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Gas-Lift Allocation under Precedence Constraints: Piecewise-Linear Formulation and K-Covers

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***Abstract*— Artificial lifting is a costly, but indispensable means to recover oil from high depth reservoirs. Continuous gas-lift works by injecting high pressure gas at the bottom of the production tubing to gasify the oil column, thereby forcing the flow of fluid to surface facilities. The problem consists in deciding which wells should produce and allocating a limited lift-gas rate to the active ones, subject to lower and upper bounds on gas injection, activation precedence constraints, and nonlinearities and discontinuities of the well performance curves. To this end, this paper develops a piecewise linear formulation of the lift-gas allocation problem that allows the application of powerful integer-programming algorithms. More specifically, it analyzes the constraint polyhedron of the piecewise linear formulation and extends cover inequalities of the knapsack polytope to the problem at hand.**

I. INTRODUCTION

Not unlike other economic sectors, the oil industry is relying on modern automation and control technology to cut costs in response to the pressure from environmental legislation and competitive markets. Artificial gas-lift techniques, for instance, are implemented to recover oil in high depth wells. It works by injecting high pressure gas at the bottom of a well to gasify the fluid column, thereby boosting the reservoir's internal pressure and forcing the flow of fluid to surface facilities, where the mixture is separated in oil, gas, and water [1]. Because the availability of gas can be limited and the gas-compressing costs are pronounced, one faces the combinatorial problem of deciding which wells should produce and allocating the lift-gas rate to the active ones. Even though the lift-gas allocation problem has appeared in the literature since the early 70's with varied emphasis [2], [3], [4], most of the literature disregard the combinatorial issues or suggest ad hoc rules that can lead to suboptimal allocations. To this end, dynamic programming algorithms have been developed for lift-gas allocation that can cope with combinatorial decisions, activation precedence constraints, and multiple well performance curves [5], [6]. This paper proposes the piecewise-linearization of the nonlinear well performance curves to render the lift-gas allocation problem a mixed-integer linear programming (MILP) problem. The advantages of the MILP approach are threefold. First, it allows the optimal allocation of lift-gas for large and very large oil fields. Second, it can be more efficiently generalized to handle multiple facility constraints. Third, the theory and algorithms from integer programming [7], [8] can be used to advantage in solving the problem of concern.

The remainder of this paper develops a piecewise-linear formulation of the lift-gas allocation problem, analyzes the constraint polyhedron, and extends cover inequalities of the precedence constrained knapsack problem to the problem at hand.

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II. PROBLEM DEFINITION

The lift-gas allocation problem can be expressed in mathematical programming as follows:

$$
\begin{aligned}
P(G): \ \text{Max} \quad & f = \textstyle\sum_{n=1}^{N} f_n(q_i^n, q_o^n) && (1.1)\\
\text{S.to}: \quad & \textstyle\sum_{n=1}^{N} q_i^n \le q_i^{max} && (1.2)\\
& y_n \le y_m, \ \forall (m,n) \in E[G] && (1.3)\\
& \text{For } n = 1, \ldots, N: && \\
& \quad q_o^n = \tilde{q}_o^n(q_i^n) && (1.4)\\
& \quad l_n y_n \le q_i^n \le u_n y_n && (1.5)\\
& \quad y_n \in \{0,1\} && (1.6)
\end{aligned}
$$

having the following given parameters and functions:

- N is the number of oil wells;
- q_i^{max} is the maximum lift-gas rate yielded by the gas-compressing station (subscript "i" indicates injection);
- l_n and u_n are lower and upper bounds respectively on the lift-gas injection into well n;
- f_n is the profit function of well n obtained by selling the hydrocarbons discounted processing and gas compression costs, typically a linear function of q_i^n and q_o^n;
- $\tilde{q}_o^n(q_i^n)$ is a nonlinear, continuous function modeling the outflow from a well n in response to the injection of a lift-gas rate q_i^n within the interval $[l_n, u_n]$;
- $G = (V, E)$ is an acyclic, directed graph that spells out the precedence constraints on the activation of wells, where $V = \{1, \ldots, N\}$;

and deciding upon the values of the variables:

- q_i^n is the lift-gas rate allocated to well n;
- q_o^n is the outflow from well n (the subscript "o" indicates output flow); and
- y_n is a binary variable taking on value 1 if the n^{th} well is activated, and 0 otherwise.

Proposition 1: [6] $P(G)$ is NP-Hard in the strong sense.

In words, the above proposition says that there does not exist a pseudo polynomial algorithm for $P(G)$ unless $\mathbb{P} = \mathbb{NP}$ [9], which is a motivation for the investigation hereafter.

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III. PIECEWISE LINEAR FORMULATION

Definition 1: $Q^n = \{(q_i^{n,1}, q_o^{n,1}), \ldots, (q_i^{n,\kappa(n)}, q_o^{n,\kappa(n)})\}$ is a set of $\kappa(n)$ pairs of gas injection and outflow for well n, such that $q_o^{n,k} = \tilde{q}_o^n(q_i^{n,k})$, $k = 1, \ldots, \kappa(n)$.

Assumption 1: For each well $n \in \mathcal{N} = \{1, \ldots, N\}$:

i) $0 < q_i^{n,k} < q_i^{n,k+1}$ for each $1 \leq k < \kappa(n)$ which implies that the probe injection points are distinct;

ii) $q_i^{n,1} = l_n$ and $q_i^{n,\kappa(n)} = u_n \leq q_i^{max}$ to leave out unnecessary probe points; and

iii) $(q_i^{n,k}, q_o^{n,k})$ cannot be expressed as a convex combination of the elements of $Q^n - \{(q_i^{n,k}, q_o^{n,k})\}$ for each k to discard redundant points.

