element method is employed as the basic scheme to spatially discretize the model equations; exploiting the modular features of the language, a new discretization scheme (mixed finite element-finite volume) is derived as well. Selected simulation results are presented in order to validate the model with respect to both theoretical predictions and literature reference results.**

I. INTRODUCTION

Many control engineering applications require the development of simulation models for flexible multibody systems (e.g., robot manipulators, helicopter rotors, aircraft wings, space structures, machining tools, etc.) both dynamically accurate and computationally affordable.

The task of developing models for generic-shaped, fully deformable bodies is usually demanded to specialized simulation tools, due to the complexity of the task. Such models are usually adequate for structural analysis and design tasks, while being far too complex for affordable dynamics analysis and control systems prototyping.

On the other hand, particular classes of deformable bodies, such as flexible beams, can be represented with less complex models which are still able to represent all the dynamically relevant deformation effects.

Flexible beams are continuous non linear dynamical systems characterized by an infinite number of degrees of freedom. Obviously, dealing directly with infinite dimensional models is impractical both for dynamic analysis and simulation purposes. Hence it is necessary to introduce methods to describe flexibility with a discrete number of parameters.

Three different approaches have been traditionally used to derive approximated finite dimensional models: lumped parameters, assumed modes and finite element method [1],[2]. In this paper, the latter one is used, since it is the most suited for a modular approach [3].

In the finite element method approach [4], the flexible beam is divided into several elements, with a local description of the deformation field by the use of element-wise basis functions.

In order to manage the complexity, a real modular approach is also needed. The main characteristics of this approach are:

- models description by stating physical principles rather than by writing calculation algorithms (*acausal* modelling);
- realization of models of large, complex and heterogeneous physical systems by basic components *connection*;
- charging the simulation environment with the task of defining the computational causality of the assembled model.

Among the various modular physical modelling languages and tools developed from the end of the 70's (e.g., OMOLA [5], gPROMS [6], MOSES [7]), *Modelica* [8] and Dymola [9] have emerged respectively as a standard for a modular, acausal modelling language and for a complete and efficient modelling and simulation environment. Dymola is also considered in this work as the development framework. In particular, a main contribution of the paper is the development of the model of a flexible thin beam, based on the finite element method, as a component of the *Modelica MultiBody* library [10]. The model is valid as long as the deformation field is small compared to the beam length, as it is the case, for example, when studying the dynamics of vibrations in machining tools. However, by exploiting the modular approach and the symbolic manipulation capabilities of the environment, several beams can be easily connected in mixed finite element-finite volume models, in order to account for large deformations.

The paper is organized as follows: Section II contains an introduction to the modular modelling techniques for multibody systems; in Section III the problem of the representation of a generic deformable body in a multibody system is introduced; in Section IV the development of the equations of motions is shown; in Section V the *Modelica* implementation is analyzed; Section VI contains selected simulation results; finally, in Section VII the main results are summarized and future developments are introduced.

II. MODULAR MODELLING FOR MULTIBODY SYSTEMS

The development of the equations of motion for a complete multibody system is a task whose complexity increases very rapidly with the number of bodies involved. Furthermore, such development of the equations of motion is not suitable for model reusability, a key requirement for advanced modelling techniques.

Modern modelling languages for physical systems such as *Modelica* [8], on the other hand, allow the development of complex multibody systems by aggregation of simple models

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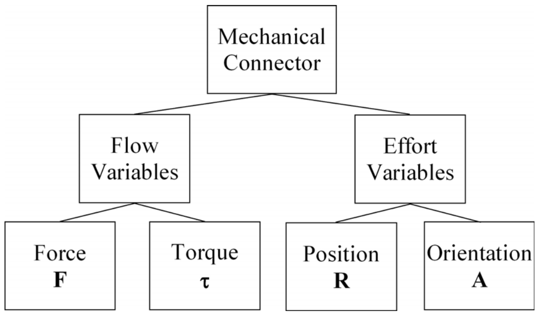

Fig. 1. Mechanical Connector Scheme

from a library containing all the basic components (e.g., rigid bodies, joints, forces, etc.).

