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Robust Sampled-Data Control of Linear Singularly Perturbed Systems

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***Abstract*— State-feedback H_∞ control problem for linear singularly perturbed systems with norm-bounded uncertainties is studied. The fast variables are sampled with fast rates, while for the slow variables both cases of slow and of fast sampling are considered. The recent 'input delay' approach to sampled-data control is applied, where the closed-loop system is represented as a continuous one with time-varying input delay. Linear Matrix Inequalities (LMIs) criteria are derived for stability and stabilization via input-output approach to stability and L_2-gain analysis of time-delay systems. Numerical example illustrates the efficiency of the method.**

Keywords: singularly perturbed systems, sampled-data control, H_∞ control, time-varying delay, LMI.

I. INTRODUCTION

Singular perturbations in control systems often occur due to the presence of small "parasitic" parameters, such as small masses, small time-delays. The main objective of singular perturbation methods is to alleviate the difficulties caused by the high dimensionality and the ill-conditioning that results from the interaction of slow and fast dynamical modes. Decomposition of the full-order problem to the ε-independent reduced-order slow and fast subproblems was started with the classical Tikhonov theorem on the asymptotic behavior of the solution to initial value problem [18] and developed further to composite controller design [2], [14] (see a survey [16] for recent references). A LMI approach to linear singularly perturbed systems was introduced in [6], where an ε-independent LMI was derived for stability analysis, while ε-dependent LMI gave a simple sufficient stability condition for the full-order system.

Two main approaches have been used to the sampled-data robust control. The first one is based on the lifting technique [1], [20] in which the problem is transformed to equivalent finite-dimensional discrete problem. This approach was applied to sampled-data nonlinear singularly perturbed systems , where the composite controller with the fast sampling in the fast variables was suggested [4]. The second approach is based on the representation of the system in the form of hybrid discrete/continuous model. This approach leads to necessary and sufficient conditions for stability and L_2-gain analysis in the form of differential equations (or inequalities) with jumps and it was applied to sampled-data H_∞ control of linear singularly perturbed systems [17], where the slow sampled-data controller was designed. The above approaches do not work in the cases with uncertain sampling times or uncertain system matrices.

A new 'input delay' approach to sampled-data control has been suggested recently in [7]. By this approach a digital control law is represented as a delayed control as follows:

$$u(t) = u_d(t_k) = u_d(t - (t - t_k)) = u_d(t - \tau(t)),\\ \tau(t) = t - t_k,\ t_k \le t < t_{k+1}, \tag{1}$$

where u_d is a discrete-time control signal and the time-varying delay $\tau(t) = t - t_k$ is piecewise linear with derivative $\dot{\tau}(t) = 1$ for $t \neq t_k$. Moreover, $\tau \le t_{k+1} - t_k$. The solution to the problem is found then by solving the problem for a continuous-time system with uncertain but bounded (by the maximum sampling interval) time-varying delay in the control input via Lyapunov technique. Given $h > 0$, the conditions obtained are robust with respect to different samplings with the only requirement that the maximum sampling interval is not greater than h.

Stability of singularly perturbed systems with *constant* delays has been studied in two cases: 1) h is proportional to ε and 2) ε and h are independent. The first case, being less general than the second one, is encountered in many publications (see e.g. [10], [9] and references therein). The second case has been studied in the frequency domain [15]. A Lyapunov-based approach to the problem leading

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to LMIs has been introduced in [6] for the general case of independent delay and ε. In the case of constant delay it was shown [6], that the necessary condition for robust stability of singularly perturbed system for all small enough values of singular perturbation parameter $\varepsilon > 0$ is the delay-independent stability of the fast subsystem, which is rather restrictive. The same is true for systems with uncertain and bounded time-varying delays, where constant delay is just a particular case of delay. Therefore, it is natural to design a delayed state-feedback controller with small delay in the fast variable $\varepsilon\tau(t)$, which corresponds to fast sampling of fast variables of [4].