Having introduced the above notation, we can follow the procedure from [7, Section I.1.4] to piecewise-linearize the well performance curves $\tilde{q}_o^n$ (as illustrated in Figure 1) and recast $P(G)$ as an MILP problem:

$P_{pl}(G)$:

$$J = \text{Max} \quad f = \sum_{n=1}^{N} \sum_{k=1}^{\kappa(n)} f^{n,k} \lambda_{n,k} \tag{2.0}$$

$$\text{S.to}: \quad \sum_{n=1}^{N} q_i^n \leq q_i^{max} \tag{2.1}$$

$$x_{m,1} \leq x_{n,1}, \ \forall (m,n) \in E[G] \tag{2.2}$$

For $n = 1, \ldots, N$:

$$\lambda_{n,0} \leq x_{n,1} \tag{2.3}$$

$$\lambda_{n,1} \leq x_{n,2} \tag{2.4}$$

For $k = 2, \ldots, \kappa(n) - 1$:

$$\lambda_{n,k} \leq x_{n,k} + x_{n,k+1} \tag{2.5}$$

$$\lambda_{n,\kappa(n)} \leq x_{n,\kappa(n)} \tag{2.6}$$

$$\sum_{k=0}^{\kappa(n)} \lambda_{n,k} = 1 \tag{2.7}$$

$$\sum_{k=1}^{\kappa(n)} x_{n,k} = 1 \tag{2.8}$$

$$q_i^n = \sum_{k=0}^{\kappa(n)} q_i^{n,k} \lambda_{n,k} \tag{2.9}$$

For $n = 1, \ldots, N, \ k = 0, \ldots, \kappa(n)$:

$$\lambda_{n,k} \geq 0 \tag{2.10}$$

For $n = 1, \ldots, N, \ k = 1, \ldots, \kappa(n)$:

$$x_{n,k} \in \{0, 1\} \tag{2.11}$$

where:

- $f^{n,k} = f_n(q_i^{n,k}, q_o^{n,k})$ is the contribution to the objective function by well n at the probe point $(q_i^{n,k}, q_o^{n,k})$;
- $(q_i^{n,0}, q_o^{n,0}) = (0, 0)$;
- (q_i^n, q_o^n) is a convex combination of the elements of $\tilde{Q}^n = \{(q_i^{n,0}, q_o^{n,0})\} \cup Q^n$, where $q_o^n = \sum_{k=0}^{\kappa(n)} q_o^{n,k} \lambda_{n,k}$;
- $\lambda_n = \{\lambda_{n,k} : k = 0, \ldots, \kappa(n)\}$ is the set of factors that induce the convex combination of the elements of $\tilde{Q}^n$;
- $x_n = \{x_{n,k} : k = 1, \ldots, \kappa(n)\}$ is the set of discrete variables that force the convex combinations to use only two consecutive points from $\tilde{Q}^n$; $x_{n,k}$ takes on value 1 if and only if (q_i^n, q_o^n) is a convex combination of $(q_i^{n,k-1}, q_o^{n,k-1})$ and $(q_i^{n,k}, q_o^{n,k})$ for $2 \leq k \leq \kappa(n)$ as illustrated in Figure 1; note that $x_{n,1} = 1$ if and only if well n is shut off, and otherwise the well is producing.

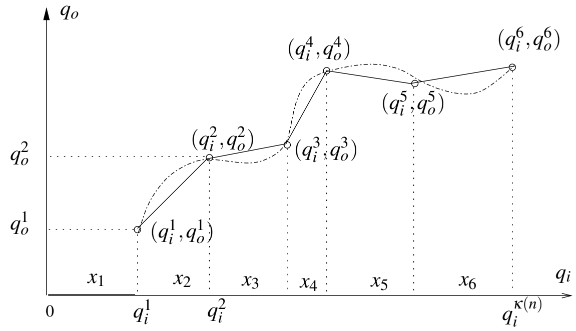

Fig. 1. Piecewise-linearization of the well performance curve.

Notice that $y_n = \sum_{k=2}^{\kappa(n)} x_{n,k}$. For the sake of brevity, additional notation is introduced: $\lambda = \bigcup_{n=1}^{N} \lambda_n$; $x = \bigcup_{n=1}^{N} x_n$; $q_i = \{q_i^n : n = 1, \ldots, N\}$; and $K = \sum_{n=1}^{N} \kappa(n)$. In terms of this notation, the space of feasible solutions can be conveniently expressed as the polyhedron $\mathscr{P}_{pl}(G) = \{(x, \lambda, q_i) \in \mathbb{B}^K \times \mathbb{R}^{K+2N} : (x, \lambda, q_i)$ meets the constraints (2.1) through (2.10)$\}$, where $\mathbb{B} = \{0, 1\}$.

Proposition 2: If:

i) $f_n(q_i^n) = f_n(q_i^n, \tilde{q}_o^n(q_i^n))$ is a concave function in $[l_n, u_n]$;

ii) $\tilde{P}_{pl}(G)$ is obtained from $P_{pl}(G)$ by removing variables $x_{n,2}, \ldots, x_{n,\kappa(n)}$, dropping the constraints (2.4)-(2.6) and (2.8), and replacing (2.3) by an equality;

iii) no element $(q_i^{n,k}, f^{n,k}) \in F^n = \{(q_i^{n,1}, f^{n,1}), \ldots, (q_i^{n,\kappa(n)}, f^{n,\kappa(n)})\}$ can be written as a convex combination of the elements of $F^n - \{(q_i^{n,k}, f^{n,k})\}$; and

iv) (x, λ, q) is an optimal solution to $\tilde{P}_{pl}(G)$;

then $\lambda_{n,k} + \lambda_{n,k+1} = 1$ for some $0 \leq k \leq \kappa(n) - 1$.

