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Stability and L_2-Gain Analysis of Systems with Time-Varying Delays: Input-Output Approach

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***Abstract*— Stability and L_2 (l_2)-gain analysis of linear (continuous-time and discrete-time) systems with uncertain bounded time-varying delays possessing non-zero nominal values and norm-bounded uncertainties. The delay derivatives (in the continuous-time) are not assumed to be less than $q < 1$. An input-output approach is applied by introducing a new input-output model, which leads to effective frequency domain and time domain criteria. The new method essentially improves the existing results for delays with derivative not greater than 1, which were treated in the past as fast-varying delays (without any constraints on the delay derivative). New BRLs are derived for systems with state and objective vector delays. Numerical examples illustrate the efficiency of the new method.**

Keywords: time-varying delay, small-gain theorem, BRL, Lyapunov-Krasovskii functional, uncertainties.

I. INTRODUCTION

The stability and control of systems with uncertain time-delay is a subject of recurring interest. Most of the works are based on application of different types of Lyapunov Krasovskii Functionals (LKFs) (see e.g. [1], [2],[6], [9]-[11]). The continuous-time systems with *fast varying delays* were for the first time treated via descriptor type LKF [2], where the derivative of the LKF along the trajectories of the system depends on the state and the *state derivative*.

Robust stability has been analyzed also via *IO* approach, which reduces the stability analysis of the uncertain system to the stability analysis of the class of systems with the same nominal part but with additional inputs and outputs. This approach was introduced for constant delays in [7]. The stability conditions for constant delays by LKFs were recovered by this approach in [12]. The method of [12] has been generalized to the case of slowly-varying delays (i.e. to delays with the derivative less than $q < 1$) in [6]. All the above works on IO approach consider the continuous-time case.

Frequency domain stability criteria for continuous-time and discrete-time systems with *fast-varying delays* have been derived in [8] in terms of transfer functions. In [4] a frequency domain stability criterion for continuous-time systems with fast-varying delays has been found via direct application of the Laplace transform.

The present paper is inspired by [6], where the IO approach was developed for the case where the delay derivative is smaller than $q < 1$. We develop here the IO approach to the continuous- and discrete-time systems with norm-bounded uncertainties and with fast varying delays from known segments. We introduce a new IO model with an output, which explicitly depends on $\dot{x}(t)$ ($x(k+1)-x(k)$). This corresponds to the term with $\dot{x}(t)$ in the derivative of descriptor LKF [2]. For the first time, we apply IO approach to $L_2(l_2)$-gain analysis. As a result, new BRLs with the delayed objective vector are obtained both, in the frequency and in the time domain. The time domain results are based on the application of the descriptor type LKF combined with the free weighting matrices technique of [11].

II. STABILITY AND BRL IN THE FREQUENCY DOMAIN

A. Robust stability: continuous-time systems

We consider the following linear system with uncertain coefficients and uncertain time-varying delays $\tau_i(t)$ (i=1,2):

$$\dot{x}(t)=(A_0+H\Delta E_0)x(t)+\sum_{i=1}^{2}(A_i+H\Delta E_i)x(t-\tau_i(t)), \quad (1)$$

where $x(t) \in \mathcal{R}^n$ is the system state, $A_i,\ E_i,\ i=0,1,2$ and H are constant matrices of appropriate dimensions and

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$\Delta(t)$ is a time-varying uncertain $n \times n$ matrix that satisfies

$$\Delta^T(t)\Delta(t) \le I_n. \tag{2}$$

The uncertain delays $\tau_i(t)$ are *piecewise-continuous* functions of the form

$$\tau_i(t) = h_i + \eta_i(t),\ i = 1, 2, \quad |\eta_i(t)| \le \mu_i \le h_i \tag{3}$$

with the known upper bounds μ_1 and μ_2.

To obtain a less conservative result in the case of sign-varying $\eta_1(t)$, we assume additionally that $t - \tau_1(t)$ is a *non-decreasing* function. The latter assumption means that $\tau_1(t)$ is differentiable almost for all $t \ge 0$ and $\dot{\tau}_1(t) = \dot{\eta}_1(t) \le 1$ almost for all $t \ge 0$. Note that this derivative constraint is less strong than $\dot{\tau}_1 \le q < 1$ of [6]. Moreover, $\tau_1(t)$ may include such delays that $t - \tau_1(t)$ is piecewise constant. This kind of delay appears [5] in sampled-data control $u(t) = Kx(t_k),\ t \in [t_k, t_{k+1})$ with the sampling times $0 = t_0 \le t_1 \le ...$ satisfying $t_{k+1} - t_k \le 2\mu_1$, where $x(t_k)$ can be represented as $x(t_k) = x(t - \mu_1 - \eta_1(t))$ with $\eta_1(t) = t - \mu_1 - t_k,\ t \in [t_k, t_{k+1})$.