In the present paper we derive a state-feedback sampled-data controller by applying the input delay approach to sampled-data control and by developing the input-output approach to singularly perturbed time-delay systems. The input-output approach was introduced for regular systems with constant delays in [12] and further developed in [11] (see also references therein), where it was generalized to the time-varying delays with the delay derivative less than $q < 1$. Recently the input-output approach has been developed to L_2-gain analysis of regular systems with time-varying bounded delays without any constraints on the delay derivative [8]. It is the objective of the present paper to develop this approach to singularly perturbed systems with time-varying delay. Two controller designs are suggested: 1) with the fast sampling in the fast variables and the slow one in the slow variables and 2) with the fast sampling in both variables.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. L_2 is the space of square integrable functions $v : [0,\infty) \to C^n$ with the norm $\|v\|_{L_2} = [\int_0^\infty \|v(t)\|^2 dt]^{1/2}$.

II. Problem Formulation

Given the following system:

$$\begin{aligned} &E_\varepsilon \dot{x}(t) = (A + H\Delta F_0)x(t) + (B_1 + H\Delta F_1)w(t) \\ &+(B_2 + H\Delta F_2)u(t), \\ &z(t) = Cx(t) + D_{12}u(t), \end{aligned} \tag{2}$$

where $x(t) = col\{x_1(t), x_2(t)\}$, $x_1(t) \in \mathcal{R}^{n_1}$, $x_2(t) \in \mathcal{R}^{n_2}$ is the system state vector, $u(t) \in \mathcal{R}^{\ell}$ is the control input, $w(t) \in R^q$ is the exogenous disturbance signal, and $z(t) \in \mathcal{R}^p$ is the state combination (objective function signal) to be attenuated. The matrix E_ε is given by

$$E_\varepsilon = diag\{I_{n_1}, \varepsilon I_{n_2}\}, \tag{3}$$

where $\varepsilon > 0$ is a small parameter.

Denote $n = n_1 + n_2$. The matrices $A, B_1, B_2, F_0, F_1, F_2$ and C are constant matrices of appropriate dimensions. The matrices in (2) have the following structures:

$$\begin{aligned} &A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}, \\ &F_0 = \begin{bmatrix} F_{01} & F_{02} \\ F_{03} & F_{04} \end{bmatrix}, B_i = \begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix}, \\ &C = [C_1\ C_2],\ F_i = \begin{bmatrix} F_{i1} \\ F_{i2} \end{bmatrix},\ i = 1, 2. \end{aligned} \tag{4}$$

We do not require A_4 to be nonsingular. Such a system is a *non-standard* singularly perturbed system [13]. In the case of singular A_4 open-loop system (2) with $\varepsilon = 0$ has index more than one and possesses an impulse solution [3].

The uncertain time-varying matrix $\Delta(t) = \begin{bmatrix} \Delta_1(t) & \Delta_2(t) \\ \Delta_3(t) & \Delta_4(t) \end{bmatrix}$ satisfies the inequality

$$\Delta^T(t)\Delta(t) \le I_n,\ t \ge 0. \tag{5}$$

We are looking for a piecewise-constant control law of two forms:

1) a multi (slow/fast) rate state-feedback

$$\begin{aligned} &u(t) = u_s(t) + u_f(t), \quad u_s(t) = K_1 x_1(t_k),\ t_k \le t < t_{k+1}, \\ &u_f(t) = K_2 x_2(\varepsilon t_k),\ \varepsilon t_k \le t < \varepsilon t_{k+1}, \end{aligned} \tag{6}$$

where $0 = t_0 < t_1 < ... < t_k < ...$ and $0 = \varepsilon t_0 < \varepsilon t_1 < ... < \varepsilon t_k < ...$ are the slow and the fast sampling instants and $\lim_{k\to\infty} t_k = \infty$.

