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Area contraction of k-dimensional surfaces and almost global asymptotic stability

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Abstract— **In this paper we will formulate sufficient conditions for the area contraction of k-dimensional surfaces under the flow of a set of differential equations. We discuss the connection with the Hausdorff dimension of invariant sets and show how the presence of first integrals of the system influences these results. We conclude with an application to almost global asymptotic stability.**

Index Terms— **k-contracting vector fields, Hausdorff dimension, first integrals**

I. INTRODUCTION

As is known from continuum mechanics, the divergence of a vector field equals the rate at which volumes increase or decrease, when flowing along the vector field. If the divergence has a fixed sign, then no sets with a finite volume exist that are invariant under the flow of the vector field. Of course a vector field can always be modified by multiplication with a positive function without changing the dynamics. When considering the divergence of the modified vector field and relating it to the rate of volume change of sets, this is equivalent to introducing a density function that associates a mass to (or redefines the volume of) a region in the state space.

This extra degree of freedom was exploited in [1] to establish a criterion for almost global asymptotic stability of an equilibrium point of a dynamical system, which means that the equilibrium point is locally stable and that the set of points in the state space that will not converge to the equilibrium point has zero volume. The criterion involved expansion of all volumes under the flow of the modified vector field (i.e. positive divergence) and the finiteness of the volume of the entire state space apart from some neighbourhood of the equilibrium point. The set of points not converging to the equilibrium point is invariant and if the equilibrium point is locally asymptotically stable this set is also bounded away from the equilibrium. It then has a finite volume and it follows that this volume must be zero. In the case that the equilibrium point is stable but not asymptotically stable, the considered set can be written as a (countable) union of invariant sets that are bounded away from the equilibrium point. It follows that each set has volume zero and therefore also its union.

To derive better bounds for the dimension of invariant sets one can consider the k-dimensional area of a k-dimensional surface in the state space ($k \leq n$, with n the dimension of the state space) and investigate how it evolves under the flow of a dynamical system. The contraction or expansion of the area of k-dimensional surfaces everywhere in the state space implies that no invariant surfaces can exist with a finite area of a(n) (integer) dimension larger than or equal to k (as we will show in this paper). Since this will only lead to results on regular surfaces, it is a restrictive result, but it can be generalised by using the concept of *Hausdorff dimension* [2], [3], [4] or *box-counting dimension* [5], [4]. A condition similar to the one for the contraction/expansion of k-dimensional surfaces can be derived to guarantee that *Hausdorff d-measures* (d not necessarily integer) decrease/increase along the flow of the vector field, implying that the *Hausdorff dimension* of a bounded invariant set cannot be larger than d.

Physical systems often exhibit symmetries and conservation laws, allowing us to derive stronger results. In this paper, we generalise a result of [6] by showing that, if a system has p conservation laws, the contraction (resp. expansion) of k-dimensional surfaces will lead to contraction (resp. expansion) of $k-p$-dimensional surfaces in an arbitrary level set of the conservation laws. The previously mentioned results can then be applied to give an upper bound for the dimension of invariant sets in this level set.

II. OUTLINE AND PRELIMINARIES

Consider a dynamical system in $\mathbb{R}^n$, given by the differential equation

$$\dot{x} = f(x),$$

and denote by $\phi_t(x)$ the solution with initial condition $\phi_0(x) = x$. (We assume that f is continuously differentiable in $\mathbb{R}^n$, and that the dynamical system has no finite escape time.) Further, assume there is a (positive definite) C^3 metric

g, taking the form

$$g = \sum_{i,j} g_{ij} \mathrm{d}x^i \mathrm{d}x^j.$$

(We let g denote both the metric and the (symmetric) matrix consisting of the elements g_{ij}.) For a vector function $v(x)$ we will use the notation $\frac{\partial v}{\partial x}$ to denote the matrix with $\frac{\partial v_i}{\partial x^j}$ on the i'th row and j'th column, while we will use $\det A_{ij}$ to denote the determinant of the matrix with A_{ij} on the i'th row and j'th column.

Although the results stated will be restricted to $\mathbb{R}^n$, they all can be generalised to arbitrary (but sufficiently smooth) orientable manifolds [7].