Proof: We only need to consider the case in which $x_{n,1} = 0$. Let $\tilde{f}_n(q_i^n)$ be the piecewise linear function approximating $f_n(q_i^n)$. By concavity of f_n [condition (i)], it follows that $\sum_{k=1}^{\kappa(n)} \lambda_{n,k} f^{n,k} \leq \tilde{f}_n(\sum_{k=1}^{\kappa(n)} \lambda_{n,k} q_i^{n,k}) = \tilde{f}_n(q_i^n)$, and from the optimality of (x, λ, q) [condition (iv)], we conclude that $\sum_{k=1}^{\kappa(n)} \lambda_{n,k} f^{n,k} = \tilde{f}_n(q_i^n)$. There must exist $\tilde{k} \in \{1, \ldots, \kappa(n) - 1\}$ and $\tilde{\lambda}_{n,\tilde{k}}, \tilde{\lambda}_{n,\tilde{k}+1} \in \mathbb{R}_+$ such that $\tilde{\lambda}_{n,\tilde{k}} + \tilde{\lambda}_{n,\tilde{k}+1} = 1$ and $q_i^n = \tilde{\lambda}_{n,\tilde{k}} q_i^{n,\tilde{k}} + \tilde{\lambda}_{n,\tilde{k}+1} q_i^{n,\tilde{k}+1}$. From condition (iii), $(q_i^n, \tilde{f}_n(q_i^n))$ can be obtained only from convex combination of $(q_i^{n,\tilde{k}}, f^{n,\tilde{k}})$ and $(q_i^{n,\tilde{k}+1}, f^{n,\tilde{k}+1})$, implying that $(q_i^n, \tilde{f}_n(q_i^n)) = \tilde{\lambda}_{n,\tilde{k}}(q_i^{n,\tilde{k}}, f^{n,\tilde{k}}) + \tilde{\lambda}_{n,\tilde{k}+1}(q_i^{n,\tilde{k}+1}, f^{n,\tilde{k}+1})$. Consequently, only $\lambda_{n,\tilde{k}}$ and $\lambda_{n,\tilde{k}+1}$ can be nonzero, which in turn demonstrates the claim. ■

From an inspection of the constraints of $P_{pl}(G)$, one realizes that some decision variables can be readily eliminated which results in a more compact formulation. The variables that can be expressed as functions of the others are:

$$x_{n,1} = 1 - \sum_{k=2}^{\kappa(n)} x_{n,k} \tag{3.1}$$

$$\lambda_{n,0} = x_{n,1} \tag{3.2}$$

$$\lambda_{n,1} = 1 - \lambda_{n,0} - \sum_{n=2}^{\kappa(n)} \lambda_{n,k} = \sum_{k=2}^{\kappa(n)} (x_{n,k} - \lambda_{n,k}) \tag{3.3}$$

$$q_i^n = \sum_{k=2}^{\kappa(n)} \left(q_i^{n,1} x_{n,k} + (q_i^{n,k} - q_i^{n,1}) \lambda_{n,k} \right) \tag{3.4}$$

By piecewise-linearizing the objective function and substituting the expressions given by (3.1)–(3.4) for the respective variables of the formulation $P_{pl}(G)$, the following equivalent but more compact formulation results:

$P_{cpl}(G)$:

$$\text{Max} \quad f = \sum_{n=1}^{N} \sum_{k=2}^{\kappa(n)} \left(f^{n,1} x_{n,k} + (f^{n,k} - f^{n,1}) \lambda_{n,k} \right) \tag{4.0}$$

S.to :

$$\sum_{n=1}^{N} \sum_{k=2}^{\kappa(n)} \left(q_i^{n,1} x_{n,k} + (q_i^{n,k} - q_i^{n,1}) \lambda_{n,k} \right) \leq q_i^{max} \tag{4.1}$$

$$\sum_{k=2}^{\kappa(m)} x_{m,k} \geq \sum_{k=2}^{\kappa(n)} x_{n,k}, \forall (m,n) \in E[G] \tag{4.2}$$

For $n = 1, \ldots, N$:

$$\sum_{k=2}^{\kappa(n)} x_{n,k} \leq 1 \tag{4.3}$$

$$\sum_{k=2}^{\kappa(n)} x_{n,k} - \sum_{k=2}^{\kappa(n)} \lambda_{n,k} \geq 0 \tag{4.4}$$

$$\sum_{k=3}^{\kappa(n)} x_{n,k} - \sum_{k=2}^{\kappa(n)} \lambda_{n,k} \leq 0 \tag{4.5}$$

For $k = 2, \ldots, \kappa(n) - 1$:

$$\lambda_{n,k} \leq x_{n,k} + x_{n,k+1} \tag{4.6}$$

$$\lambda_{n,\kappa(n)} \leq x_{n,\kappa(n)} \tag{4.7}$$

For $n = 1, \ldots, N, \ k = 2 \ldots, \kappa(n)$:

$$\lambda_{n,k} \geq 0 \tag{4.8}$$

$$x_{n,k} \in \{0, 1\} \tag{4.9}$$

whose variables and interpretations are identical to those of the formulation $P_{pl}(G)$. The vectors x and λ can aggregate arrays of variables as they did in $P_{pl}(G)$, this way allowing the feasible space of $P_{cpl}(G)$ to be more compactly represented by the polyhedron $\mathscr{P}_{cpl}(G) = \{(x, \lambda) \in \mathbb{B}^{K-N} \times \mathbb{R}^{K-N} : (x, \lambda)$ meets constraints (4.1) through (4.8)$\}$.

IV. CUTTING PLANES

Here and further along the text, we introduce notation to support our developments:

- $A_n^+ = \{m : \exists \text{ path from } m \text{ to } n \text{ in } G\}$ and $A_n = A_n^+ - \{n\}$;
- $root(G) = \{n : |A_n| = 0\}$ are the root nodes of G;
- $\sigma_n = \sum_{m \in A_n} q_i^{m,1}$ is the minimum amount of gas necessary to activate the wells that precede n;
- $\sigma_n^{max} = \sigma_n + q_i^{n,\kappa(n)}$ is the minimum amount of gas required to activate well n at its highest injection level.

Proposition 3: $\mathscr{P}_{cpl}(G)$ is full dimensional if $\max\{\sigma_n^{max} : n \in \mathscr{N}\} \leq q_i^{max}$.