2) a single (fast) rate state-feedback $u(t) = \bar{K}x(\varepsilon t_k)$, $\varepsilon t_k \le t < \varepsilon t_{k+1}$, where $0 = \varepsilon t_0 < \varepsilon t_1 < ... < \varepsilon t_k < ...$ are the fast sampling instants and $\lim_{k\to\infty} t_k = \infty$.

Given $\gamma > 0$ our objective is to find a piecewise constant controller which internally stabilizes the system and leads to L_2-gain less than γ

$$J = \|z\|^2_{L_2} - \gamma^2 \|w\|^2_{L_2} < 0 \tag{7}$$

for $x(0) = 0$ and for all non-zero $w \in L_2$.

We represent a piecewise-constant control law as a continuous-time control with a time-varying piecewise-continuous (continuous from the right) delay $\tau(t)=t-t_k$ as given in (1), corresponding to the slow sampling, and with small delay $\varepsilon\tau(t)=\varepsilon(t-t_k)$, corresponding to the fast sampling. We will thus look for state-feedback controllers of two forms:

$$u(t)=K\begin{bmatrix} x_1(t-\tau(t)) \\ x_2(t-\varepsilon\tau(t))\end{bmatrix}, \quad K=[K_1\ K_2] \tag{8}$$

and

$$u(t)=Kx(t-\varepsilon\tau(t)). \tag{9}$$

We assume that

A1. $t_{k+1}-t_k\le h\ \forall k\ge 0$.

From A1 it follows that $\tau(t)\le h$ since $\tau(t)\le t_{k+1}-t_k$.

To guarantee that for all small enough ε the full-order system is stabilizable-detectable we assume [19]:

A2. Both pencils $[sE_0-A;B_2]$ and $[sE_0-A^T;C^T]$ are of full row rank for all s with nonnegative real parts, where E_0 is given by (3) with $\varepsilon=0$.

A3. The triple $\{A_4,\ B_{22},\ C_2\}$ is stabilizable-detectable.

III. Multiple rate H_∞ Control

A. Input-output model

Substituting (8) into (2), we obtain the following closed-loop system:

$$\begin{aligned}&E_\varepsilon\dot x(t)=(A+H\Delta F_0)x(t)\\&+(B_2+H\Delta F_2)K\begin{bmatrix} x_1(t-\tau(t)) \\ x_2(t-\varepsilon\tau(t))\end{bmatrix}+(B_1+H\Delta F_1)w(t),\\&z(t)=Cx(t)+D_{12}K\begin{bmatrix} x_1(t-\tau(t)) \\ x_2(t-\varepsilon\tau(t))\end{bmatrix}.\end{aligned} \tag{10}$$

We will further consider (10) as the system with uncertain and bounded delay $\tau(t)\in[0,h]$.

We represent (10) in the form:

$$\begin{aligned}&E_\varepsilon\dot x(t)=(A+B_2K+H\Delta(F_0+F_2K))x(t)\\&-(B_2+H\Delta F_2)K\begin{bmatrix} \int_{-\tau(t)}^0\dot x_1(t+s)ds \\ \int_{-\varepsilon\tau(t)}^0\dot x_2(t+s)ds\end{bmatrix}\\&+(B_1+H\Delta F_1)w(t),\\&z(t)=(C+D_{12}K)x(t)-D_{12}K\begin{bmatrix} \int_{-\tau(t)}^0\dot x_1(t+s)ds \\ \int_{-\varepsilon\tau(t)}^0\dot x_2(t+s)ds\end{bmatrix}.\end{aligned} \tag{11}$$