Models are described in *Modelica* in a declarative form, i.e. by stating physical principles rather than by writing calculation algorithms. This results in acausal models, described by DAE (Differential-Algebraic Equations) systems, which represent in the most natural and physically consistent way each system component, while the task of defining the computational causality of the assembled model is charged to the simulation environment. It is important to recall that in *Modelica* the interaction among (sub)models is implemented through connectors only, whose design is of paramount importance. A connector is defined by a set of effort variables and by a set of flow variables and the simulation environment implements a connection by equating the effort variables and by balancing the flow variables.

Modelling and simulation of multibody systems can be dealt with in *Modelica* through a standard library, which however can handle only rigid bodies. A reference frame is associated to the connector of the *MultiBody* library and a connection is equivalent to a rigid overlapping of the two connector frames (it is the abstraction of an ideal "welding" realized at the frame location). The multibody connector assumes the cut force and torque as flow variables, while the effort variables are given by the position of the origin of the connector frame with respect to the world frame and by an "orientation object", describing the relative orientation between the world frame and the connector frame (Fig. 1).

The body reference system orientation can be efficiently represented with a rotation matrix having a different parametrization depending on the body specific dynamic conditions; however, the library has been designed so that the choice of any specific rotation matrix parametrization is automatic and completely transparent to the user.

The library provides several utility functions to operate on instances of the object, for example to rotate vectors and to compute the angular velocity, while the linear velocity and acceleration and the angular acceleration are simply computed by applying the derivative operator `der()` to the corresponding variables. The library innovative features allow the possibility connecting the components in any arbitrary fashion, along with automatic analytic handling of kinematic loops by mean of advanced symbolic manipulation techniques [11].

The generation of a numerically efficient procedural form from the description of the model in the *Modelica* language is carried out in several steps. First, the object-oriented code is "flattened" in a set of constants, variables, functions and equations and the connection equations are generated. Then, after the conversion of the system of equations in BLT form, an algebraic simplification follows, removing the trivial equations and resulting in a minimal set of equations. If needed index reduction is also performed [12].

III. Deformable Body Degrees of Freedom

In a generic multibody system, the position, in local coordinates, of a point on a specific deformable body has the following expression:

$$\overline{u} = \overline{u}_0 + \overline{u}_f \,, \tag{1}$$

where $\overline{u}_0$ is the "undeformed" (i.e., rigid) position vector and $\overline{u}_f$ is the deformation contribution to position (i.e., the deformation field).

The description of the generic deformation of a body requires the deformation field to belong to an infinite dimensional functional space, requiring, in turn, an infinite number of deformation degrees of freedom.

In this paper, the deformation field is described by an approximation of the functional basis space it belongs to, supposing such space has a finite dimension, say M, so that the vector u_f can be expressed by the following finite dimensional product:

$$\overline{u}_f = S q_f \,, \tag{2}$$

where S is the $[3 \times M]$ shape functions matrix (i.e., a matrix of functions defined over the body domain and used as a basis to describe the deformation field of the body itself) and q_f is the M-dimensional vector of deformation degrees of freedom.

The position of a generic body point can then be expressed in world reference as follows:

$$r = R + A\overline{u} = R + A(\overline{u}_0 + Sq_f) = R + A\overline{u}_0 + ASq_f \,, \tag{3}$$

where R is the vector identifying the origin of the body local reference system and A is the rotation matrix for such reference system.

The representation of a generic deformable body in world reference requires then $6+M$ d.o.f. (i.e., 6 corresponding to rigid displacements and rotations and M to deformation fields):

$$q = \left[q_r \;\; q_f\right]^T = [R \;\; \theta \;\; q_f]^T \,, \tag{4}$$

where θ represents the undeformed body orientation angles and q_r is a vector containing the 6 rigid degrees of freedom.