We assume

A1 The nominal system

$$\dot{x}(t) = A_0x(t) + A_1x(t - h_1) + A_2x(t - h_2), \tag{4}$$

is asymptotically stable.

The results are easily generalized to the case of any finite number of the delays.

We represent (1) in the form:

$$\begin{array}{l}\dot{x}(t) = A_0x(t) + \sum_{i=1}^{2} A_ix(t - h_i) - \sum_{i=1}^{2} A_i \int_{-h_i-\eta_i}^{-h_i} \\ \dot{x}(t + s)ds + H\Delta[E_0x(t) + \sum_{i=1}^{2} E_ix(t - h_i) \\ - \sum_{i=1}^{2} E_i \int_{-h_i-\eta_i}^{-h_i} \dot{x}(t + s)ds].\end{array} \tag{5}$$

Following the idea of [7], [12], [6] to embed the perturbed system (5) into a class of systems with additional inputs and outputs, the stability of which guarantees the stability of (5), we introduce the following forward system:

$$\begin{array}{l}\dot{x}(t) = A_0x(t) + \sum_{i=1}^{2} A_ix(t - h_i) + \sum_{i=1}^{2} \sqrt{\mu_i}A_iu_i(t) \\ +Hu_3(t), \\ y_1(t) = \sqrt{\mu_1}\dot{x}(t), \quad y_2(t) = \sqrt{2\mu_2}\dot{x}(t), \\ y_3(t) = E_0x(t) + \sum_{i=1}^{2} E_ix(t - h_i) + \sum_{i=1}^{2} \sqrt{\mu_i}E_iu_i(t),\end{array} \tag{6a-d}$$

with feedback

$$\begin{array}{l}u_1(t) = -\frac{1}{\mu_1}\int_{-h_1-\eta_1}^{-h_1} y_1(t+s)ds, \\ u_2(t) = -\frac{1}{\sqrt{2\mu_2}}\int_{-h_2-\eta_2}^{-h_2} y_2(t+s)ds,\ u_3(t) = \Delta y_3(t).\end{array} \tag{7}$$

Note that $y_1(t)$ and $y_2(t)$ differ from the output of [7], [12], [6], and correspond to the term with $\dot{x}(t)$ in $\dot{V}_n$ by descriptor approach [2].

Let $u^T = [u_1^T\ u_2^T\ u_3^T]$, $y^T = [y_1^T\ y_2^T\ y_3^T]$. Then the forward system (6) can be written as $y = Gu$ with transfer function matrix

$$\begin{array}{l}G(s) = \begin{bmatrix}\sqrt{\mu_1}sI \\ \sqrt{2\mu_2}sI \\ E_0 + \sum_{i=1}^{2} E_ie^{-h_is}\end{bmatrix}(sI - A_0 - \sum_{i=1}^{2} A_ie^{-h_is})^{-1} \\ \begin{bmatrix}\sqrt{\mu_1}A_1 & \sqrt{\mu_2}A_2 & H\end{bmatrix} + \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{\mu_1}E_1 & \sqrt{\mu_2}E_2 & 0\end{bmatrix}.\end{array} \tag{8}$$

Assume that $y_i(t) = 0,\ \forall t \le 0,\ i = 1, 2, 3.$ the following result is proved.

Lemma 2.1: The following holds:

$$\|u_i\|_{L_2} \le \|y_i\|_{L_2},\ i = 1, 2, 3. \tag{9}$$

Proof. For $i = 1$ we have by Cauchy-Schwartz inequality for all $t \ge 0$

$$\begin{array}{l}\mu_1^2\|u_1(t)\|^2 = \| \int_{-h_1-\eta_1(t)}^{-h_1} y_1(t + s)ds\|^2 \le \\ \eta_1(t) \int_{t-h_1-\eta_1(t)}^{t-h_1} \|y_1(s)\|^2ds.\end{array} \tag{10}$$

Denote $s = p(t) = t - h_1 - \eta_1(t)$. Since $p(t)$ is a non-decreasing function, the set of segments $t \in [t_1, t_2]$, where $p(t)$ is constant, is countable, while out of these segments $p(t)$ is increasing. Hence, for almost all s the inverse $t = p^{-1}(s) = q(s)$ is well-defined and satisfies $|q(s) - s - h_1| \le \mu_1$. Then, integrating (10) in t from 0 to ∞, changing the order of integration and taking into account that $y_1(t) = 0,\ t \le 0$, we find that

$$\begin{array}{l}\mu_1^2\|u_1\|_{L_2}^2 \le \int_0^\infty \eta_1(t) \int_{t-h_1-\eta_1(t)}^{t-h_1} \|y_1(s)\|^2dsdt \\ = \int_0^\infty \int_{q(s)}^{s+h_1} \eta_1(q(s))\|y_1(s)\|^2dtds \le \mu_1^2\|y_1\|_{L_2}^2.\end{array}$$

For $i = 2$ the proof is similar. For $i = 3$ (9) is immediate.