Following the idea of [12], [11] to embed the perturbed system (11) into a class of systems with additional inputs and outputs, the stability of which guarantees the stability of (11), we introduce the following forward system:

$$\begin{aligned}&E_\varepsilon\dot x(t)=(A+B_2K)x(t)\\&+hB_2K\begin{bmatrix} v_1(t) \\ v_2(t)\end{bmatrix}+B_1w(t)+Hv_3(t),\\&z(t)=(C+D_{12}K)x(t)+hD_{12}K\begin{bmatrix} v_1(t) \\ v_2(t)\end{bmatrix},\\&\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix}=E_\varepsilon\dot x(t)=(A+B_2K)x(t)\\&+hB_2K\begin{bmatrix} v_1(t) \\ v_2(t)\end{bmatrix}+B_1w(t)+Hv_3(t),\\&y_3(t)=(F_0+F_2K)x(t)+hF_2K\begin{bmatrix} v_1(t) \\ v_2(t)\end{bmatrix}+F_1w(t),\end{aligned} \tag{12a-d}$$

with feedback

$$\begin{aligned}&v_1(t)=-\tfrac{1}{h}\int_{-\tau(t)}^0 y_1(t+s)ds,\\&v_2(t)=-\tfrac{1}{\varepsilon h}\int_{-\varepsilon\tau(t)}^0 y_2(t+s)ds,\\&v_3(t)=\Delta y_3(t).\end{aligned} \tag{13}$$

Note that for $h\to 0$ the above model (12), (13) corresponds to the closed-loop system (2) with the continuous state-feedback $u(t)=Kx(t)$.

Let $v^T=[v_1^T\ v_2^T\ v_3^T]$, $y^T=[y_1^T\ y_2^T\ y_3^T]$. Assume that $y_i(t)=0,\ \forall t\le 0,\ i=1,2,3$. The following holds for $n_i\times n_i$-matrices $R_i>0,\ i=1,2$ and a scalar $r>0$ [11]:

$$\begin{aligned}&\|\sqrt{R}_iv_i\|_{L_2}\le\|\sqrt{R}_iy_i\|_{L_2},\ i=1,2,\\&\|\sqrt{r}v_3\|_{L_2}\le\|\sqrt{r}y_3\|_{L_2}.\end{aligned} \tag{14}$$

For $\varepsilon\to 0$ inequality (14) is valid and y_2 given by (12c) vanishes. Thus, for $\varepsilon\to 0$ (12), (13) corresponds to the descriptor system without delay in x_2:

$$\begin{aligned}&E_0\dot x(t)=(A+H\Delta F_0)x(t)+(B_1+H\Delta F_1)w(t)\\&+(B_2+H\Delta F_2)u(t),\\&u(t)=K_1x_1(t_k)+K_2x_2(t),\quad t\in[t_k,t_{k+1}),\\&0\le t_{k+1}-t_k\le h.\end{aligned} \tag{15}$$

Remark 3.1: Descriptor system can be destabilized by arbitrary fast sampling in the fast variable of the feedback even if it is stable without the sampling. Consider the following simple example

$$E_0\dot x(t)=\begin{bmatrix} -1 & 0 \\ 1 & 1\end{bmatrix}x(t)+\begin{bmatrix} 0 \\ 1\end{bmatrix}u(t),\ x(t)\in R^2. \tag{16}$$

It is clear that the closed-loop system is stable with the continuous state-feedback $u(t)=-2x_2(t)$, while it is unstable with $u(t)=-2x(t_k),\ t\in[t_k,t_{k+1})$, for any sampling t_k. Really, the closed-loop triangular system is stable if equation $x_2(t) + u(t) = 0$ is stable. However, this equation in the sampled-data case $x_2(t) = 2x_2(t_k)$, $t \in [t_k, t_{k+1})$ is unstable.

B. L_2-gain analysis

Consider the Lyapunov function $V(t) = x^T(t)E_\varepsilon P_\varepsilon x(t)$, where P_ε has the structure of

$$P_\varepsilon = \begin{bmatrix} P_1 & \varepsilon P_2^T \\ P_2 & P_3 \end{bmatrix}, \; P_1 > 0, \; P_3 > 0. \tag{17}$$

Note that P_ε is chosen to be of the form of (17) (as e.g. in [19]), such that for $\varepsilon = 0$, the function V with $E_\varepsilon = E_0$ and $P_\varepsilon = P_0$, corresponds to the descriptor case).