IV. Equations of Motion

The equations of motion for a generic flexible body in a multibody system can be developed applying the principle of virtual work [1], which states that the virtual work of the inertial forces δW_i must counterbalance the sum of the virtual work of the *continuum* elastic forces δW_s and of the external ones δW_e:

$$\delta W_i = \delta W_s + \delta W_e \,. \tag{5}$$

The terms of equation (5) are defined as follows:

$$\delta W_i = \int_V \rho \delta r^T \ddot{r} dV \, , \quad (6)$$

$$\delta W_s = -\int_V \delta \varepsilon^T \sigma dV \, , \quad (7)$$

$$\delta W_e = \int_V \delta r^T F_e \, dV + \int_\Omega \delta r^T f_e \, d\Omega \, , \quad (8)$$

where V is the body volume, ρ is the body density, δr is an infinitesimal virtual displacement, $\ddot{r}$ is the body acceleration (in world reference), $\delta\varepsilon$ is a vector of virtual infinitesimal internal strains, σ is the internal stresses vector, F_e is the vector of external volume forces, Ω is the body surface and f_e is the vector of external surface forces.

The exploitation of the principle of virtual work lead to the following expression for the *generalized Newton-Euler* equations of motion in body axes [3]:

$$\begin{bmatrix} m_{RR} & \tilde{\bar{S}}_t^T & \bar{S} \\ \tilde{\bar{S}}_t & \bar{I}_{\theta\theta} & \bar{I}_{\theta f} \\ \bar{S}^T & \bar{I}_{\theta f}^T & m_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\bar{R}} \\ \bar{\alpha} \\ \ddot{q}_f \end{bmatrix} = \\ = \begin{bmatrix} 0_{3,1} \\ 0_{3,1} \\ -K_{ff} q_f - D_{ff} \dot{q}_f \end{bmatrix} + \begin{bmatrix} Q_v^R \\ Q_v^\theta \\ Q_v^f \end{bmatrix} + \begin{bmatrix} \overline{Q}_e^R \\ \overline{Q}_e^\theta \\ Q_e^f \end{bmatrix} , \quad (9)$$

where $\ddot{\bar{R}}$ and $\ddot{\bar{\alpha}}$ are the body linear and angular acceleration, K_{ff} is the structural stiffness matrix [4], D_{ff} is a damping term modelling the dissipative properties of the material, Q_e^R, Q_e^θ and Q_e^f are the generalized components of the active forces associated to translational, rotational and deformation coordinates, respectively, and the other quantities are defined as follows:

$$m_{RR} = \int_V \rho \, dV \, , \quad (10)$$

$$m_{R\theta} = \int_V \rho A \, (\bar{u}\times)^T A^T dV \, , \quad (11)$$

$$m_{Rf} = \int_V \rho A S \, dV \, , \quad (12)$$

$$m_{\theta\theta} = -\int_V \rho A \bar{u} \times \bar{u} \times A^T \, dV \, , \quad (13)$$

$$m_{\theta f} = \int_V \rho A \bar{u} \times S \, dV \, , \quad (14)$$

$$m_{ff} = \int_V \rho S^T S \, dV \, , \quad (15)$$

$$\bar{S} = \int_V \rho S dV = A^T m_{Rf} \, , \quad (16)$$

$$\bar{S}_t = \int_V \rho \bar{u} dV \, , \quad (17)$$

$$\tilde{\bar{S}}_t = \int_V \rho \, (\bar{u}\times) \, dV = A m_{R\theta} A^T \, , \quad (18)$$

$$\bar{I}_{\theta\theta} = \int_V \rho \, (\bar{u}\times)^T (\bar{u}\times) \, dV = A^T m_{\theta\theta} A \, , \quad (19)$$

$$\bar{I}_{\theta f} = \int_V \rho \, (\bar{u}\times) S dV = A^T m_{\theta f} \, , \quad (20)$$

$$Q_v^R = -\bar{\omega} \times \bar{\omega} \times \bar{S}_t - 2\bar{\omega} \times \bar{S} \dot{q}_f \, , \quad (21)$$