In the following section we will derive an expression for the area of a k-dimensional parallelepiped with respect to a time-dependent metric, and we will give an upper bound for its time-derivative. In section IV we will apply these results to give an expression for the area of a k-dimensional surface and an upper bound for its time-derivative when evolving under the flow of the dynamical system. This results in an upper bound for the dimension of regular bounded invariant sets. In order to extend this result to arbitrary bounded invariant sets, we will introduce the concept of Hausdorff measure in section V, after which we will discuss its evolution under the flow of the dynamical system and the consequences for the Hausdorff dimension of invariant sets.

In section VI we assume that the dynamical system has p first integrals and we show how the evolution of k-dimensional surfaces is related to the evolution of $k-p$-dimensional surfaces in the level set of the first integrals. We conclude with some applications of the stated results, one of which will be treated in more detail and clarified with an example.

III. Evolution of the volume of a parallelepiped

Consider a parallelepiped P_k spanned by k ($k \leq n$) linearly independent vectors $w_1, \ldots, w_k$ in a vector space $\mathbb{R}^n$ that is equipped with a metric, represented by the symmetric, positive definite matrix G. Then the length of the vector w_i equals

$$\sqrt{\langle w_i, w_i \rangle} = \sqrt{w_i^T G w_i}.$$

First we will assume a standard metric: $G = I_n$. Let B_k be an orthonormal basis in the k-dimensional subspace spanned by the w_i's. Define $W_n \in \mathbb{R}^{n \times k}$ and $W_k \in \mathbb{R}^{k \times k}$ by

$$\begin{aligned} W_n &= \begin{bmatrix} w_1 & \cdots & w_k \end{bmatrix}, \\ W_k &= \begin{bmatrix} [w_1]_{B_k} & \cdots & [w_k]_{B_k} \end{bmatrix}, \end{aligned}$$

where $[w_i]_{B_k}$ is the column vector containing the coordinates of w_i with respect to the basis B_k. Then the k-dimensional area/volume $\sigma_{k,s}(P_k)$ (with respect to the standard metric) of the aforementioned parallelepiped can be written as

$$\begin{aligned} \sigma_{k,s}(P_k) &= |\det W_k| \\ &= \sqrt{\det(W_k^T W_k)}, \end{aligned}$$

and since the element on row i, column j equals $[w_i]_{B_k}^T [w_j]_{B_k} = \langle w_i, w_j \rangle = w_i^T w_j$,

$$\begin{aligned} \sigma_{k,s}(P_k) &= \sqrt{\det(\langle w_i, w_j \rangle)} \\ &= \sqrt{\det(W_n^T I_n W_n)}. \end{aligned}$$

From now on we let the metric be arbitrary. The expression $\sqrt{\det(\langle w_i, w_j \rangle)}$ is not only coordinate independent, but also defines the k-dimensional area for a general metric G:

$$\begin{aligned} \sigma_k(P_k) &= \sqrt{\det(\langle w_i, w_j \rangle)} \\ &= \sqrt{\det(W_n^T G W_n)}. \end{aligned}$$

Assume that $G = G(t)$ is time-varying and consider the time-derivative of $(\sigma_k(P_k))^2$ for the case $k = 1$ ($W_n = w$):

$$\begin{aligned} \frac{\mathrm{d}(\sigma_1(P_1))^2}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(w^T G w) \\ &= w^T \frac{\mathrm{d}G}{\mathrm{d}t} w, \end{aligned}$$

which we rewrite as

$$\frac{\mathrm{d}(\sigma_1(P_1))^2}{\mathrm{d}t} = w^T G \mathcal{L} w,$$

with $\mathcal{L} = G^{-1}\frac{\mathrm{d}G}{\mathrm{d}t}$, because we only want w to appear in the combination $w^T G w$. So now we will show that we can bound this expression by the product of $\sigma_1(P_1)^2 = w^T G w$ and the largest eigenvalue of $\mathcal{L}$. First note that $G^{-\frac{1}{2}}\frac{\mathrm{d}G}{\mathrm{d}t}G^{-\frac{1}{2}}$ is symmetric ($G^{\frac{1}{2}}$ is the positive definite matrix satisfying $(G^{\frac{1}{2}})^2 = G$), such that there exists a $Q_0 \in \mathbb{R}^{n \times n}$ with

$$\begin{aligned} &G^{-\frac{1}{2}} \frac{\mathrm{d}G}{\mathrm{d}t} G^{-\frac{1}{2}} Q_0 = Q_0 \Lambda, \\ &Q_0^T Q_0 = I_n, \end{aligned}$$

where Λ is diagonal (and real) with $\Lambda_{11} \geq \cdots \geq \Lambda_{nn}$. Setting $Q_1 = G^{-\frac{1}{2}} Q_0$ we get