Given $\varepsilon > 0$, from (14) it follows that the following condition along (12a)

$$\begin{array}{l} \mathcal{W} = \dot{V}(t) + y^T(t)diag\{hR_1, hR_2, r\}y(t) \\ -v^T(t)diag\{hR_1, hR_2, r\}v(t) + \|z(t)\|^2 - \|w(t)\|^2 \\ < -\alpha(\|x(t)\|^2 + \|u(t)\|^2 + \|w(t)\|^2), \; \alpha > 0 \end{array} \tag{18}$$

guarantees the internal stability of (10) and that L_2-gain of (10) less than γ. Moreover, since $y(t)$ depends on $\dot{x}(t)$, we consider the derivative condition $\dot{V}(t) \le -\beta(\|x(t)\|^2 + \|\dot{x}(t)\|^2), \; \beta > 0$. Such derivative condition corresponds to the descriptor model transformation introduced in [5].

We have similarly to [5]

$$\dot{V}(t) = 2x^T(t)P_\varepsilon^T E_\varepsilon \dot{x}(t) = 2\begin{bmatrix} x(t) \\ E_\varepsilon \dot{x}(t) \end{bmatrix}^T \mathcal{P}_\varepsilon^T \begin{bmatrix} E_\varepsilon \dot{x}(t) \\ 0 \end{bmatrix}, \tag{19}$$

where

$$\mathcal{P}_\varepsilon = \begin{bmatrix} P_\varepsilon & 0 \\ \Phi_2 & \Phi_3 \end{bmatrix}, \; \Phi_j = \begin{bmatrix} \Phi_{j1} & 0 \\ \Phi_{j2} & \Phi_{j3} \end{bmatrix}, \; j = 2, 3. \tag{20}$$

Setting in the right side of (19)

$$\begin{array}{l} 0 = (A + B_2K)x(t) + hB_2K\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \\ +B_1w(t) + Hv_3(t) - E_\varepsilon \dot{x}(t) \end{array}$$

and applying the Schur complements to the term $y^T(t)diag\{hR_1, hR_2, r\}y(t) + \|z(t)\|^2$ we conclude that (18) is satisfied if

$$\left[\begin{array}{cccc} \Gamma_\varepsilon & h\mathcal{P}_\varepsilon^T\begin{bmatrix} 0 \\ B_2K \end{bmatrix} & \mathcal{P}_\varepsilon^T\begin{bmatrix} 0 \\ H \end{bmatrix} & \mathcal{P}_\varepsilon^T\begin{bmatrix} 0 \\ B_1 \end{bmatrix} \\ * & -h\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} & 0 & 0 \\ * & * & -rI_n & 0 \\ * & * & * & -\gamma^2 I_q \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right.$$
$$\left.\begin{array}{ccc} r(F_0 + F_2K)^T & 0 & C^T + K^T D_{12}^T \\ 0 & h\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} & 0 \\ hrK^T F_2^T & 0 & hK^T D_{12}^T \\ 0 & 0 & 0 \\ rF_1^T & 0 & 0 \\ -rI & 0 & 0 \\ * & -h\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} & 0 \\ * & * & -I_p \end{array}\right] < 0,$$
$$\Gamma_\varepsilon = \mathcal{P}_\varepsilon^T\begin{bmatrix} 0 & I_n \\ A + B_2K & -I_n \end{bmatrix} + \begin{bmatrix} 0 & A^T + K^T B_2^T \\ I_n & -I_n \end{bmatrix}\mathcal{P}_\varepsilon. \tag{21}$$