$$Q_v^\theta = -\bar{\omega} \times \bar{I}_{\theta\theta} \bar{\omega} - \dot{\bar{I}}_{\theta\theta} \bar{\omega} - \bar{\omega} \times \bar{I}_{\theta f} \dot{q}_f \, , \quad (22)$$

$$Q_v^f = -\int_V \rho S^T \left(\tilde{\bar{\omega}}^2 \bar{u} + 2\tilde{\bar{\omega}} S \dot{q}_f \right) dV \, . \quad (23)$$

Equations (9) are valid for a general deformable body, though many of the quantities involved (e.g., the matrix K_{ff}) depend on specific body characteristics such as the shape or the material properties.

From now on, the case of a *thin beam* will be considered. In detail, it will be assumed that the body is a 1D elastic *continuum* with constant cross-sectional properties. Furthermore, it will be assumed that the beam constitutive material is homogeneous, isotropic and perfectly elastic (i.e., the elastic internal forces are conservative). Finally, it will be assumed that the deformation field is restricted to lie within the xy plane of the beam local reference system.

These assumptions do not restrict the model validity or generality, since the model remains still representative for a large number of dynamic simulation applications (e.g., many of the flexible robots commonly studied have flexible links which can be represented by such model).

A. The element point of view

The finite element method is based upon a discretization of the beam into N elements. A single element can itself be viewed as a thin beam characterized by a planar deformation field. It is then possible to define the local dimensionless *abscissa* as $\xi = x/\ell$, where x is the longitudinal local coordinate and ℓ is the element length.

In [4] it is shown that the partial differential equations associated with the deformation problem at hand, under the hypothesis of elastic constitutive law for the material, require, for a consistent finite element formulation, the use of linear and Hermite cubic polynomials for the approximation of the axial and transversal deformation field, respectively. Thus, for a single element, the generic equations of motion (9) can be expanded as follows:

$$\bar{u}_{f,el} = \begin{bmatrix} \bar{u}_{f1,el} \\ \bar{u}_{f2,el} \\ \bar{u}_{f3,el} \end{bmatrix} = S_{el} \, q_{f,el} \, ,$$

$$S_{el} = \begin{bmatrix} 1-\xi & 0 & 0 \\ 0 & 1-3\xi^2+2\xi^3 & \ell(\xi-2\xi^2+\xi^3) & \cdots \\ 0 & 0 & 0 \end{bmatrix} \quad (24)$$

$$\begin{bmatrix} \xi & 0 & 0 \\ \cdots \; 0 & 3\xi^2-2\xi^3 & \ell(\xi^3-\xi^2) \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} S_{el1} \\ S_{el2} \\ S_{el3} \end{bmatrix} ,$$

$$q_{f,el} = \begin{bmatrix} q_{f1,el} & q_{f2,el} & q_{f3,el} & q_{f4,el} & q_{f5,el} & q_{f6,el} \end{bmatrix}^T ,$$

where the subscript $_{el}$ is used to refer the quantities to a single element.

Fig. 2 depicts the element coordinate systems associated with the deformation degrees of freedom: $q_{f1,el}$ and $q_{f4,el}$ are associated with axial compression, $q_{f2,el}$ and $q_{f5,el}$ with transversal displacement and $q_{f3,el}$ and $q_{f6,el}$ with extremities rotation.