$$\begin{aligned} &\mathcal{L} Q_1 = Q_1 \Lambda, \\ &Q_1^T G Q_1 = I_n, \end{aligned}$$

and the columns of Q_1 form a basis of orthonormal (with respect to G) eigenvectors of $\mathcal{L}$. By writing w as a linear combination of these eigenvectors we get

$$\begin{aligned} \frac{\mathrm{d}(\sigma_1(P_1))^2}{\mathrm{d}t} &= w^T G \mathcal{L} w \\ &= w'^T Q_1^T G \mathcal{L} Q_1 w' \qquad (\text{with } w' = Q_1^{-1} w) \\ &= w'^T Q_1^T G Q_1 \Lambda w' \\ &= w'^T \Lambda w' = \sum_i 2\lambda_i w_i'^2 \qquad (\text{with } \lambda_i = \frac{1}{2}\Lambda_{ii}) \\ &\leq \sum_i 2\lambda_1 w_i'^2 = 2\lambda_1 w'^T w' \\ &= 2\lambda_1 w'^T Q_1^T G Q_1 w' \\ &= 2\lambda_1 w^T G w \\ &= 2\lambda_1 \sigma_1^2(P_1), \end{aligned}$$

and thus

$$\frac{\mathrm{d}\sigma_1(P_1)}{\mathrm{d}t} \le \lambda_1 \sigma_1(P_1).$$

For general k-values one can prove that ([8])

$$\frac{\mathrm{d}\sigma_k(P_k)}{\mathrm{d}t} \le (\lambda_1 + \cdots + \lambda_k)\sigma_k(P_k).$$

IV. Evolution of the area of k-dimensional surfaces

In the standard metric, the length $\sigma_{1,s}$ of a curve $\psi(V)$ in $\mathbb{R}^n$, represented by the function $\psi : V \to \mathbb{R}^n$, $V \subset \mathbb{R}$, is given by the well-known expression

$$\sigma_{1,s}(\psi(V)) = \int_{\psi(V)} \sqrt{\sum_i (\mathrm{d}x^i)^2} = \int_V \sqrt{\sum_i \left(\frac{\partial \psi^i}{\partial y}\right)^2} \mathrm{d}y.$$

For a general metric the length σ_1 equals

$$\begin{aligned}\sigma_1(\psi(V)) &= \int_{\psi(V)} \sqrt{\sum_{i,j} g_{ij} \mathrm{d}x^i \mathrm{d}x^j} \\ &= \int_V \sqrt{\sum_{i,j} g_{ij}(\psi(y)) \frac{\partial \psi^i}{\partial y}\frac{\partial \psi^j}{\partial y}} \mathrm{d}y.\end{aligned}$$

This formula can be extended to an expression for the area of surfaces of larger dimensions in the following way. Let V be a region in $\mathbb{R}^k$ such that the function $\psi : V \to \mathbb{R}^n$ defines a (smooth) k-dimensional surface in $\mathbb{R}^n$. Then the k-dimensional area $\sigma_k(U)$ (with $U = \psi(V)$) can be found by replacing W_n by $\frac{\partial \psi}{\partial y}\mathrm{d}y$ and G by $g(\psi(y))$ in the previous section and integrating over V:

$$\sigma_k(U) = \int_V \sqrt{\det\left(\frac{\partial \psi}{\partial y}^T g(\psi(y)) \frac{\partial \psi}{\partial y}\right)} \mathrm{d}y.$$

Now we let U evolve under the flow of the given dynamical system to obtain the time-variant surface $\phi_t(U) = \phi_t \circ \psi(V)$ and we consider its area:

$$\sigma_k(\phi_t(U)) = \int_V \sqrt{\det\left(\frac{\partial \psi}{\partial y}^T \frac{\partial \phi_t}{\partial x}^T g \frac{\partial \phi_t}{\partial x}\frac{\partial \psi}{\partial y}\right)} \mathrm{d}y$$