We thus obtained the following

Lemma 3.1: (i) Given $\gamma > 0$ and $m \times n$-matrix K, (10) is internally stable and has L_2-gain less than γ for all small enough $\varepsilon > 0$ and $0 \le \tau(t) \le h$, if there exist $n_1 \times n_1$ matrices $P_1 > 0, \; R_1 > 0, \Phi_{21}, \Phi_{31}$, $n_2 \times n_2$ matrices $P_3 > 0, \; R_2 > 0, \Phi_{23}, \Phi_{33}$, $n_1 \times n_2$-matrices $P_2, \Phi_{22}, \Phi_{32}$ and a scalar $r > 0$ such that LMI (21) is feasible for $\varepsilon = 0$, where $\mathcal{P}_0$ is given by (17) and (20).
(ii) Given $\varepsilon > 0$, $m \times n$-matrix K and $\gamma > 0$, (10) is internally stable and has L_2-gain less than γ for all $0 \le \tau(t) \le h$, if there exist $n_1 \times n_1$ matrices $P_1 > 0, \; R_1 > 0$, $\Phi_{21}, \; \Phi_{31}$, $n_2 \times n_2$ matrices $P_3 > 0, \; R_2 > 0$, Φ_{23}, Φ_{33}, $n_1 \times n_2$-matrices $P_2, \Phi_{22}, \Phi_{32}$ and a scalar $r > 0$ such that LMI (21) is feasible and $E_\varepsilon P_\varepsilon > 0$, where $\mathcal{P}_\varepsilon$ is given by (17) and (20).

If (21) is feasible for $\varepsilon = 0$, then the *slow* (descriptor) (15) is internally stable and has L_2-gain less than $\gamma > 0$. Moreover, the following *fast* LMI

$$\left[\begin{array}{cccc} \Gamma_f & h\mathcal{P}_f^T\begin{bmatrix} 0 \\ B_{22}K_2 \end{bmatrix} & \mathcal{P}_f^T\begin{bmatrix} 0 \\ H_4 \end{bmatrix} & \mathcal{P}_f^T\begin{bmatrix} 0 \\ B_{12} \end{bmatrix} \\ * & -hR_2 & 0 & 0 \\ * & * & -rI_{n_2} & 0 \\ * & * & * & -\gamma^2 I_q \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right.$$
$$\left.\begin{array}{ccc} r(F_{04} + F_{22}K_2)^T & 0 & C_2^T + K_2^T D_{12}^T \\ 0 & hR_2 & 0 \\ hrK_2^T F_{22}^T & 0 & hK_2^T D_{12}^T \\ 0 & 0 & 0 \\ rF_{12}^T & 0 & 0 \\ -rI_{n_1} & 0 & 0 \\ * & -hR_2 & 0 \\ * & * & -I_p \end{array}\right] < 0,$$

$$\Gamma_f = \mathcal{P}_f^T \begin{bmatrix} 0 & I_{n_2} \\ A_4 + B_{22}K_2 & -I_{n_2} \end{bmatrix} + \begin{bmatrix} 0 & A_4^T + K_2^T B_{22}^T \\ I_{n_2} & -I_{n_2} \end{bmatrix} \mathcal{P}_f,\ \mathcal{P}_f = \begin{bmatrix} P_3 & 0 \\ \Phi_{23} & \Phi_{33} \end{bmatrix} \tag{22}$$

is feasible. The latter LMI guarantees that the *fast*

$$\begin{array}{l} \dot{x}_2(t) = (A_4 + H_4\Delta_4 F_4)x_2(t) + (B_{12} + H_4\Delta_4 F_{12})w(t) \\ +(B_{22} + H_4\Delta_4 F_{22})u(t), \\ u(t) = K_2 x_2(t_k),\ t \in [t_k, t_{k+1}),\ 0 \le t_{k+1} - t_k \le h, \end{array} \tag{23}$$

system is internally stable and has L_2-gain less than $\gamma > 0$. Thus the feasibility of ε-independent LMI (21), where $\varepsilon = 0$, implies that the fast subproblem is solvable by a sampled-data controller, while the slow subproblem is solvable by a mixed controller (continuous in the fast variable and sampled-data in the slow one).