From Lemma 2.1 it follows by the small gain theorem (see e.g. [6]) that the system (1) is IO stable (and thus asymptotically stable, since the nominal system is time-invariant) if

$$\|G\|_\infty < 1. \tag{11}$$

Theorem 2.1: Consider (1) with delays given by (3), where $\eta_i(t),\ i = 0, 1$ are piecewise-continuous functions and $\dot{\eta}_1(t) \le 1$ for almost all $t \ge 0$. Under A1 the system is asymptotically stable if (11) holds, where G is given by (8).

Remark 2.1: A stronger result is obtained by scaling G:

$$G_X(s) = diag\{X_1, X_2, \rho I_n\}G(s)diag\{X_1^{-1}, X_2^{-1}, \rho^{-1}I_n\},$$

where $X_i, i = 1,2$ are non-singular $n \times n$ matrices and $\rho \neq 0$ is a scalar. G is not scaled by $\sqrt{\mu_i}$ so as substituting of $\mu_i = 0$ leads to G_X, which corresponds to the case of known constant delays $\tau_i \equiv h_i$ and norm-bounded uncertainties.

Hence, under A1 (1) is asymptotically stable for all delays satisfying (3) if there exist X_i and ρ such that $\|G_X\|_\infty < 1$.

B. BRL: continuous-time systems

We consider the following linear system with uncertain coefficients and uncertain time-varying delays $\tau_i(t)$ (i=1,2) as above:

$$\begin{aligned}&\dot{x}(t) = (A_0 + H\Delta E_0)x(t) + \textstyle\sum_{i=1}^2 (A_i + H\Delta E_i)x(t-\tau_i(t))\\ &+(B_1 + H\Delta E_3)w(t),\\ &z(t) = C_0 x(t) + \textstyle\sum_{i=1}^2 C_i x(t-\tau_i(t)), \quad x(s) = 0,\ s \le 0,\end{aligned} \tag{12}$$

where $w(t) \in \mathcal{R}^q$ is an arbitrary disturbance vector in $L_2[0\ \infty)$ and $z(t) \in \mathcal{R}^p$ is the objective vector. Given $\gamma > 0$, we search a condition which guarantees that L_2-gain of (5) is less than γ, i.e. that the following inequality holds:

$$\|z\|_{L_2}^2 < \gamma^2 \|w\|_{L_2}^2, \quad \forall\ 0 \neq w \in L_2. \tag{13}$$

Consider an auxiliary system

$$\begin{aligned}&\dot{x}(t) = (A_0 + H\Delta E_0)x(t) + \textstyle\sum_{i=1}^2 (A_i + H\Delta E_i)x(t-\tau_i(t))\\ &+\gamma^{-1}(B_1 + H\Delta E_3)\bar{w}(t),\\ &z(t) = C_0 x(t) + \textstyle\sum_{i=1}^2 C_i x(t-\tau_i(t)),\\ &x(t) = 0, \quad t \le 0.\end{aligned} \tag{14}$$

It is clear that

$$\|z\|_{L_2}^2 < \|\bar{w}\|_{L_2}^2, \quad \forall\ 0 \neq \bar{w} \in L_2 \tag{15}$$

for (14) is equivalent to (13) for (5).

To derive 'scaled conditions' consider the following forward system

$$\begin{aligned}&\dot{x}(t) = A_0 x(t) + \textstyle\sum_{i=1}^2 A_i x(t-h_i) + \sum_{i=1}^2 \sqrt{\mu_i} A_i X_i^{-1} u_i(t)\\ &+\rho^{-1} H u_3(t) + \gamma^{-1} B_1 \bar{w}(t),\\ &y_1(t) = \sqrt{\mu_1} X_1 \dot{x}(t), \quad y_2(t) = \sqrt{2\mu_2} X_2 \dot{x}(t),\\ &y_3(t) = \rho[E_0 x(t) + \textstyle\sum_{i=1}^2 E_i x(t-h_i)\\ &+\textstyle\sum_{i=1}^2 \sqrt{\mu_i} E_i X_i^{-1} u_i(t) + \gamma^{-1} E_3 \bar{w}(t)],\\ &z(t) = C_0 x(t) + \textstyle\sum_{i=1}^2 C_i x(t-h_i) + \sum_{i=1}^2 \sqrt{\mu_i} C_i X_i^{-1} u_i(t),\end{aligned} \tag{16a-d}$$

with the feedback of (7).