(where the argument of $\frac{\partial \psi}{\partial y}$ is y, that of $\frac{\partial \phi_t}{\partial x}$ is $\psi(y)$ and that of g is $\phi_t(\psi(y))$). To calculate the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_k(\phi_t(U))$, we first consider the matrix

$$\begin{aligned}\mathcal{L}(f,g) &= g^{-1}(x)\, \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{\partial \phi_t}{\partial x}(x)\right)^T g(\phi_t(x)) \frac{\partial \phi_t}{\partial x}(x)\right)\Bigg|_{t=0} \\ &= g^{-1}\frac{\partial f}{\partial x}^T g + g^{-1}\sum_i f^i \frac{\partial g}{\partial x^i} + \frac{\partial f}{\partial x},\end{aligned}$$

and denote the eigenvalues of $\frac{1}{2}\mathcal{L}(f,g)$ in $x \in \mathbb{R}^n$ by $\lambda_1(x) \ge \cdots \ge \lambda_n(x)$. Then it follows from section III (with $W_n = \frac{\partial \psi}{\partial y}\mathrm{d}y$ and $G(t) = (\frac{\partial \phi_t}{\partial x}(x))^T g(\phi_t(x))\frac{\partial \phi_t}{\partial x}(x))$ that

$$\begin{aligned}&\frac{\partial}{\partial t}\sqrt{\det\left(\frac{\partial \psi}{\partial y}^T \frac{\partial \phi_t}{\partial x}^T g \frac{\partial \phi_t}{\partial x}\frac{\partial \psi}{\partial y}\right)}\Bigg|_{t=t_0} \\ &\qquad \le (\lambda_1(x) + \cdots + \lambda_k(x))\sqrt{\det\left(\frac{\partial \psi}{\partial y}^T g(x) \frac{\partial \psi}{\partial y}\right)},\end{aligned}$$

where $x = \psi(y)$. In the expression of $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_k(\phi_t(U))$ we can bring $\frac{\mathrm{d}}{\mathrm{d}t}$ into the integrand (as $\frac{\partial}{\partial t}$) and after applying the previous formula we can put the sum of λ_i's in front of the integral by taking the supremum over the surface:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_k(\phi_t(U))\Big|_{t=t_0} \le \sup_{x\in U}\left(\lambda_1(x) + \cdots + \lambda_k(x)\right)\sigma_k(U).$$

This means that the supremum of the sum $\lambda_1(x)+\cdots+\lambda_k(x)$ gives an upper bound for the rate at which k-dimensional surfaces can increase.

Remark 1: Note that $\frac{1}{2}\mathcal{L}(f, I_n)$ is the symmetric part of $\frac{\partial f}{\partial x}$ and corresponds to the strain (resp. rate of deformation) tensor in continuum mechanics if we let f represent the displacement (resp. velocity) of an elastic medium (resp. fluid). This shows that $\mathcal{L}(f, I_n)$ contains all information about deformation under the flow of the vector field f.

Remark 2: For a general manifold M one can choose an arbitrary coordinate system on M to calculate $\frac{1}{2}\mathcal{L}(f,g)$ and its eigenvalues λ_i. Although $\frac{1}{2}\mathcal{L}(f,g)$ is dependent on the chosen coordinate system, its eigenvalues are not. The results in this paper remain valid on general manifolds [7].

Assume that for some region $\Omega \subset \mathbb{R}^n$, it is true that

$$\lambda_1(x) + \cdots + \lambda_k(x) \le 0, \qquad \forall\, x \in \Omega.$$

Then the area of any k-dimensional surface lying in Ω cannot increase under ϕ_t. So if the k-dimensional surface under consideration is invariant under the flow of the dynamical system, then on this surface we must have that $\lambda_1(x)+\cdots+\lambda_k(x) = 0$. Thus under some extra conditions on the set $\{x \in \Omega : \lambda_1(x) + \cdots + \lambda_k(x) = 0\}$ (e.g. demanding that its dimension is smaller than k) we can conclude that there can be no invariant k-dimensional surfaces in Ω with a finite area.

If we also have that

$$\sup_{x\in\Omega} \lambda_1(x) + \cdots + \lambda_k(x) < 0,$$

then we have uniform contraction of k-dimensional surfaces and then the previous result can be extended to arbitrary (but still bounded) sets by using a result of Reitmann [3] (that is based on an article by Douady and Oesterlé [2]). To explain this result, we first need to recall the definition of the Hausdorff dimension.