Proof: (*Sketch*) We only outline the main steps of the demonstration. The claim can be proven by obtaining a unit vector from linear combination of the elements of $\mathscr{P}$ for each variable[1]. Let $X = \{x_{n,k} : n \in \mathscr{N}, k = 2, \ldots, \kappa(n)\}$ denote the set of x variables and $\Lambda = \{\lambda_{n,k} : n \in \mathscr{N}, k = 2, \ldots, \kappa(n)\}$ denote the set of λ variables, but not their values. Let also $z(U) \in \mathscr{P}$ be a solution obtained by setting the variables from $U \subseteq X \cup \Lambda$ to unit (while the remaining variables are set to zero)[2] and let $\eta(y) \in \mathbb{B}^{K-N} \times \mathbb{R}^{K-N}$ be the unit vector whose positive entry corresponds to variable $y \in X \cup \Lambda$. The proof can be divided in two main parts.

(*Part I*) Unit vectors are obtained for all variables associated with root nodes. Take any $n \in root(G)$. Clearly $z(x_{n,2}) \in \mathscr{P}$ and $\eta(x_{n,2}) = z(x_{n,2})$. By noticing that $z(x_{n,2}, \lambda_{n,2}) \in \mathscr{P}$, a unit vector can be obtained for $\lambda_{n,2}$ by making $\eta(\lambda_{n,2}) = z(x_{n,2}, \lambda_{n,2}) - \eta(x_{n,2})$. Following this pattern, $\eta(x_{n,k})$ and $\eta(\lambda_{n,k})$ can be produced for $k = 3, \ldots, \kappa(n)$.

(*Part II*) Unit vectors are synthesized for all variables associated with non-root nodes. Let $T = (n_1, \ldots, n_t)$ be a topological order of $G[V - root(G)]$, i.e., for all $1 \leq i < j \leq t$, $n_j \notin A_{n_i}$. For $i = 1, \ldots, t$ (in topological order), unit vectors are produced for all variables associated with vertex n_i. Let $X' = \{x_{j,2} : j \in A_{n_i}^+\}$. Because $\sigma_{n_i}^{max} \leq q_i^{max}$, $z(X') \in \mathscr{P}$ and from induction in the topological order, it follows that $\eta(x_{n_i,2}) = z(X') - \sum_{j \in A_{n_i}} \eta(x_{j,2})$. With $X'' = X' \cup \{\lambda_{n_i,2}\}$, one can verify that $z(X'') \in \mathscr{P}$ and $\eta(\lambda_{n_i,2}) = z(X'') - \sum_{j \in A_{n_i}^+} \eta(x_{j,2})$. In the same manner of Part I, one can follow this pattern to get $\eta(z_{n_i,k})$ and $\eta(\lambda_{n_i,k})$ for $k = 3, \ldots, \kappa(n)$.

At this point, unit vectors have been obtained for all elements of $X \cup \Lambda$. Because the null vector is a valid solution, $\mathscr{P}_{cpl}(G)$ has $|X \cup \Lambda| + 1$ affinely independent vectors and, hence, $dim(\mathscr{P}_{cpl}(G)) = 2\sum_{n=1}^{N} (\kappa(n) - 1) = 2(K - N)$. ■

A. K-Covers

One of the most successful techniques to design MILP algorithms rests on the identification of valid inequalities from the combinatorial structure of the problem, preferably facet inducing inequalities. Families of valid inequalities and separation procedures or heuristics are key to implement effective branch-and-cut algorithms [8]. Henceforth we extend the cover inequalities of the constrained knapsack polyhedron [10] to $\mathscr{P}_{cpl}(G)$. But, before defining K-cover and its induced inequalities, we augment the notation:

- $\Omega = \{(n,k) : n = 1, \ldots, N \text{ and } k = 2, \ldots, \kappa(n)\}$ is the set of pairs of wells and activation levels;
- $\Omega_n = \{(n,k) \in \Omega\}$ is Ω restricted to well n;
- $\Omega(U) = \bigcup_{n \in U} \Omega_n$ where $U \subseteq \mathscr{N}$;
- $N(S) = \{n : (n,k) \in S\}$ for $S \subseteq \Omega$;
- $\Omega(S) = \bigcup_{n \in N(S)} \Omega_n$ where $S \subseteq \Omega$;
- $S(n,k) = S - \{(n,k)\}$ for $S \subseteq \Omega$;
- $\gamma(S) = \sum_{(n,k) \in S} q_i^{n,k-1}$ for $S \subseteq \Omega$;
- $m \preceq n$ means that m precedes n in a topological order of G, i.e., $m \in A_n^+$;
- $m \prec n$ means that m strictly precedes n in a topological order of G, i.e., $m \in A_n$;
- $l(U) = \bigcup_{n \in U} A_n^+$ is the set of ancestors of $U \subseteq \mathscr{N}$;
- $l(S) = \bigcup_{n \in N(S)} A_n^+$ is the set of ancestors of the nodes appearing in $N(S)$, where $S \subseteq \Omega$;
- $H(U) = \{n \in U : \nexists m \in U \text{ such that } n \prec m\}$ for $U \subseteq \mathscr{N}$;
- $H(S) = \{(n,k) \in S : n \in H(N(S))\}$ where $S \subseteq \Omega$; and
- $\Gamma(S) = \{(n,j) : (n,k) \in S \text{ and } 2 \leq j \leq k\}$ for $S \subseteq \Omega$.

[1]Hereafter, $\mathscr{P}$ will be a shorthand for $\mathscr{P}_{cpl}(G)$

[2]The curly brackets that delimit the elements of a set will be omitted to simplify notation.

Definition 2: $C \subseteq \Omega$ is a cover if:

i) $m \neq n$ for all distinct pairs $(m,i),(n,j) \in C$;
ii) $l(C) = N(C)$;
iii) $k = 2$ for all $(n,k) \in C - H(C)$; and
iv) $\gamma(C) > q_i^{max}$.

For a cover C, the wells appearing in $N(H(C))$ cannot be simultaneously activated at the levels specified in $H(C)$.