The forward system (16) can be written as

$$\begin{bmatrix} y \\ z \end{bmatrix} = G_\gamma \begin{bmatrix} u \\ \bar{w} \end{bmatrix}, \quad u^T = [u_1^T\ u_2^T\ u_3^T\ ],\ y^T = [y_1^T\ y_2^T\ y_3^T] \tag{17}$$

with transfer matrix

$$\begin{aligned}G_\gamma(s) = &\begin{bmatrix} \sqrt{\mu_1} s X_1 \\ \sqrt{2\mu_2} s X_2 \\ \rho E_0 + \sum_{i=1}^2 \rho E_i e^{-h_i s} \\ C_0 + \sum_{i=1}^2 C_i e^{-h_i s} \end{bmatrix} (sI - A_0\\ &- \textstyle\sum_{i=1}^2 A_i e^{-h_i s})^{-1} [\sqrt{\mu_1} A_1 X_1^{-1}\ \sqrt{\mu_2} A_2 X_2^{-1}\ \frac{H}{\rho}\ \frac{B_1}{\gamma}]\\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho\sqrt{\mu_1} E_1 X_1^{-1} & \rho\sqrt{\mu_2} E_2 X_2^{-1} & 0 & \frac{\rho}{\gamma} E_3 \\ \sqrt{\mu_1} C_1 X_1^{-1} & \sqrt{\mu_2} C_2 X_2^{-1} & 0 & 0 \end{bmatrix}.\end{aligned} \tag{18}$$

Theorem 2.2: Assume A1. Given $\gamma > 0$, (12) is internally stable and has L_2-gain less than γ for all delays satisfying (3), if there exist non-singular $n \times n$-matrices X_1, X_2 and a scalar $\rho \neq 0$ such that

$$\|G_\gamma\|_\infty < 1. \tag{19}$$

Proof: Eq. (17) and (19) imply that

$$\|y\|_{L_2}^2 + \|z\|_{L_2}^2 < \|u\|_{L_2}^2 + \|\bar{w}\|_{L_2}^2.$$

The latter inequality together with $\|u\|_{L_2}^2 \le \|y\|_{L_2}^2$ yield (15) and (13).

C. Extension to the discrete-time delay systems

We consider the following linear discrete system with uncertain coefficients and uncertain time-varying delays $\tau_i(k)$ (i=1,2):

$$\begin{aligned}&x(k+1) = (A_0 + H\Delta E_0)x(k) + \textstyle\sum_{i=1}^2 (A_i + H\Delta E_i)\\ &x(k-\tau_i(k)) + (B_1 + H\Delta E_3)w(k),\\ &z(k) = C_0 x(k) + \textstyle\sum_{i=1}^2 C_i x(k-\tau_i(k)), \quad x(l) = 0,\ l \le 0,\end{aligned} \tag{20}$$

where $x(k) \in \mathcal{R}^n$ is the system state, $A_1,\ A_2, H,\ E_i$ and B_1 are constant matrices of appropriate dimensions and $\Delta(k)$ is a time-varying uncertain matrix that satisfies

$$\Delta^T(k)\Delta(k) \le I. \tag{21}$$

The uncertain delays $\tau_i(k)$ are supposed to have the following form:

$$\begin{aligned}&\tau_i(k) = h_i + \eta_i(k),\ i = 1,2,\\ &-h_i \le -\mu_{i-} \le \eta_i(k) \le \mu_{i+} \le h_i, \quad |\mu_{i-} - \mu_{i+}| \le 1\end{aligned} \tag{22}$$

with the known bounds $\mu_{i+} \ge 0$ and $\mu_{i-} \ge 0$. Note that similarly to the continuous-time case we choose h_i in the 'middle' of the delay interval. Denote $\mu_i = \max\{\mu_{i-}, \mu_{i+}\},\ i = 1,2$.

We assume additionally that $k - \tau_1(k)$ is increasing function, i.e. that τ_1 satisfies the following constraint: $\tau_1(k+$

$1) - \tau_1(k) \le 0$. Note that the constraint on τ_1 is more restrictive, than in the continuous-time case, where $t - \tau_1$ is supposed to be non-decreasing.

We assume

A1d The nominal system

$$x(k+1) = A_0x(k) + A_1x(k-h_1) + A_2x(k-h_2), \tag{23}$$

is asymptotically stable.