V. The Hausdorff dimension and the evolution of Hausdorff measures

Consider a bounded set S in $\mathbb{R}^n$. Cover S with a countable number of balls of radius $r_i < \epsilon$, with $\epsilon > 0$. For a given $d \in$

$[0, n]$ and $\epsilon > 0$, the *Hausdorff outer measure* $\mu_H(S, d, \epsilon)$ is defined as follows:

$$\mu_H(S, d, \epsilon) = \inf \sum_i r_i^d,$$

where the infimum is taken over all possible covers of S that satisfy $r_i < \epsilon$, $\forall\, i$. Keeping d fixed, $\mu_H(S, d, \epsilon)$ as a function of ϵ is decreasing and non-negative. Therefore, the *Hausdorff d-measure*, equal to

$$\mu_H(S, d) = \lim_{\epsilon \to 0} \mu_H(S, d, \epsilon) \in \mathbb{R}^+ \cup \{+\infty\},$$

is well-defined. If S is a smooth k-dimensional surface, this measure has the property that $\mu_H(S, k)$ is proportional to the k-dimensional area of the surface and therefore it can be considered as an extension to the notion of length, (k-dimensional) area and volume. It also follows that for a general set S there exists a d^* such that

$$\begin{aligned} d < d^* &\Rightarrow \mu_H(S, d) = +\infty, \\ d > d^* &\Rightarrow \mu_H(S, d) = 0. \end{aligned}$$

By definition, $d^* = \dim_H S$, the *Hausdorff dimension* of S. For instance, a two-dimensional surface in $\mathbb{R}^3$ will have $d^* = 2$ and the above inequalities can be interpreted by stating that it has an infinite length and zero (3-dimensional) volume.

For the evolution of Hausdoff d-measures we will split d in an integer part k and a fractional part s and consider the linear interpolation between $\lambda_1 + \cdots + \lambda_k$ and $\lambda_1 + \cdots + \lambda_{k+1}$. From results in [3] and [4] one can then obtain the following:

Theorem 1: Let Ω be a subset of a manifold M with

$$\sup_{x \in \Omega} \lambda_1(x) + \cdots + \lambda_k(x) + s\lambda_{k+1}(x) < 0,$$

where $k \in \{1, \ldots, n-1\}$ and $s \in [0, 1]$, and let S be a bounded set, satisfying $\phi_t(S) \subset \Omega$, $\forall\, t \in \mathbb{R}$. Then, if we set $d = k + s$, for each $c > 0$, there exists a $T > 0$ and a $\epsilon_0 > 0$, such that for all $t > T$ and $\epsilon \in (0, \epsilon_0)$

$$\mu_H(\phi_t(S), d, \epsilon) \le c\mu_H(S, d, \epsilon),$$

implying that

$$\mu_H(\phi_t(S), d) \le c\mu_H(S, d).$$

Since we can choose c as small as we want, this means that, under similar conditions as for the contraction of k-dimensional surfaces, we also have that the d-dimensional Hausdorff outer measure will decrease under the flow of the dynamical system (for sufficiently large values of T). So if S is invariant under ϕ_t, then we can choose $c < 1$ to obtain that for sufficiently small values of ϵ

$$\mu_H(S, d, \epsilon) = 0 \text{ and thus } \mu_H(S, d) = 0,$$

implying that

$$\dim_H S \le d.$$

Therefore there can be no bounded invariant sets in Ω with a Hausdorff dimension larger than d.

Remark 3: Although the condition of S being bounded and the definition of Hausdorff measure will depend on the chosen metric, under some mild conditions the Hausdorff dimension will not. This allows for deriving better upper bounds for the Hausdorff dimension by choosing an appropriate metric.

VI. The presence of first integrals

Assume there are p first integrals of the dynamical system, denoted by the column vector h, such that

$$\sum_i f^i \frac{\partial h}{\partial x^i} = 0,$$

and the matrix $\frac{\partial h}{\partial x}$ has full row rank everywhere in some region $\Omega \subset \mathbb{R}^n$. Then the level set

$$L_C = \{x : h(x) = C\},$$

with $C \in \mathbb{R}^p$, is invariant under ϕ_t and we can consider the restriction of the dynamical system to L_C. Let $\hat{g}$ be a metric in $L_C \cap \Omega$ which has to be determined yet.

In a neighbourhood $U_x \subset L_C \cap \Omega$ of some $x \in L_C \cap \Omega$ we can choose a coordinate system and calculate the corresponding $(n-p) \times (n-p)$-matrix $\mathcal{L}(f, \hat{g})$ for $\hat{g}$. The matrix $\mathcal{L}(f, \hat{g})$ determines how the area of higher dimensional surfaces evolves under ϕ_t in U_x with respect to $\hat{g}$. Let $\hat{\lambda}_1(x) \ge \cdots \ge \hat{\lambda}_{n-p}(x)$ denote the eigenvalues of $\frac{1}{2}\mathcal{L}(f, \hat{g})$ in x. (They are independent of the chosen coordinate system.) Choose an integer k with $p \le k < n$ and an $s \in (0, 1]$. Then we can prove the following.