The third row of the shape matrix S_{el} is composed only by zeros, according to the hypothesis of planar deformation. Such hypothesis and the assumption of a homogeneous, isotropic and elastic material for the beam, allow to exploit the Euler-Bernoulli theory and to calculate the elastic potential energy U_{el}, neglecting the contribution of shear stresses and considering only the work of the resulting axial force

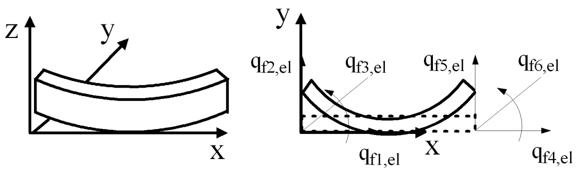

Fig. 2. Element coordinate systems

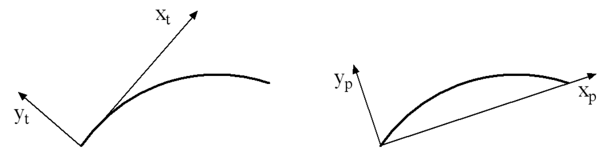

Fig. 3. Tangent (left) and pinned (right) reference systems

N_{el} and bending moment M_{el}, as follows [4]:

$$U_{el} = \frac{1}{2}\int_{\ell}\left(\frac{N_{el}}{Ea}\frac{\partial N_{el}}{\partial x} + \frac{M_{el}}{EJ}\frac{\partial M_{el}}{\partial x}\right)dx = \frac{1}{2}\int_{\ell}\left(EJ\left(\frac{\partial^2 \overline{u}_{f2,el}}{\partial x^2}\right)^2 + Ea\left(\frac{\partial \overline{u}_{f1,el}}{\partial x}\right)^2\right)dx = \frac{1}{2}q^T_{f,el}K_{ff,el}q_{f,el}\,, \tag{25}$$

where E is the material Young's modulus, a is the (constant) cross-sectional area and J is the (constant) cross-sectional second moment of area. The analytical expression for the matrix $K_{ff,el}$, usually known as the structural stiffness matrix, can be found in reference books on structural mechanics (e.g., in [4]).

B. Finite Element Method Equations Assembly

The equations of motion for the entire beam can be obtained by assembling the equations of motion for beam elements as the one defined in the previous subsection. The body reference system will be the local reference system located at the root of the first element, so that the rigid degrees of freedom, common to all the elements, will be referred to such coordinate system.

Let then m and L be the mass and length of the entire beam, and N the number of elements to be used, so that $\ell = L/N$. Indicating with $\widehat{\overrightarrow{X}}$ the reference system unit vector along the beam axis, the expression of the generic position $\overline{u}_j$ of a point of element j is

$$\overline{u}_j = \overline{u}_{0j} + S_{el}B_j q_f = \left[\xi_j\ell + (j-1)\ell\right]\widehat{\overrightarrow{X}} + S_{el}B_j q_f\,, \tag{26}$$

where $\overline{u}_{0j}$ is the position of the root of the j^{th} element, S_{el} is the shape functions matrix defined by (24), B_j is the so-called *connectivity matrix* and q_f is a vector containing the deformation degrees of freedom for the whole beam.

The matrices B_j have the following form:

$$B_j = \left[\begin{array}{c|c|c} 0_{6,3(j-1)} & I_6 & 0_{6,3(N-j)} \end{array}\right], \forall j = 1,\cdots,N\,. \tag{27}$$

The connectivity matrices are used to relate the vector q_f, which contains the deformation degrees of freedom for the whole beam, to the corresponding j^{th} element, according to the expression:

$$q_{f,el_j} = B_j q_f\,. \tag{28}$$

The dynamics of the complete flexible beam can then be described by equation (9), using the following expressions:

$$\begin{aligned}
\overline{S} &= \sum_{j=1}^{N}\frac{m}{L}\int_{V_j} S_{el}B_j dV_j\,,\\
\overline{S}_t &= \sum_{j=1}^{N}\frac{m}{L}\int_{V_j} \overline{u}_j dV_j\,,\\
\overline{I}_{\theta\theta} &= \sum_{j=1}^{N}\frac{m}{L}\int_{V_j}\begin{pmatrix} \overline{u}^2_{2f_j} & -\overline{u}_{2f_j}\overline{u}_{1_j} & 0\\ 0 & \overline{u}^2_1 & 0\\ 0 & 0 & \overline{u}^2_{1_j} + \overline{u}^2_{2f_j}\end{pmatrix} dV_j\,,\\
\overline{I}_{\theta f} &= \sum_{j=1}^{N}\frac{m}{L}\int_{V_j}\begin{pmatrix} O_{(3N,1)}\\ O_{(3N,1)}\\ \overline{u}_{1_j}S_{el2} - \overline{u}_{2_j}S_{el1}\end{pmatrix} dV_j\,,\\
m_{ff} &= \sum_{j=1}^{N}\frac{m}{L}B_j^T\left(\int_{V_j} S^T_{el}S_{el}dV_j\right)B_j\,,\\
K_{ff} &= \sum_{j=1}^{N} B_j^T K_{ff,el}B_j\,,\\
Q_v^f &= -\sum_{j=1}^{N}\frac{m}{L}\int_{V_j}\left[B_j^T S_{el}^T\left(\tilde{\tilde{\omega}}^2\overline{u}_j + 2\tilde{\tilde{\omega}}S_{el}B_j\dot{q}_f\right)\right]dV_j\,.
\end{aligned} \tag{29}$$

C. Deformation Boundary Conditions

The equations of motion for the whole beam must be completed by enforcing suitable boundary conditions for the finite element approximation of the deformation partial differential equations. That means assuming prescribed values for some of the deformation displacements, rotations and velocities (linear or angular) at the body boundaries which are, for the case at hand, the beam root and tip.

The most commonly used boundary conditions for flexible beams are of two kinds, commonly associated with two different reference system: the *tangent* frame and the *pinned* frame condition (Fig. 3).

In both cases six conditions are given: the *tangent* one enforce null deformation at the beam root (i.e., q_{f1}, q_{f2}, q_{f3}, $\dot{q}_{f1}$, $\dot{q}_{f2}$, $\dot{q}_{f3}$ equal to zero for the first element), while the *pinned* one enforce null axial and transversal displacement at the beam root (i.e., q_{f1}, q_{f2}, $\dot{q}_{f1}$, $\dot{q}_{f2}$ equal to zero for the first element) and transversal displacement at the beam tip (i.e., q_{f5} and $\dot{q}_{f5}$ equal to zero for the last element).

The choice of which of the two set of conditions has to be used largely depends on the problem at hand.

Boundary conditions can be enforced into equations (9) with suitable modifications of the connectivity matrices B_1 and B_N, by zeroing some entries. For example, for the *tangent* reference conditions, B_N remains unvaried and B_1 becomes

$$B_1 = \left[\begin{array}{c|c} \begin{array}{c|c} 0_{3,3} & 0_{3,3}\\ \hline 0_{3,3} & I_3\end{array} & 0_{6,3(N-1)}\end{array}\right]. \tag{30}$$

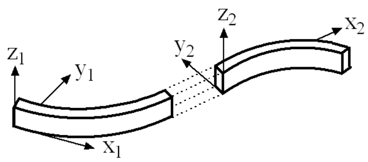

Fig. 4. Volume coordinate systems

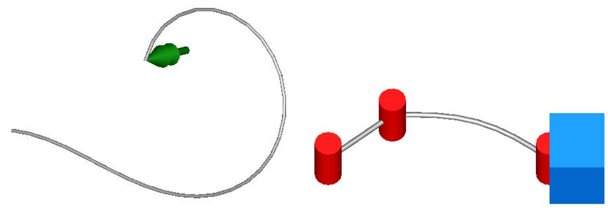

Fig. 5. Large beam deformation (left) and Slider-crank mechanism (right)

D. Extended Formulation of the Equation of Motion

In the finite element formulation for the equation of motion for a flexible beam, the reference directions of the internal actions are the same for all the elements. Such representation is acceptable as long as the deformation field is small compared to the beam length, as it is the case, for example, when studying the dynamics of vibrations in machining tools.

On the other hand, when large deformations are involved, the internal actions reference directions should change accordingly to the deformation field. That means that it is necessary to define a local reference system for each element (Fig. 4). This corresponds to the application of the finite volume method to assemble the equations of motion solved over each element (i.e., over each volume). This representation is valid also for large beam deformation, as long as the deformation field is small compared to the volumes length.