Given $\gamma > 0$, we are looking for a condition which guarantees that (20) is internally stable (i.e. asymptotically stable for $w = 0$) and has l_2-gain less than γ, i.e. that the following inequality holds:

$$\|z\|_{l_2}^2 < \gamma^2\|w\|_{l_2}^2, \quad \forall\ 0 \neq w \in l_2. \tag{24}$$

We represent (20) in the form:

$$\begin{array}{l} x(k+1) = A_0x(k) + \sum_{i=1}^2 A_ix(k-h_i) - \sum_{i=1}^2 A_i \\ \sum_{j=k-h_i-\eta_i}^{k-h_i-1}(x(j+1)-x(j)) \\ +H\Delta[E_0x(k) + \sum_{i=1}^2 E_ix(k-h_i) - \sum_{i=1}^2 E_i \\ \sum_{k-h_i-\eta_i}^{k-h_i-1}(x(j+1)-x(j))] + (B_1 + H\Delta E_3)w(k), \\ z(k) = C_0x(k) + \sum_{i=1}^2 C_ix(k-h_i) \\ -\sum_{i=1}^2 C_i\sum_{k-h_i-\eta_i}^{k-h_i-1}(x(j+1)-x(j))]. \end{array} \tag{25}$$

Consider the following forward system

$$\begin{array}{l} x(k+1) = A_0x(k) + \sum_{i=1}^2 A_ix(t-h_i) + \sum_{i=1}^2 \sqrt{\mu_i}A_i \\ X_i^{-1}u_i(k) + \rho^{-1}Hu_3(k) + \gamma^{-1}B_1w(k), \\ y_1(k) = \sqrt{\mu_1}X_1[x(k+1)-x(k)], \\ y_2(k) = \sqrt{\mu_{2-}+\mu_{2+}}X_2[x(k+1)-x(k)], \\ y_3(k) = \rho[E_0x(k) + \sum_{i=1}^2 E_ix(k-h_i) \\ +\sum_{i=1}^2\sqrt{\mu_i}E_iX_i^{-1}u_i(k) + \gamma^{-1}E_3w(k)], \\ z(k) = C_0x(k) + \sum_{i=1}^2 C_ix(k-h_i) + \sum_{i=1}^2\sqrt{\mu_i}C_iX_i^{-1}u_i(k), \end{array} \tag{26a-d}$$

with the feedback of

$$\begin{array}{l} u_1(k) = -\frac{1}{\mu_1}\sum_{j=k-h_1-\eta_1}^{k-h_1-1} y_1(j), \\ u_2(k) = -\frac{1}{\sqrt{\mu_{2-}+\mu_{2+}}}\sum_{j=k-h_2-\eta_2}^{k-h_2-1} y_2(j) \\ u_3(k) = \Delta y_3(k). \end{array} \tag{27}$$

The forward system (26) can be written as

$$\begin{bmatrix} y \\ z \end{bmatrix} = G_d \begin{bmatrix} u \\ \bar{w} \end{bmatrix}, \quad u^T = [u_1^T\ u_2^T\ u_3^T],\ y^T = [y_1^T\ y_2^T\ y_3^T] \tag{28}$$

with the transfer matrix

$$\begin{array}{l} G_d(z) = \begin{bmatrix} \sqrt{\mu_1}(z-1)X_1 \\ \sqrt{\mu_{2-}+\mu_{2+}}(z-1)X_2 \\ \rho E_0 + \sum_{i=1}^2 \rho E_iz^{-h_i} \\ C_0 + \sum_{i=1}^2 C_iz^{-h_i} \end{bmatrix}(zI - A_0 - \\ \sum_{i=1}^2 A_iz^{-h_i})^{-1}[\sqrt{\mu_1}A_1X_1^{-1}\ \sqrt{\mu_2}A_2X_2^{-1}\ \frac{H}{\rho}\ \frac{B_1}{\gamma}] \\ +\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho\sqrt{\mu_1}E_1X_1^{-1} & \rho\sqrt{\mu_2}E_2X_2^{-1} & 0 & \frac{\rho}{\gamma}E_3 \\ \sqrt{\mu_1}C_1X_1^{-1} & \sqrt{\mu_2}C_2X_2^{-1} & 0 & 0 \end{bmatrix}. \end{array} \tag{29}$$

Theorem 2.3: Assume A1d. Given $\gamma > 0$, (20) is internally stable and has l_2-gain less than γ for all delays satisfying (22) and $\tau_1(k+1) - \tau_1(k) \le 0$, if there exist non-singular $n \times n$-matrices $X_i,\ i = 1, 2$ and a scalar $\rho \neq 0$ such that

$$\|G_d\|_\infty < 1. \tag{30}$$

Proof is similar to the continuous-time case.