Definition 3: A cover C is a K-cover if for every $S \subseteq H(C)$ with $|S| = K$ it is true that $\gamma(\Phi_S) > q_i^{max}$ for $\Phi_S = \Omega(l(S)) \cap C$, but for any $(n,k) \in S$, $\gamma(\Phi_S - \{(n,k)\}) \leq q_i^{max}$.

Definition 4: A cover C is a strictly K-cover if for every $S \subseteq H(C)$ with $|S| = K$ it is true that $\gamma(\Phi_S) > q_i^{max}$ for $\Phi_S = \Omega(l(S)) \cap C$, but for any $(n,k) \in S$, $\gamma(\Phi_S - \{(n,k)\}) < q_i^{max}$ if $k = 2$ or else $\gamma((\Phi_S - \{(n,k)\}) \cup \{(n,k-1)\}) < q_i^{max}$.

Proposition 4: If C is a K-cover then

$$\sum\nolimits_{(n,k)\in H(C)} x_{n,k} \leq K - 1 \qquad (5)$$

is a valid inequality of $\mathscr{P}_C = \{(x,\lambda) \in \mathscr{P}_{cpl}(G) : x_{n,k} = \lambda_{n,k} = 0, \forall (n,k) \in \Omega - \Gamma(C)\}$.

For a K-cover C, there does not exist enough resources to simultaneously activate any subset of $H(C)$ with cardinality K, which leads to inequality (5). The terminology presented below will simplify forthcoming demonstrations.

Definition 5: Given $S \subseteq \Omega$, define $z(S) = (x(S), \lambda(S))$ as:

$$\begin{cases} x(S)_{n,k} = 1, \forall (n,k) \in S \\ x(S)_{n,k} = 0, \forall (n,k) \in \Omega - S \\ \lambda(S)_{n,k} = 1, \forall (n,k) \in Q = \{(n,j) : (n,j+1) \in S, j \geq 2\} \\ \lambda(S)_{n,k} = 0, \forall (n,k) \in \Omega - Q \end{cases}$$

Proposition 5: Given a strictly K-cover C, inequality (5) induces a maximal face $F_C = \{(x,\lambda) \in \mathscr{P}_C : \sum_{(n,k)\in H(C)} x_{n,k} = K - 1\}$ of $\mathscr{P}_C$ if and only if

$$\bigcap\nolimits_{\{S\subseteq H(C):|S|=K-1\}} \Omega(l(S)) \cap C = \emptyset \qquad (6)$$

Proof: Inequality (5) is valid for $\mathscr{P}_C$ from the definition of K-cover.

(*Necessity*) Suppose that (6) does not hold and let $(n',k') \in \bigcap_{\{S\subseteq H(C):|S|=K-1\}} \Omega(l(S)) \cap C$. For any $T \subseteq \Gamma(C)$ such that $z(T) = (x(T), \lambda(T)) \in F_C$, (n',k') must appear in T. It then follows that all $z = (x,\lambda) \in F_C$ satisfy $x_{n',k'} = 1$ and, hence, $F_C \subseteq F = \{(x,\lambda) \in \mathscr{P}_C : x_{n',k'} = 1\}$. If $F_C \neq F$, then F_C is not maximal. Thus, for F_C to be maximal, $F_C = F$, but (5) is not a scalar multiple of $x_{n',k'} \leq 1$, contradicting the claim.

(*Sufficiency*) To see that (5) induces a facet, let F_π be a maximal face of $\mathscr{P}_C$ containing F_C and induced by:

$$\sum\nolimits_{(n,k)\in\Gamma(C)} \pi_{n,k} x_{n,k} + \sum\nolimits_{(n,k)\in\Gamma(C)} \mu_{n,k}\lambda_{n,k} \leq \pi_0 \qquad (7)$$

We only outline the main steps to demonstrate that (5) and (7) differ only by a multiplicative constant, which in turn proves the claim.

(I) Take any $(n,k) \in A(C) = C - H(C)$. There must exist $T \subseteq H(C)$ with $|T| = K - 1$ such that $n \in l(T)$. Let $\Phi = \Omega(l(T)) \cap C$. Clearly $z(\Phi) \in F_C$. Let $z(\Phi)'$ be identical to $z(\Phi)$ except that $\lambda(\Phi)'_{n,k} = \varepsilon$. For $\varepsilon > 0$ sufficiently small, $z(\Phi)' \in F_C$ because C is a strictly K-cover. Thus, for $z(\Phi)$ and $z(\Phi)'$ to belong to F_π, $z(\Phi)$ and $z(\Phi)'$ must meet (7) at equality, which leads us to conclude that $\mu_{n,k} = 0$. Consequently, (7) becomes

$$\sum\nolimits_{(n,k)\in\Gamma(C)} \pi_{n,k} x_{n,k} + \sum\nolimits_{(n,k)\in\Gamma(C)-A(C)} \mu_{n,k}\lambda_{n,k} \leq \pi_0$$

For the remaining steps, we only indicate the key points:

(II) show that $\mu_{n,k} = 0$ for all $(n,k) \in \Gamma(C) - A(C)$;
(III) show that $\pi_{n,k} = 0$ for all $(n,k) \in A(C)$;
(IV) show that $\pi_{n,k} = 0$ for all $(n,k) \in \Gamma(C) - A(C) - H(C)$, proving that (7) is $\sum_{(n,k)\in H(C)} \pi_{n,k} x_{n,k} \leq \pi_0$;
(V) show that $\pi_{n,i} = \pi_{m,j}$ for all $(n,i),(m,j) \in H(C)$;
(VI) conclude that (5) is a scalar multiple of (7).

The steps (I)–(VI) show that (5) is a facet of F_C. ■

A K-cover C where $K = |H(C)|$ is referred to as a *minimal cover*. Clearly, any subset $S \subseteq H(C)$ of a K-cover C yields a minimal cover $C_S = \Omega(l(S)) \cap C$ if $|S| = K$. The inequality (5) induced by a non-minimal K-cover C is stronger than the inequality yielded by a minimal cover $C_S \subset C$.