Theorem 2: Under the above conditions, one can choose $\hat{g}$ in such a way that

$$\begin{aligned} &\hat{\lambda}_1(x) + \cdots + \hat{\lambda}_{k-p}(x) + s\hat{\lambda}_{k-p+1}(x) \\ &\qquad \le \lambda_1(x) + \cdots + \lambda_k(x) + s\lambda_{k+1}(x), \end{aligned}$$

$\forall\, x \in L_C \cap \Omega$, $\forall\, C \in \mathbb{R}^p$.

This means that, in the presence of p first integrals, the contraction of k-dimensional surfaces (resp. Hausdorff d-measures) leads to contraction of $k-p$-dimensional surfaces (resp. Hausdorff $d-p$-measures) in any level set of the p first integrals (but with respect to another metric).

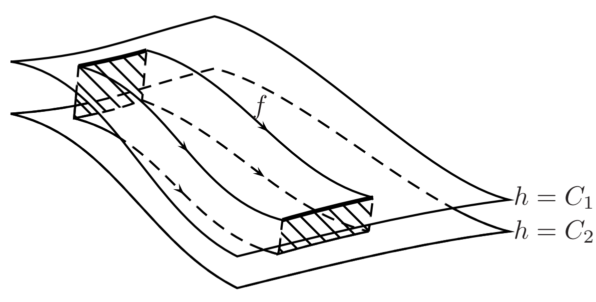

Fig. 1. Area contraction in a 3D system with one first integral.

We will not give a proof of the previous theorem (a proof can be found in [7] or [8]), but in this paragraph we will try to provide some intuition. In figure 1 a system is shown that contracts the area of 2-dimensional surfaces with respect to the standard metric. In this standard metric though, 1-dimensional curves (such as the thicker lines) are not contracted. However the system has a first integral h and we can define a new metric in the level surface $h = C_1$ by setting the length equal to (or proportional to) the area of the 2-dimensional surface that is formed by extending the curve in the direction of ∇h (for the thicker lines, these are the hatched surfaces). Since this area decreases under the flow of the system, so will the (newly defined) length of 1-dimensional curves lying in the level sets of h. By making the extensions infinitesimally small (and dividing the resulting area by some infinitesimally small factor — in figure 1 this could be $C_2 - C_1$) the length definition is local and can be represented by a metric $\hat{g}$.

We were also able to prove a converse theorem. If the system contracts $k-p$-dimensional surfaces in each level set (and some extra conditions are fulfilled), then the full system contracts k-dimensional surfaces (and analogously for Hausdorff measures) [7].

VII. Application to almost global asymptotic stability

The criterion from [1] mentioned in the introduction was generalised in [9] to include almost global asymptotical stability of an invariant set S, i.e. S is stable and the set R of points not converging to S has measure zero. The condition involved the expansion of n-dimensional volumes. (With our notation, this comes down to $\lambda_1(x) + \cdots + \lambda_n(x) > 0$ in some region.) Another condition guaranteed that R has a finite volume. Since R is invariant under the vector field it follows that it must have zero volume. This conclusion is equivalent to $\mu_H(R, n) = 0$. If the system also expands k-dimensional surfaces and if it has p first integrals, then (under an additional condition) we can use the previous results to provide more information about the set R and to bound the dimension of R.

So assume that f is a C^1 vector field with flow ϕ_t and with p first integrals h_i, such that $\frac{\partial h}{\partial x}$ has full row rank everywhere in $\Omega \subset \mathbb{R}^n$. Choose a $C \in \mathbb{R}^p$ and assume that the level set $L_C = \{x : h(x) = C\}$ is compact and lies entirely in Ω. Let d denote the distance function associated with the standard metric and let S denote a closed set, invariant under f and such that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in S_\delta : \phi_t(x) \in S_\epsilon, \quad \forall t > 0,$$

with $S_\epsilon = \{x \in \mathbb{R}^n : d(x, S) < \epsilon\}$. In other words: S is stable. Denote by R_C the set

$$R_C = \{x \in L_C : \limsup_{t\to\infty} d(\phi_t(x), S) \neq 0\}.$$

Then we can prove the following.