Furthermore, it is possible to assemble the equation of motion for a mixed (finite element-finite volume) formulation by dividing every volume into several elements.

It is not necessary to go into the detailed calculations for the finite volume or the mixed formulation since, as it will be shown in Section V, the equations of motion for such extensions can be automatically calculated with the aid of symbolic manipulation algorithms applied to the finite element formulation.

V. Modelica Implementation

The finite element formulation for the model has been implemented using the *Modelica* language, creating thus a new component, called *FlexBeamFEM*. The component interfaces are two standard mechanical flanges from the new *MultiBody* library [10]. The connectors choice makes the component fully compatible with the library, so that it is possible to connect directly the flexible beam with the pre-defined models such as mechanical constraints (revolute joints, prismatic joints, etc.), parts (3D rigid bodies) and forces elements (springs, dampers, forces, torques).

In detail, the flexible beam component uses two mechanical flanges as physical representation of the two ends of the beam while the motion is ruled by equations (9).

The terms Q_e^R, Q_e^θ, Q_e^f (i.e., the external actions) are computed on the basis of the forces and torques exchanged at the two connectors.

The model parameters include the beam length and cross sectional area, the material density and Young modulus, the cross sectional inertia, the damping factor and the number of elements.

Particular care has been put into the realization of a 3D interface for the model to visualize the simulation results (Fig. 5), implemented by exploiting the features of the graphical environment of the multibody library.

The finite volume model and the mixed one can be easily obtained by connecting several finite element beams composed by one or more elements, respectively. The achievement of such results, which significantly simplify the models implementation, is based on the modular approach adopted in the finite element model development. The assembly of the equations of motion for these cases is demanded to *Modelica*-based simulation environments, which usually employ advanced symbolic manipulation techniques and index reduction algorithms.

The dynamical properties of the latter models are significantly complex and accurate, featuring a displacement description which is fully non-linear and allowing the simulation of large displacement due to deformation (Fig. 5) at the cost, though, of a significant increase of the computational complexity with respect to the “pure” finite element model.

VI. Simulations

The different flexible beam models have been validated by several simulation analysis performed within the Dymola simulation environment [9]. The most significative ones are reported in the following subsections.

A. Free Vibration

In this simulation the free vibration of a flexible beam is analyzed. The test-case has been set up in order to investigate the models properties with respect to theoretical predictions.

The beam component is connected to the world reference system, so that no rigid motion is allowed; furthermore, no gravity field is considered.

At the initial time instant the beam is standing still with a non-null tip displacement, then it evolves, vibrating, towards steady state.

The vibration frequencies of a flexible beam clamped at the root can be calculated by solving the following partial differential equation:

$$\rho a \frac{\partial^2 y(x,t)}{\partial t^2} + EJ \frac{\partial^4 y(x,t)}{\partial x^4} = 0 \tag{31}$$

with the following boundary and initial conditions:

$$\begin{cases} y(0,t),\ \dfrac{\partial y}{\partial x}(0,t),\ \dfrac{\partial^2 y}{\partial x^2}(L,t),\ \dfrac{\partial^3 y}{\partial x^3}(L,t) = 0 \\ y(x,0) = f(x),\ \dfrac{\partial y}{\partial t}(x,0) = 0 \end{cases} \tag{32}$$

where x is the axial coordinate, y is the transversal displacement and $f(x)$ is the initial deformation field.