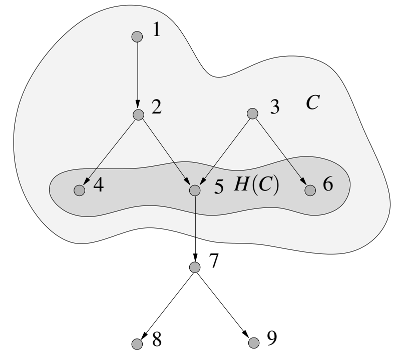

Fig. 2. Activation precedence graph

B. Illustrative Example of K-Cover

Take the activation precedence constraint graph G depicted in Figure 2 for a cluster of 9 wells. The injection rate available is $q_i^{max} = 6$ and the values $q_i^{n,k}$ appear in Table I.

An example of cover is $C = \{(1,2),(2,2),(3,2),(4,3),(5,3),(6,3)\}$. Condition (i) of Definition 2 is verified. Condition (ii) holds since $l(C) = \{1,2,3,4,5,6\} = N(C)$. Notice that $H(C) = \{(4,3),(5,3),(6,3)\}$ and $C - H(C) = \{(1,2),(2,2),(3,2)\}$ which implies condition (iii). Finally, $\gamma(C) = 0.5+1+1+2+2+2 = 8.5 > q_i^{max}$ and condition (iv) holds.

For $K = 2$, C is also a K-cover. The subsets of $H(C)$ with cardinality K are $S_{4,5} = \{(4,3),(5,3)\}$, $S_{4,6} = \{(4,3),(6,3)\}$, and $S_{5,6} = \{(5,3),(6,3)\}$, which induce $l(S_{4,5}) = \{1,2,3,4,5\}$, $l(S_{4,6}) = \{1,2,3,4,6\}$, and $l(S_{5,6}) = \{1,2,3,5,6\}$. Further, $\Phi_{4,5} = \Omega(l(S_{4,5})) \cap C = \{(1,2),(2,2),(3,2),(4,3),(5,3)\}$, $\Phi_{4,6} = \Omega(l(S_{4,6})) \cap C = \{(1,2),(2,2),(3,2),(4,3),(6,3)\}$, and $\Phi_{5,6} = \Omega(l(S_{5,6})) \cap C = \{(1,2),(2,2),(3,2),(5,3),(6,3)\}$. Notice that $\gamma(\Phi_{4,5}) = \gamma(\Phi_{4,6}) = \gamma(\Phi_{5,6}) = 6.5 > q_i^{max}$. However, for all $S_{n,m} \in \{S_{4,5}, S_{4,6}, S_{5,6}\}$, $\gamma(\Phi_{n,m} - \{(l,t)\}) = 4.5 \leq q_i^{max}$ for every

$(l,t) \in S_{n,m}$. Consequently, the K-cover inequality

$$\sum\nolimits_{(n,k)\in H(C)} x_{n,k} = x_{4,3} + x_{5,3} + x_{6,3} \leq 1 = K-1 \qquad (9)$$

is valid for $\mathscr{P}_C$ where $\Gamma(C) = \{(1,2),(2,2),\ (3,2),\ (4,2),\ (4,3),(5,2),\ (5,3),(6,2),(6,3)\}$. Actually, C is a strictly K-cover. For example, in the case of $\Phi_{4,5}$, $\gamma((\Phi_{4,5}-\{(4,3)\})\cup\{(4,2)\}) = 5.5 < q_i^{max}$ and $\gamma((\Phi_{4,5}-\{(5,3)\})\cup\{(5,2)\}) = 5.5 < q_i^{max}$. This property holds for $\Phi_{4,6}$ and $\Phi_{5,6}$ as well.

Because $\bigcap_{\{S\subseteq H(C):|S|=K-1\}} \Omega(l(S)) \cap C = [\Omega(l(\{(4,3)\})) \cap \Omega(l(\{(5,3)\})) \cap \Omega(l(\{(6,3)\}))] \cap C = \emptyset$, C satisfies condition (6) and consequently it induces a facet of $\mathscr{P}_C$.

TABLE I

INJECTION RATES FOR THE OIL WELLS

$n\backslash k$	Injection Levels $q_i^{n,k}$: 1	2	3	4	5	6	$\kappa(n)$	σ_n
1	0.5	2	3	4	5	6	6	0
2	1	2	3	4	5	5.5	6	0.5
3	1	2	3	4	5	6	6	0
4	1	2	3	4	4.5		5	1.5
5	1	2	3	3.5			4	2.5
6	1	2	3	4	5		5	1
7	1	2	2.5				3	3.5
8	1	1.5					2	4.5
9	1	1.5					2	4.5

C. Lifting K-Covers

Thus far we have shown that a strictly K-cover induces a maximal face of $\mathscr{P}_C$ which is a projection of $\mathscr{P}_{cpl}(G)$. Although (5) is valid for the feasible set of $P_{cpl}(G)$, in principle it can be "*lifted*" [8], [7], [11] to obtain a stronger and possibly facet-inducing inequality for $\mathscr{P}_{cpl}(G)$:

$$\sum_{(n,k)\in H(C)} x_{n,k} + \sum_{(n,k)\in\Omega-\Gamma(C)} \beta_{n,k}(C')x_{n,k} \leq K-1 \qquad (10)$$

where $\beta_{n,k}(C')$ are the lifting factors. Besides C, the lifting factors depend on the order $C' = \{(n_1,k_1),\ldots,(n_T,k_T)\}$ of $\Omega - \Gamma(C)$ in which they are lifted. These factors can be computed by solving a sequence $\{K_t(C') : t = 1,\ldots,T\}$ of problems akin to $P_{cpl}(G)$ defined recursively by:

$K_t(C')$:

$$\varepsilon_t \ = \ \text{Max} \sum_{(n,k)\in H(C)-\Omega_{n_t}} x_{n,k} + \sum_{j=1}^{t-1} \beta_{n_j,k_j}(C')x_{n_j,k_j} \qquad (11.0)$$

$$\text{S. to}: \sum_{(n,k)\in S_{t,n}} x_{n,k} \leq 1,\ \forall n \in N_t \qquad (11.1)$$

$$\sum_{(n,k)\in\Lambda_t} q_i^{n,k-1} x_{n,k} \leq q_i^{max} - q_i^{n_t,k_t-1} \qquad (11.2)$$

$$\sum_{(m,k)\in S_{t,m}} x_{m,k} \geq \sum_{(n,k)\in S_{t,n}} x_{n,k}, \forall (m,n) \in E[G[N_t]] \qquad (11.3)$$

$$\sum_{(m,k)\in S_{t,m}} x_{m,k} \geq 1, \forall m \in l(\{n_t\}) - \{n_t\} \qquad (11.4)$$

$$x_{n,k} \in \{0,1\},\ \forall (n,k) \in \Lambda_t \qquad (11.5)$$

for $t = 1,\ldots,T$, where:

- $N_t = N(C) \cup l(\{n_1,\ldots,n_t\}) - \{n_t\}$,
- $S_{t,n} = (C \cup \{(n_1,k_1),\ldots,(n_t,k_t)\}) \cap \Omega_n$,
- $\Lambda_t = \bigcup_{n\in N_t} S_{t,n}$.

and the coefficients $\beta_{n_j,k_j}(C')$ are defined by:

$$\beta_{n_j,k_j}(C') = K - 1 - \varepsilon_j,\ t = 1,\ldots,T-1 \qquad (12)$$

Proposition 6: If C' obeys a topological order of G ($\forall i,j \in \{1,\ldots,T\}, i<j$, either $n_j \not\prec n_i$ or $k_i < k_j$ if $n_i = n_j$), then inequality (10) is valid for $\mathscr{P}_{cpl}(G)$.

Proof: (*By induction in t*) For the basis, $t=1$, there are two possibilities. If $x_{n_1,k_1} = 0$, then (10) becomes the cover inequality (5) which is valid by definition. If $x_{n_1,k_1} = 1$, then (10) is valid if $\beta_{n_1,k_1}(C') \leq K-1-\max\{\sum_{(n,k)\in H(C)} x_{n,k}$: s.to (i) at most 1 pair (n,k) of each $n \in N(C)$ is active; (ii) the total gas allocated does not surpass the available rate discounted minimum required to activate pair (n_1,k_1); (iii) the precedence constraints are respected and all levels of activation appearing in $S_{t,n}$ are permitted for each $n \in N_t$; and (iv) pairs are either selected or not$\} = K-1-\max\{\sum_{(n,k)\in H(C)} x_{n,k}$: s.to (11.1)–(11.5)$\} = K-1-\varepsilon_1 = \beta_{n_1,k_1}(C')$ because C' is topologically ordered.

For the induction step, $t>1$, there are two possibilities. If $x_{n_t,k_t} = 0$, then (10) is valid by induction. If $x_{n_t,k_t} = 1$, then (10) is valid if $\beta_{n_t,k_t}(C') \leq K-1-\max\{\sum_{(n,k)\in H(C)} x_{n,k} + \sum_{j=1}^{t-1}\beta_{n_j,k_j}(C')x_{n_j,k_j}$: s.to (i) at most 1 pair (n,k) of each $n \in N_t$ is active; (ii) the total minimum gas allocated does not surpass the available rate discounted the minimum required to activate pair (n_t,k_t); (iii) the precedence constraints are respected and all levels of activation appearing in $S_{t,n}$ are permitted for each the $n \in N_t$; and (iv) pairs are either selected or not$\} = K-1-\max\{\sum_{(n,k)\in H(C)} x_{n,k} + \sum_{j=1}^{t-1}\beta_{n_j,k_j}(C')x_{n_j,k_j}$: s.to (11.1)–(11.5)$\} = K-1-\varepsilon_t = \beta_{n_t,k_t}(C')$ as C' is topologically ordered. ■

D. Pseudo-Lifting K-Covers

The task of computing exact lifting factors is comparable to solving $P_{cpl}(G)$ which may render the procedure impractical. One may otherwise attempt to compute approximate factors $\alpha_{n_j,k_j}(C)$ in place of $\beta_{n_j,k_j}(C')$, hereafter called *pseudo-lifting factors*. In introducing these approximate lifting factors, the notation below will be handy:

- $\overline{\Gamma}(C) = \Omega - \Gamma(C)$ has the variable indexes to be lifted;
- $\overline{\Gamma}_H(C) = \{(n,j) : (n,k) \in H(C), j = k+1,\ldots,\kappa(n)\}$ has the indexes to be lifted that have a counterpart in $H(C)$;
- $\overline{\Gamma}_A(C) = \{(n,j) : (n,k) \in C - H(C), j = k+1,\ldots,\kappa(n)\}$ has the indexes with counterpart in $C - H(C)$;
- $\overline{\Gamma}_B(C) = \overline{\Gamma}(C) - \overline{\Gamma}_H(C) - \overline{\Gamma}_A(C)$ has the other indexes;
- for any $n \in \mathscr{N}$ and $(m,j) \in \Omega - \Omega_n$, let:

$$\sigma(n)_{m,j} = \begin{cases} q_i^{m,j-1} + \sum_{l\in A_m - A_n^+} q_i^{l,1} & \text{if } m \notin A_n \\ q_i^{m,j-1} - q_i^{m,1} & \text{if } m \in A_n \end{cases}$$

  be the minimum amount of resources necessary to activate well m at level j, in the worst case, given that well n has been activated at some level;
- $H(C,n,k) = H(C) - \{(n,k)\}$;

- $H(C,n,k)_h \subseteq H(C,n,k)$ is such that:
  (i) $|H(C,n,k)_h| = h$ and
  (ii) $\min\{\sigma(n)_{m,j} : (m,j) \in H(C,n,k)_h\} \geq \max\{\sigma(n)_{m,j} : (m,j) \in H(C,n,k) - H(C,n,k)_h\}$,

  in other words, $H(C,n,k)_h$ is a subset of $H(C,n,k)$ with h pairs that need the highest amount of resources to be activated if a pair (n,j), $n \in N(H(C))$, is active;
- $H(C,n)_h \subseteq H(C)$ is such that:
  (i) $|H(C,n)_h| = h$ and
  (ii) $\min\{\sigma(n)_{m,j} : (m,j) \in H(C,n)_h\} \geq \max\{\sigma(n)_{m,j} : (m,j) \in H(C) - H(C,n)_h\}$,

  that is, $H(C,n)_h$ is a subset of $H(C)$ with h pairs that require the highest amount of resources to be activated if a pair (n,j) is active.