C. State-feedback design

In order to obtain an LMI in (21) we have to restrict ourselves to the case of block-diagonal $\Phi_2 = diag\{\Phi_{21}, \Phi_{23}\}$ and to $\Phi_3 = \rho\Phi_2$, where $\rho \neq 0$ is a scalar parameter. Note that Φ_2 is non-singular due to the fact that the only matrix which can be negative definite in the second block on the diagonal of (21) is $-\rho(\Phi_2 + \Phi_2^T)$. Defining:

$$\Psi = \Phi_2^{-1} = diag\{\Phi_{21}^{-1}, \Phi_{23}^{-1}\},\ \bar{P} = \Psi^T P_0 \Psi,\ \bar{R} = \Psi^T R \Psi$$

$\bar{r} = r^{-1}$ and $Y = K\Psi$, multiplying LMI (21) by $diag\{\Psi,\ \Psi,\ \Psi,\ \bar{r}I_n, I_q, \bar{r}I_n, \Psi, I_p\}$ and its transpose, from the right and the left, respectively, we obtain the following LMI with a tuning parameter $\rho > 0$:

$$\left[\begin{array}{cccccccc} \Sigma_1 & \Sigma_2 & hB_2Y & \bar{r}H & B_1 & \Psi^T F_0 + Y^T F_2^T & 0 & \Psi^T C^T + Y^T D_{12}^T \\ * & -\rho(\Psi + \Psi^T) & h\rho B_2 Y & \bar{r}\rho H & \rho B_1 & 0 & h\bar{R} & 0 \\ * & * & -h\bar{R} & 0 & 0 & hY^T F_2^T & 0 & hY^T D_{12}^T \\ * & * & * & -\bar{r}I_n & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I_q & F_1^T & 0 & 0 \\ * & * & * & * & * & -\bar{r}I_n & 0 & 0 \\ * & * & * & * & * & * & -h\bar{R} & 0 \\ * & * & * & * & * & * & * & -I_p \end{array}\right] < 0, \tag{24}$$

$$\Sigma_1 = A\Psi + \Psi^T A^T + B_2 Y + Y^T B_2^T,$$
$$\Sigma_2 = \bar{P}^T - \Psi + \rho\Psi^T A^T + \rho Y^T B_2^T.$$

Note that $\bar{P}$ and $\bar{R}$ have the same, block-triangular and block-diagonal structures, as P_0 and R correspondingly.

Theorem 3.1: Given $\gamma > 0$, consider the system of (2) and the multi-rate state-feedback law of (8). Assume A1-A3.

(i) The state-feedback (8) internally stabilizes (2) and guarantees L_2-gain less than γ for all small enough $\varepsilon \ge 0$, if for some prescribed scalar $\rho \neq 0$ there exist $n_1 \times n_1$ matrices $\bar{P}_1 > 0,\ \bar{R}_1 > 0, \Psi_1$, $n_2 \times n_2$ matrices $\bar{P}_3 > 0,\ \bar{R}_2 > 0, \Psi_3$, an $n_1 \times n_2$-matrix $\bar{P}_2$, a $p \times n$ matrix Y and a scalar $r > 0$ such that LMI (24) with

$$\Psi = \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_3 \end{bmatrix},\ \bar{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ \bar{P}_2 & \bar{P}_3 \end{bmatrix},\ \bar{R} = \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & \bar{R}_2 \end{bmatrix} \tag{25}$$

is feasible. The state-feedback ε-independent gain is given by $K = Y\Psi^{-1}$;

(ii) The gain $K = [K_1\ K_2]$ obtained in (i) solves the slow (15) and the fast (23) subproblems;

(iii) Given $\varepsilon > 0$ the gain obtained in (i) internally stabilizes (2) and guarantees L_2-gain less than γ if there exist $n_1 \times n_1$ matrices $P_1 > 0,\ R_1 > 0,\ \Phi_{21},\ \Phi_{31}$, $n_2 \times n_2$ matrices $P_3 > 0,\ R_2 > 0,\ \Phi_{23},\ \Phi_{33}$, $n_1 \times n_2$-matrices $P_2, \Phi_{22}, \Phi_{32}$ and a scalar $r > 0$ such that LMI (21) is feasible and $E_\varepsilon P_\varepsilon > 0$, where P_ε is given by (17).