III. Stability and BRL in the time domain

In the continuous-time case, let V_n be Lyapunov-Krasovskii Functional (LKF), which guarantees the stability of the nominal system (4). The following condition along (16)

$$\begin{array}{l} \mathcal{W} \triangleq \dot{V}_n(t) + \|y(t)\|^2 + \|z(t)\|^2 - \|u(t)\|^2 - \|\bar{w}(t)\|^2 \\ < -\varepsilon(\|x(t)\|^2 + \|u(t)\|^2 + \|\bar{w}(t)\|^2),\ \varepsilon > 0 \end{array} \tag{31}$$

guarantees that the H_∞-norm of (16) is less than 1. Therefore, (31) is a sufficient condition for the feasibility of the frequency domain condition (19) of Theorem 2.2.

In the discrete-time case the corresponding condition along (26) has a form

$$\begin{array}{l} \mathcal{W}_d \triangleq V_n(k+1) - V_n(k) + \|y(k)\|^2 + \|z(k)\|^2 - \|u(k)\|^2 \\ -\|\bar{w}(k)\|^2 < -\varepsilon(\|x(k)\|^2 + \|u(k)\|^2 + \|\bar{w}(k)\|^2). \end{array} \tag{32}$$

Note that in this section we will derive conditions, which will be sufficient for the feasibility of the frequency domain conditions of Theorems 2.2 and 2.3. We choose the descriptor type V_n [2].

A. Discrete-time results

We combine the discrete-time descriptor LKF (see e.g. [1]):

$$\begin{array}{l} V_n(k) = x^T(k)P_1x(k) + \sum_{i=1}^2\sum_{m=-h_i}^{-1}\sum_{j=k+m}^{k-1}\bar{y}(j)^TR_i\bar{y}(j) \\ +\sum_{i=1}^2\sum_{j=k-h_i}^{k-1}x(j)^TS_ix(j),\ \bar{y}(k) = x(k+1) - x(k), \\ P_1 > 0,\ R > 0,\ S > 0. \end{array} \tag{33}$$

with the free weighting matrices technique of [11].

Lemma 3.1: The nominal system (23) is asymptotically stable if there exist $n\times n$ matrices $0<P_1,\ P_2,\ P_3,\ S_i>0$, $Y_{i1},\ Y_{i2}, Z_{i1},\ Z_{i2}, Z_{i3}, R_i>0, W_{i1}, W_{i2}, W_{i3}, T_i$ such that the following LMIs are feasible:

$$\Gamma_n=\begin{bmatrix} \Psi_n & P^T\begin{bmatrix}0\\A_1\end{bmatrix}-Y_1^T+\begin{bmatrix}T_1\\0\end{bmatrix}+h_1W_1 & P^T\begin{bmatrix}0\\A_2\end{bmatrix}-Y_2^T+\begin{bmatrix}T_2\\0\end{bmatrix}+h_2W_2 \\ * & -S_1-T_1-T_1^T+h_1W_{13} & 0 \\ * & * & -S_2-T_2-T_2^T+h_2W_{23} \end{bmatrix}<0,$$

$$\Xi_i=\begin{bmatrix} Z_i & W_i & Y_i^T \\ * & W_{i3} & T_i^T \\ * & * & R_i \end{bmatrix}\geq 0,\quad i=1,2. \tag{34a,b}$$

where

$$P=\begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix},\quad Y_i=[Y_{i1}\ Y_{i2}],\quad W_i=\begin{bmatrix} W_{i1} \\ W_{i2} \end{bmatrix},$$

$$Z_i=\begin{bmatrix} Z_{i1} & Z_{i2} \\ * & Z_{i3} \end{bmatrix},$$

$$\Psi_n=P^T\begin{bmatrix} 0 & I \\ A_0-I & -I \end{bmatrix}+\begin{bmatrix} 0 & A_0^T-I \\ I & -I \end{bmatrix}P+\sum_{i=1}^2 h_iZ_i$$

$$+diag\{\sum_{i=1}^2 S_i,\ P_1+\sum_{i=1}^2 h_iR_i\}+\sum_{i=1}^2\begin{bmatrix} Y_i \\ 0 \end{bmatrix}+\sum_{i=1}^2\begin{bmatrix} Y_i \\ 0 \end{bmatrix}^T. \tag{35a-e}$$