Taking advantage of the above notation, we can propose pseudo-lifting factors for the elements of $\overline{\Gamma}(C)$ that are tailored for the subsets $\overline{\Gamma}_H(C)$, $\overline{\Gamma}_A(C)$, and $\overline{\Gamma}_B(C)$.

Definition 6: The pseudo-lifting factor $\alpha_{n,k}$ for $(n,k) \in \overline{\Gamma}_H(C)$ is $\alpha_{n,k} = 1 + \max\{h : \sum_{(m,j)\in H(C,n,\delta(n))_h} \sigma(n)_{m,j} \leq q_i^{n,k-1} - q_i^{n,\delta(n)-1}\}$, where $(n,\delta(n)) \in H(C)$.

Definition 7: The pseudo-lifting factor $\alpha_{n,k}$ for $(n,k) \in \overline{\Gamma}_A(C)$ is $\alpha_{n,k} = \max\{h : \sum_{(m,j)\in H(C,n)_h} \sigma(n)_{m,j} \leq q_i^{n,k-1} - q_i^{n,1}\}$.

With respect to the elements of $\overline{\Gamma}_B(C)$, there is some choice on how the approximate lifting factors can be calculated. Let $\Theta_0, \Theta_1, \ldots, \Theta_R$ be a partition of $\overline{\Gamma}_B(C)$ such that for all $r, 1 \leq r \leq R$, the following holds:

(i) $H(\Theta_r) = \{(n_r,2),\ldots,(n_r,\kappa(n_r))\}$ for some $n_r \in \mathscr{N}$;
(ii) $m \prec n_r$ and $i = 2$ for all $(m,i) \in \Theta_r - H(\Theta_r)$.

Definition 8: Given a partition $\{\Theta_r\}_0^R$ of $\overline{\Gamma}_B(C)$, the pseudo-lifting factor $\alpha_{n,k}$ for $(n,k) \in \Theta_r$ is defined as follows:

- $\alpha_{n,k} = \max\{h : \sum_{(m,j)\in H(C,n)_h} \sigma(n)_{m,j} \leq q_i^{n,k-1} - q_i^{n,1}\}$ if $r = 0$;
- $\alpha_{n,k} = \max\{h : \sum_{(m,j)\in H(C,n)_h} \sigma(n)_{m,j} \leq q_i^{n,k-1} + \sum_{(m,j)\in\Theta_r - H(\Theta_r)} q_i^{m,j-1}\}$ if $r \geq 1$ and $(n,k) \in H(\Theta_r)$; and
- $\alpha_{n,k} = 0$ if $r \geq 1$ and $(n,k) \in \Theta_r - H(\Theta_r)$.

Proposition 7: Given a K-cover C and pseudo-lifting factors $\alpha_{n,k}$ obtained as above, the pseudo-lifted inequality:

$$\sum_{(n,k)\in H(C)} x_{n,k} + \sum_{(n,k)\in\Gamma(C)} \alpha_{n,k} x_{n,k} \leq K - 1 \qquad (13)$$

is valid for $\mathscr{P}_{cpl}(G)$.

E. Illustrative Example of Lifting and Pseudo-Lifting

The computation of lifting factors can be exemplified for the instance given in Table I with precedence constraint graph depicted in Figure 2. We take C' as the lexicographic ordering of the elements of $\overline{\Gamma}(C)$, $(m,i) \prec (n,j)$ in C' if $m < n$ or else $m = n$ and $i < j$, which induces a topological order of $\overline{\Gamma}(C)$. The application of the lifting and pseudo-lifting procedures yield the coefficients given in Table II. In computing the pseudo-lifting factors, the partition of $\overline{\Gamma}_B(C)$ was taken as $\Theta_0 = \emptyset$, $\Theta_1 = \Omega_7$, $\Theta_2 = \Omega_8$, and $\Theta_3 = \Omega_9$.

TABLE II
LIFTING AND PSEUDO-LIFTING COEFFICIENTS

	Coefficients $\beta_{n,k}(C')/\alpha_{n,k}(C)$				
$n \backslash k$	2	3	4	5	6
1		0 / 0	0 / 0	1 / 0	1 / 1
2		0 / 0	0 / 0	1 / 1	1 / 1
3		0 / 0	0 / 0	0 / 0	1 / 1
4			1 / 1	1 / 1	
5			1 / 1		
6			1 / 1	1 / 1	
7	0 / 0	1 / 1			
8	1 / 0				
9	1 / 0				

V. CONCLUSIONS AND FUTURE WORKS

This paper presented key concepts of the gas-lifted operation of oil wells and the lift-gas allocation problem under precedence constraints. A piecewise-linear reformulation was proposed to render the problem a mixed-integer linear programming problem, thereby allowing the application of the theory and algorithms from the domain of integer programming. Specifically, we simplified the piecewise-linear formulation to obtain a full dimensional polyhedron and extended K-covers for the problem at hand. The face induced by a K-cover was shown to be maximal for the projection of the polyhedron over the space of variables spanned by the cover. We also proposed exact and approximate lifting procedures to strengthen the original inequalities, both of which were illustrated in a simple scenario.

Future work will be focused on the design of procedures to separate K-covers, the extension of 1-configuration inequalities to $P_{cpl}(G)$, and the computational analysis of the developments heretofore in solving large problem instances.

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