Example [17]: Consider (10) with

$$A_0 = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix},\quad B_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$
$$H = 0,\ C = \begin{bmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{\frac{1}{2}} \\ 0 \quad 0 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{26}$$

Given $\gamma = 3$ and the uniform sampling $t_{k+1} - t_k = 0.1$, it was shown in [17] that the slow state-feedback $u(t) = -1.1618x_1(t_k),\ t \in [t_k, t_{k+1})$ solves the H_∞-control problem for the full-order system for all small enough $\varepsilon > 0$. The slow controller of [17] can not achieve $J < 0$ for $\gamma < 2.85$. Applying Theorem 3.1 with the smaller $\gamma = 2.8$ and the same $h = 0.1$, we find that the multi-rate state-feedback of (8) with ε-independent gain $K = [-1.5888\ \ -0.4433]$ achieves $J < 0$ for $\gamma = 2.8$ and for all small enough $\varepsilon > 0$ and all samplings with $t_{k+1} - t_k \le 0.1$. Moreover, by applying Lemma 3.1 to the resulting closed-loop system for a greater value of $h = 0.16$ and for different values of $\varepsilon > 0$ ($\varepsilon = 0.01, 0.02, ...$) we verify that this state-feedback solves H_∞-control problem with $\gamma = 2.8$ for the full-order system with all samplings $0 \le t_{k+1} - t_k \le 0.16$ and for $0 < \varepsilon \le 0.49$. The estimates on the upper values of sampling and of ε, for which the H_∞ problem is still solvable, are the advantages of the LMI approach.

IV. On fast sample-rate H_∞ control

Substituting (9) into (2), we obtain (similarly to the previous case) the forward system (12), where v_1 should be changed to εv_1 and the feedback system (13), where v_1 should be changed by the following expression:

$$v_1(t) = -\frac{1}{\varepsilon h}\int_{-\varepsilon\tau(t)}^{0} y_1(t+s)ds.$$

Inequalities (14) are valid here. This leads to the corresponding changes in the LMIs. Thus the fast subproblem has the same form, while the slow one corresponds to the continuous-time state-feedback problem:

$$\begin{aligned} &E_0\dot{x}(t) = (A + H\Delta F)x(t) + (B_1 + H\Delta F_1)w(t) \\ &+(B_2 + H\Delta F_2)u(t),\ u(t) = Kx(t). \end{aligned} \tag{27}$$

The fast-rate controller naturally leads to better performance, than the multi-rate one. In the above example it achieves the smaller value of $\gamma = 2.6$ for all small enough values of $\varepsilon > 0$.

V. Conclusions

Sampled-data state-feedback H_∞ control problem for singularly perturbed system with norm-bounded uncertainties has been solved via input delay approach to sampled-data control. The only assumption on the sampling that the distance between the sequel sampling times is not greater than some $h > 0$. Two kinds of controllers have been designed (both with the fast sampling in the fast variable): the multi-rate state-feedback (slow rate in the slow variables) and the fast-rate state-feedback. The ε-independent gains of the controllers are found from ε-independent LMIs. ε-dependent LMIs are derived which give sufficient conditions for the solvability of the full-order system. An illustrative example shows that the fast-rate controller leads to better performance, than the multi-rate one. The trade-off is in the fast sampling of the slow variables and not only of the fast ones.

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