Thus, along the trajectories of (26) we have

$$V_n(k+1)-V_n(k)\leq[\xi_1^T(k)\ x^T(k-h_2)]\Gamma_n\begin{bmatrix} \xi_1(k) \\ x(k-h_2) \end{bmatrix}$$
$$+2\bar{x}^T(k)P^T\begin{bmatrix} 0 \\ \sum_{i=1}^2 \sqrt{\mu}_i A_iX_i^{-1}u_i(k)+\frac{H}{\rho}u_3(t)+\frac{B_1}{\gamma}\bar{w}(k) \end{bmatrix}$$

and

$$\mathcal{W}_d\leq\zeta^T(k)\Gamma\zeta(k)+\|y(k)\|^2+\|z(k)\|^2,$$

where

$$\zeta^T(k)=[\xi_1^T(k)\ x(k-h_2)^T\ u_1^T(k)\ u_2^T(k)\ u_3^T(k)\ \bar{w}^T(k)]$$

and

$$\Gamma=\begin{bmatrix} \Gamma_n & P^T\begin{bmatrix}0\\A_1X_1^{-1}\\0\\0\end{bmatrix} & P^T\begin{bmatrix}0\\A_2X_2^{-1}\\0\\0\end{bmatrix} & P^T\begin{bmatrix}0\\\frac{H}{\rho}\\0\\0\end{bmatrix} & P^T\begin{bmatrix}0\\\frac{B_1}{\gamma}\\0\\0\end{bmatrix} \\ * & -\mu_1^{-1}I_n & 0 & 0 & 0 \\ * & * & -\mu_2^{-1}I_n & 0 & 0 \\ * & * & * & -I_n & 0 \\ * & * & * & * & -I_q \end{bmatrix},$$

By applying Schur complements to the terms $\|y(k)\|^2+\|z(k)\|^2$ and multiplying the resulting matrix by $diag\{I_{4n},\mu_1X_1,\mu_2X_2,\rho I,\gamma I_q,I_{2n},\rho I_n,I_p\}$ and its transpose from the right and from the left we conclude that (32) is satisfied if

$$\begin{bmatrix} \Gamma_1 & \begin{matrix} 0 \\ \mu_1R_{1a} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ (\mu_{2-}+\mu_{2+})R_{2a} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} rE_0^T \\ 0 \\ rE_1^T \\ rE_2^T \\ r\mu_1E_1^T \\ r\mu_2E_2^T \\ 0 \\ rE_3^T \end{matrix} & \begin{matrix} C_0^T \\ 0 \\ C_1^T \\ C_2^T \\ \mu_1C_1^T \\ \mu_2C_2^T \\ 0 \\ 0 \end{matrix} \\ * & -\mu_1R_{1a} & 0 & 0 & 0 \\ * & * & -(\mu_{2-}+\mu_{2+})R_{2a} & 0 & 0 \\ * & * & * & -rI_n & 0 \\ * & * & * & * & -I_p \end{bmatrix}<0, \tag{36}$$

where $R_{ia}=X_i^TX_i$, $r=\rho^2$ and

$$\Gamma_1=\begin{bmatrix} \Gamma_n & \mu_1P^T\begin{bmatrix}0\\A_1\\0\\0\end{bmatrix} & \mu_2P^T\begin{bmatrix}0\\A_2\\0\\0\end{bmatrix} & P^T\begin{bmatrix}0\\H\\0\\0\end{bmatrix} & P^T\begin{bmatrix}0\\B_1\\0\\0\end{bmatrix} \\ * & -\mu_1R_{1a} & 0 & 0 & 0 \\ * & * & -\mu_2R_{2a} & 0 & 0 \\ * & * & * & -rI_n & 0 \\ * & * & * & * & -\gamma^2I_q \end{bmatrix},$$

We thus obtained the following.

Theorem 3.1: Given $\gamma>0$, (20) is internally stable and has L_2-gain less than γ for all delays satisfying (22), if there exist $n\times n$ matrices $0<P_1,\ P_2,\ P_3,\ S_i>0, Y_{i1},\ Y_{i2}$, $Z_{i1},\ Z_{i2}, Z_{i3},\ R_i, W_{i1}, W_{i2}, W_{i3}, T_i,\ R_{ia},\ i=1,2$ and a scalar $r>0$ such that LMIs (34b) and (36) are feasible.