Theorem 3: If there exists a C^3 metric g defined on $\Omega \setminus S$ such that

$$\inf_{x\in L_C\setminus S} s\lambda_{n-k}(x) + \lambda_{n-k+1}(x) + \cdots + \lambda_n(x) > 0,$$

for some integer $k \in [p, n-1]$ and some $s \in (0, 1]$, ($\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ are the eigenvalues of $\frac{1}{2}\mathcal{L}(f, g)$ in x), then

$$\mu_H(R_C, k + s - p) = 0,$$

implying that $\dim_H R_C \leq k + s - p$.

Proof: Denote by $\lambda'_1(x) \geq \cdots \geq \lambda'_n(x)$ the eigenvalues of $\frac{1}{2}\mathcal{L}(-f, g)$ in x. Then $\lambda'_i(x) = -\lambda_{n+1-i}(x)$ and

$$\sup_{x\in L_C\setminus S} \lambda'_1(x) + \cdots + \lambda'_k(x) + s\lambda'_{k+1}(x) < 0.$$

By theorem 2 there exists a metric $\hat{g}$ such that

$$\sup_{x\in L_C\setminus S} \hat{\lambda}'_1(x) + \cdots + \hat{\lambda}'_{k-p}(x) + s\hat{\lambda}'_{k-p+1}(x) < 0,$$

where $\hat{\lambda}'_1(x) \geq \cdots \geq \hat{\lambda}'_n(x)$ are the eigenvalues of $\frac{1}{2}\mathcal{L}(-f, \hat{g})$ in x with respect to some coordinate system on L_C in a neighbourhood of x.

Now define $R_{C,\epsilon}$ ($\epsilon > 0$) by

$$R_{C,\epsilon} = \{x \in L_C : \limsup_{t\to\infty} d(\phi_t(x), S) \geq \epsilon\}.$$

Note that $R_C = \cup_{\epsilon>0} R_{C,\epsilon}$. The set $R_{C,\epsilon}$ is invariant under $-f$ and because of the stability of S (under f) there exists a $\delta > 0$, such that $R_{C,\epsilon} \subset L_C \setminus S_\delta$. Therefore $R_{C,\epsilon}$ is bounded (with respect to $\hat{g}$) and we can apply theorem 1 on the vector field $-f$ and the invariant set $R_{C,\epsilon}$ for arbitrary ϵ, resulting in

$$\mu_H(R_{C,\epsilon}, k + s - p) = 0.$$

If we choose a sequence of $\epsilon_i > 0$ for which $\epsilon_i \to 0$ as i tends to infinity, then we can write

$$\mu_H(R_C, k + s - p) \leq \sum_{i\in\mathbb{N}} \mu_H(R_{C,\epsilon_i}, k + s - p) = 0,$$

and thus $\dim_H R_C \leq k + s - p$. ■

Example 1: Consider the following vector field f in $\mathbb{R}^3$:

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_2x_3^2 - x_1x_3^2, \\ f_2(x_1, x_2, x_3) &= -x_1x_3^2 - x_2x_3^2, \\ f_3(x_1, x_2, x_3) &= (x_1^2 + x_2^2)x_3. \end{aligned}$$

One can easily verify that the function h, with

$$h(x) = x_1^2 + x_2^2 + x_3^2,$$

is a first integral for the system $\dot{x} = f(x)$, and $\frac{\partial h}{\partial x}$ has full row rank everywhere in $\Omega = \mathbb{R}^3 \setminus \{0\}$. The level sets $L_C = \{x \in \mathbb{R}^3 : h(x) = C\}$, with $C > 0$, are compact and lie entirely in Ω. From the expression for f_1 and f_2 it follows that the set $S = \{x \in \mathbb{R}^3 : (x_1, x_2) = (0, 0)\}$ is stable. With the metric

$$g = \frac{1}{x_1^2 + x_2^2} I_3$$

one can derive that the eigenvalues of $\frac{1}{2}\mathcal{L}(f, g)$ satisfy

$$\lambda\left(\lambda^2 - (x_1^2 + x_2^2 + x_3^2)\lambda - (x_1^2 + x_2^2)x_3^2\right) = 0,$$

and that, with $k = 2$ and $s > s_0 = 3 - 2\sqrt{2} \approx 0.17$,

$$\inf_{x\in L_C\setminus S} s\lambda_1(x) + \lambda_2(x) + \lambda_3(x) > 0.$$

So we can conclude that R_C has a Hausdorff dimension smaller than or equal to $k + s_0 - p = 4 - 2\sqrt{2} \approx 1.17$.