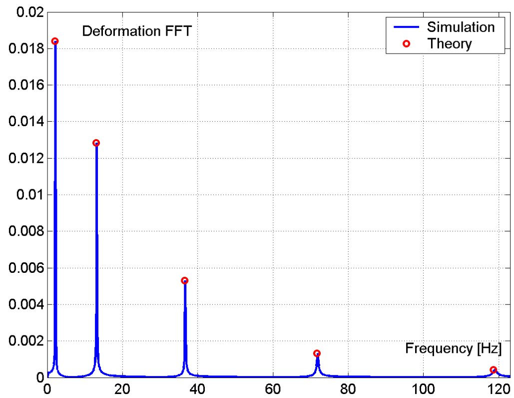

Fig. 6. Tip displacement frequency spectrum

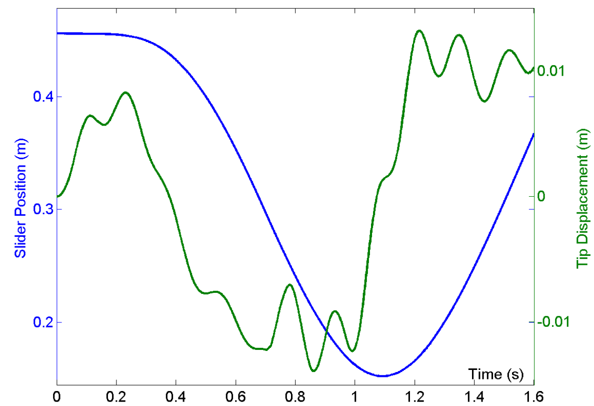

Fig. 7. Slider block position and tip transverse displacement of the connecting rod

The beam, made by aluminium, has square cross section $a = 1\,cm^2$, length $L = 2\,m$, density $\rho = 2700\,kg/m^3$, Young's modulus $E = 72 \cdot 10^9 N/m^2$ and has been discretized with $N = 10$ elements. The initial tip displacement is $1\,cm$.

Table I contains a comparison between the results for for the first five vibrational modes obtained by simulation and by solving numerically equation (31). The results are in good accordance, as it is shown also in Fig. 6, depicting the tip displacement frequency spectrum.

Mode	Frequency* [Hz]	Frequency† [Hz]	Error [%]
1	2.0854733	2.0854750	8.418e-005
2	13.0694381	13.0698705	3.308e-003
3	36.5948052	36.6041219	2.545e-002
4	71.7112127	71.7795490	9.529e-002
5	118.543772	118.842591	2.521e-001

* Theoretical prediction † Simulation result

TABLE I

THEORETICAL AND MODEL NATURAL FREQUENCIES

B. Elastic Slider-crank Mechanism

The simulation of an elastic slider-crank mechanism, reported as a reference test-case also in [13], has been performed to validate the models for use within closed-loop mechanical chains. The simulation set up involves a slider, a rod and a crankshaft connected by revolute joints (Fig. 5).

The crank has length $L = 0.152\,m$, cross sectional area $a = 0.7854\,cm^2$ and second moment of area $J = 4.909 \cdot 10^{-10}\,m^4$, density $\rho = 2770\,kg/m^3$ and modulus of elasticity $E = 10^9\,N/m^2$. The connecting rod has the same physical parameters of the crank, apart from the lenght $L = 0.304\,m$ and the Young's modulus $E = 5 \cdot 10^7 N/m^2$. The crank and the connecting rod have been discretized with 3 and 8 elements, respectively. Finally, the slider block has been assumed to be a massless rigid body.

During the simulation, the crankshaft is driven by a torque with the following law:

$$\begin{cases} M(t) = [0.01\,(1 - e^{-t/0.167})]Nm & , \quad t \leq 0.7\,sec \\ 0 & , \quad t > 0.7\,sec \end{cases} \tag{33}$$

The slider position and the connecting rod tip transverse displacement are depicted in Fig. 7. The results are in perfect accordance with those reported in [13].

VII. Conclusion and Future Work

In this paper, a new model for flexible thin beams in *Modelica* is introduced. The model, fully compatible with the *MultiBody* library, is based on the application of the finite element method. Selected simulation results have been presented in order to validate the model properties with respect to scientific literature reference cases.

Future work will include the model extension to handle full 3D deformation and distributed loads. The model will also be employed for the development of applications in the field of robot control and satellite attitude control.

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