B. Continuous-time results

By combining the descriptor model transformation and the corresponding LKF [2]

$$V_n=x^T(t)P_1x(t)+\sum_{i=1}^2\Big[\int_{-h_i}^0\int_{t+\theta}^t\dot{x}^T(s)R_i\dot{x}(s)dsd\theta$$
$$+\int_{t-h_i}^t x^T(s)S_ix(s)ds\Big],\ R_i>0,\ S_i>0,$$

with the technique of [11] and the arguments as above we obtain:

Theorem 3.2: Given $\gamma>0$, (12) is internally stable and has L_2-gain less than γ for all delays satisfying (22), if there exist $n\times n$ matrices $0<P_1,\ P_2,\ P_3,\ S_i>0, Y_{i1},\ Y_{i2}$, $Z_{i1},\ Z_{i2}, Z_{i3},\ R_i, W_{i1}, W_{i2}, W_{i3}, T_i,\ R_{ia},\ i=1,2$ and a scalar $\rho>0$ such that LMIs (34b) and (36) are feasible, where $\mu_{2-}+\mu_{2+}$ should be changed by $2\mu_2$ and Ψ_n should be substituted by Ψ_{nc} given by

$$\Psi_{nc}=P^T\begin{bmatrix}0 & I\\ A_0 & -I\end{bmatrix}+\begin{bmatrix}0 & A_0^T\\ I & -I\end{bmatrix}P+\sum_{i=1}^2 h_iZ_i$$
$$+\begin{bmatrix}\sum_{i=1}^2 S_i & 0\\ 0 & \sum_{i=1}^2 h_iR_i\end{bmatrix}+\sum_{i=1}^2\begin{bmatrix}Y_i\\ 0\end{bmatrix}+\sum_{i=1}^2\begin{bmatrix}Y_i\\ 0\end{bmatrix}^T.$$

C. Examples

In order to verify the conditions of Theorems 2.1 and 2.2 for the continuous case, or the one of Theorem 2.3 for discrete-time, a constant nonsingular matrix D of a specific diagonal block structure is sought that satisfies, for say G of Theorem 2.1, the following inequality:

$$D^{-T}G^T(-j\omega)D^TDG(j\omega)D^{-1}<I,\quad \forall\omega\in[0\ \infty).$$

By Remark 2.1, the matrix D should possess the structure $diag\{X_1,\ X_2,\ \rho I\}$.

Denoting $Q=D^TD$, the latter inequality becomes:

$$G^T(j\omega)QG(j\omega)<Q,\quad Q>0,\quad \forall\omega\in[0\ \infty)\qquad (37a,b)$$

which can be solved for Q of the above structure for preselected values of ω in $[0\ \infty)$.

Example 1 [11]: (*continuous-time*). Consider (1) with

$$A_0=\begin{bmatrix}-0.5 & -2\\ 1 & -1\end{bmatrix},\ A_1=\begin{bmatrix}-0.5 & -1\\ 0 & 0.6\end{bmatrix},\ A_2=E_2=0,$$
$$H=I,\ E_i=0.2I,\ i=0,1.$$

In this example, for $\dot\tau_1\le 0.9$, the following stability interval was obtained in [11]: $\tau_1(t)\in[0,0.242]$. The LMIs of Theorem 3.2 are feasible for all fast varying delays, where $h_1=\mu_1=0.146$. The LMIs of Theorem 3.2, where W and T with corresponding indices are taken to be zero, are feasible for smaller values: $h_1=\mu_1=0.133$. Hence, the system is stable for all fast varying delays from a larger interval: $\tau_1(t)\in[0,0.292]$. For delays $\dot\tau_1\le 1$ the corresponding interval is $[0,0.388]$. By applying the frequency domain result of Theorem 2.2, it is found that the system is asymptotically stable for delays from slightly wider intervals: [0 0.298] in the fast varying case and [0 0.4] if $\dot\tau_1\le 1$.

In the case of constant delay $\tau_1=h_1$ by Theorem 3.2 without free weighting matrices W and T with indices we find $h_1\le 0.68$, while the above matrices improve the result till $h_1\le 0.84$. In the case of known system matrices ($H=0$) the free weighting matrices do not improve the result. Thus in the fast varying case we have $h_1=\mu_1=0.34$.

IV. CONCLUSIONS

Stability and L_2 (l_2)-gain analysis of linear retarded type continuous-time and discrete-time systems with uncertain time-varying delays from given segments and norm-bounded uncertainties is studied. IO approach is applied to stability of systems with fast varying delays (without any constraints on the delay derivative) and is extended, for the first time, to BRL. A new IO model is introduced. New BRLs are derived in the case of delayed state and *objective* vector, which allows to solve delayed state-feedback H_∞ control problem. Both, frequency domain and time domain criteria are derived. Equivalent LMIs may be derived in the fast-varying delay case by direct application of LKF [3], where the same nominal LKF is added by additional terms which compensate the uncertainties. However, when the delay τ is not too fast ($\dot\tau(t)\le 1$ for almost all $t\ge 0$), the results of [3] are improved.

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