Indeed, from the differential equations it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1^2 + x_2^2) = -(x_1^2 + x_2^2)x_3^2,$$

such that the only points in $\mathbb{R}^3$ that will not converge to S lie in the plane $\{x \in \mathbb{R}^3 : x_3 = 0\}$, and thus R_C is the circle in this plane around the origin with radius $\sqrt{C}$ and has a Hausdorff dimension of 1. In figure 2 10 different trajectories belonging to the same level set ($C = 1$) are shown. They all start near the circle R_C in the (x_1, x_2)-plane and converge to the x_3-axis. Since the trajectories are symmetric about the (x_1, x_2)-plane one clearly sees that the points not converging to the x_3-axis must be lying on the aforementioned circle.

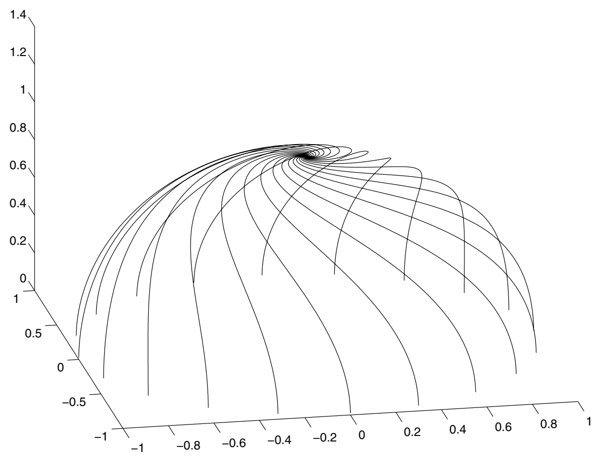

Fig. 2. A plot of 10 different trajectories in the same level set.

VIII. Other Applications

- For a dynamical system that contracts k-dimensional surfaces, every bounded orbit will converge to an ω-limit set with a Hausdorff dimension smaller than or equal to k.
- In [4] it was proven that the contraction of 2-dimensional surfaces in the neighbourhood of a limit cycle is a sufficient condition for the asymptotic stability of the limit cycle. The expansion of n-dimensional volumes is sufficient to conclude that the limit cycle is unstable.
- Length contraction ($k = 1$) leads to interesting properties and can be used for instance to analyse synchronisation phenomena [10].

When the dynamical system has known first integrals, these results can be made stronger or the conditions can be weakened.

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References

[1] A. Rantzer, “A dual to Lyapunov’s stability theorem.” *Systems and Control Letters*, vol. 42, pp. 161–168, 2001.

[2] A. Douady and J. Oesterlé, “Dimension de Hausdorff des attracteurs.” *C.R. Acad. Sci. Paris, Sér. A*, vol. 290, pp. 1135–1138, 1980.

[3] V. Reitmann and U. Schnabel, “Hausdorff dimension estimates for invariant sets of piecewise smooth maps.” *Zeitschrift für angewandte mathematik und mechanik*, vol. 80, no. 9, pp. 623–632, 2000.

[4] A. Noack, “Dimensions- und Entropieabschätzungen sowie Stabilitätsuntersuchungen für nichtlineare Systeme auf Mannigfaltigkeiten.” Ph.D. dissertation, TU Dresden, 1998.

[5] K. Gelfert, “Maximum local Lyapunov dimension bounds the box dimension. direct proof for invariant sets on Riemannian manifolds.” *Zeitschrift für analysis und ihre anwendungen*, vol. 22, no. 3, pp. 553–568, 2003.

[6] M. Fečkan, “A generalization of Bendixson’s criterion.” *Proceedings of the American Mathematical Society*, vol. 129, no. 11, pp. 3395–3399, 2001.

[7] D. Aeyels, F. De Smet, and B. Langerock, “Dimension estimates of invariant sets in the presence of first integrals and symmetry,” in preparation.

[8] ——, “Area contraction of multi-dimensional surfaces and almost global convergence in systems with first integrals,” in preparation.

[9] A. Rantzer and F. Ceragioli, “Smooth blending of nonlinear controllers using density functions,” in *Proceedings of European Control Conference*, 2001.

[10] J. Slotine, W. Wang, and K. El Rifai, “Contraction analysis of synchronization in networks of nonlinearly coupled oscillators.” in *Proceedings Sixteenth International Symposium on Mathematical Theory of Networks and Systems*, 2004.